Minimum Spanning Trees

CSE 373
Data Structures

Recall Spanning Tree

- Given (connected) graph G(V,E), a spanning tree T(V',E'):
  - › Spans the graph (V' = V)
  - › Forms a tree (no cycle);
  - › E' has |V| -1 edges

Minimum Spanning Tree

- Edges are weighted: find minimum cost spanning tree
- Applications
  - › Find cheapest way to wire your house
  - › Find minimum cost to send a message on the Internet

Strategy for Minimum Spanning Tree

- For any spanning tree T, inserting an edge e_{new} not in T creates a cycle
- But
  - › Removing any edge e_{old} from the cycle gives back a spanning tree
  - › If e_{new} has a lower cost than e_{old} we have progressed!

Strategy

- Strategy for construction:
  - › Add an edge of minimum cost that does not create a cycle (greedy algorithm)
  - › Repeat |V| -1 times
  - › Correct since if we could replace an edge with one of lower cost, the algorithm would have picked it up

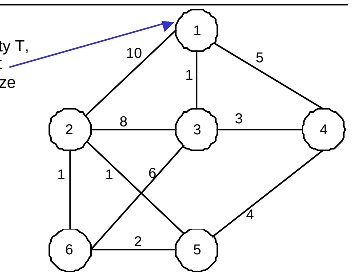
Two Algorithms

- Prim: (build tree incrementally)
  - › Pick lower cost edge connected to known (incomplete) spanning tree that does not create a cycle and expand to include it in the tree
- Kruskal: (build forest that will finish as a tree)
  - › Pick lower cost edge not yet in a tree that does not create a cycle and expand to include it somewhere in the forest

Prim's algorithm

Starting from empty T, choose a vertex at random and initialize

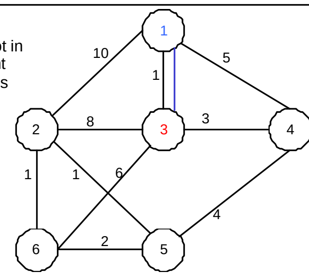
V = {1), E' ={}



Prim's algorithm

Choose the vertex u not in V such that edge weight from u to a vertex in V is minimal (greedy!)

V={1,3} E'= {(1,3) }



Prim's algorithm

Repeat until all vertices have been chosen

Choose the vertex u not in V such that edge weight from v to a vertex in V is minimal (greedy!)

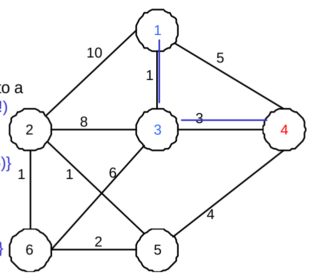
V= {1,3,4} E'= {(1,3),(3,4)}

V={1,3,4,5} E'={(1,3),(3,4),(4,5)}

….

V={1,3,4,5,2,6}

E'={(1,3),(3,4),(4,5),(5,2),(2,6)}



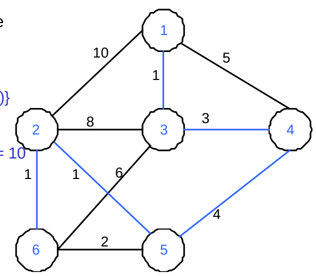
Prim's algorithm

Repeat until all vertices have been chosen

V={1,3,4,5,2,6}

E'={(1,3),(3,4),(4,5),(5,2),(2,6)}

Final Cost: 1 + 3 + 4 + 1 + 1 = 10



Prim's Algorithm Implementation

- Assume adjacency list representation

Initialize connection cost of each node to "inf" and "unmark" them
Choose one node, say v and set cost[v] = 0 and prev[v] =0
While they are unmarked nodes
    Select the unmarked node **u** with minimum cost; mark it
    For each unmarked node **w** adjacent to **u**
        if cost(u,w) < cost(w) then cost(w) := cost (u,w)
        prev[w] = u

- Looks a lot like Dijkstra's algorithm!

Prim's algorithm Analysis

- Like Dijkstra's algorithm
- If the "Select the unmarked node u with minimum cost" is done with binary heap then O((n+m)logn)

Kruskal's Algorithm

- Select edges in order of increasing cost
- Accept an edge to expand tree or forest only if it does not cause a cycle
- Implementation using adjacency list, priority queues and disjoint sets

Kruskal's Algorithm

Initialize a forest of trees, each tree being a single node
Build a priority queue of edges with priority being lowest cost
Repeat until |V| -1 edges have been accepted {
  Deletemin edge from priority queue
  If it forms a cycle then discard it
  else accept the edge – It will join 2 existing trees yielding a larger tree and reducing the forest by one tree
}
The accepted edges form the minimum spanning tree

Detecting Cycles

- If the edge to be added (u,v) is such that vertices u and v belong to the same tree, then by adding (u,v) you would form a cycle
  - Therefore to check, Find(u) and Find(v). If they are the same discard (u,v)
  - If they are different Union(Find(u),Find(v))

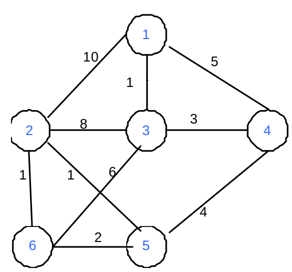
Properties of trees in K's algorithm

- Vertices in different trees are disjoint
  - True at initialization and Union won't modify the fact for remaining trees
- Trees form equivalent classes under the relation "is connected to"
  - u connected to u (reflexivity)
  - u connected to v implies v connected to u (symmetry)
  - u connected to v and v connected to w implies a path from u to w so u connected to w (transitivity)

K's Algorithm Data Structures

- Adjacency list for the graph
  - To perform the initialization of the data structures below
- Disjoint Set ADT's for the trees (recall Up tree implementation of Union-Find)
- Binary heap for edges

Example

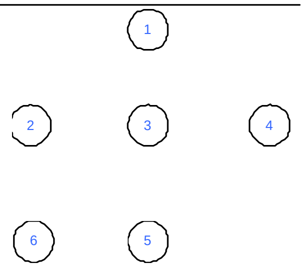


Initialization

Initially, Forest of 6 trees

F= {{1},{2},{3},{4},{5},{6}}

Edges in a heap (not shown)



Step 1

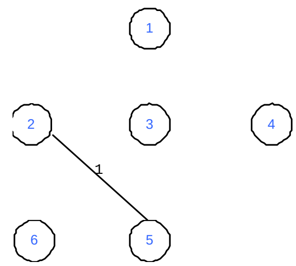
Select edge with lowest cost (2,5)

Find(2) = 2, Find (5) = 5

Union(2,5)

F= {{1},{2,5},{3},{4},{6}}

1 edge accepted



Step 2

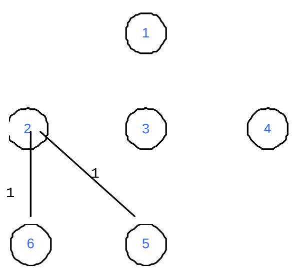
Select edge with lowest cost (2,6)

Find(2) = 2, Find (6) = 6

Union(2,6)

F= {{1},{2,5,6},{3},{4}}

2 edges accepted



Step 3

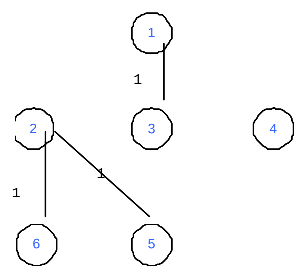
Select edge with lowest cost (1,3)

Find(1) = 1, Find (3) = 3

Union(1,3)

F= {{1,3},{2,5,6},{4}}

3 edges accepted



Step 4

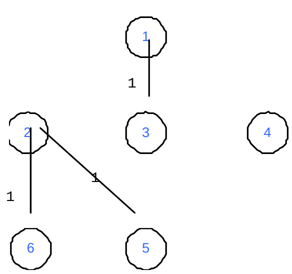
Select edge with lowest cost (5,6)

Find(5) = 2, Find (6) = 2

Do nothing

F= {{1,3},{2,5,6},{4}}

3 edges accepted



Step 5

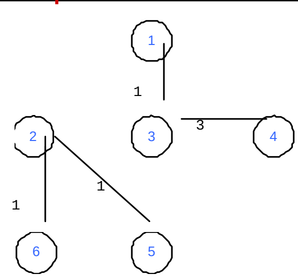
Select edge with lowest cost (3,4)

Find(3) = 1, Find (4) = 4

Union(1,4)

F= {{1,3,4},{2,5,6}}

4 edges accepted



Step 6

Select edge with lowest cost (4,5)

Find(4) = 1, Find (5) = 2

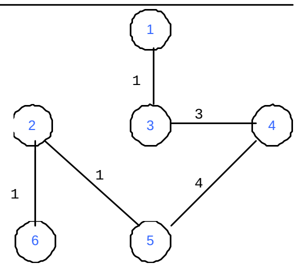
Union(1,2)

F= {{1,3,4,2,5,6}}

5 edges accepted : end

Total cost = 10

Although there is a unique spanning tree in this example, this is not generally the case



Kruskal's Algorithm Analysis

- Initialize forest O(n)
- Initialize heap O(m), m = |E|
- Loop performed m times
  - › In the loop one Deletemin O(logm)
  - › Two Find, each O(logn)
  - › One Union (at most) O(1)
- So worst case O(mlogm) = O(mlogn)

Time Complexity Summary

- Recall that $m = |E| = O(V^2) = O(n^2)$
- Prim's runs in O((n+m) log n)
- Kruskal's runs in O(mlogm) = O(mlogn)
- In practice, Kruskal has a tendency to run faster since graphs might not be dense and not all edges need to be looked at in the Deletemin operations