AVL Trees

CSE 373
Data Structures
Winter 2006

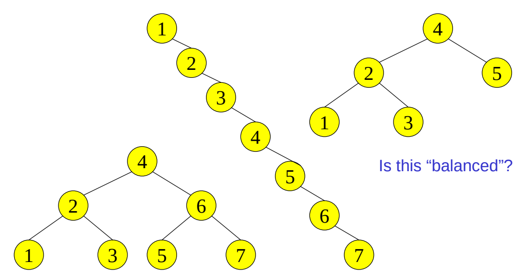
Binary Search Tree - Best Time

- All BST operations are O(d), where d is tree depth
- minimum d is $d = \lfloor \log_2 N \rfloor$ for a binary tree with N nodes
  - › What is the best case tree?
  - › What is the worst case tree?
- So, best case running time of BST operations is O(log N)

Binary Search Tree - Worst Time

- Worst case running time is O(N)
  - › What happens when you Insert elements in ascending order?
    - Insert: 2, 4, 6, 8, 10, 12 into an empty BST
  - › Problem: Lack of "balance":
    - compare depths of left and right subtree
  - › Unbalanced degenerate tree

Balanced and unbalanced BST



Approaches to balancing trees

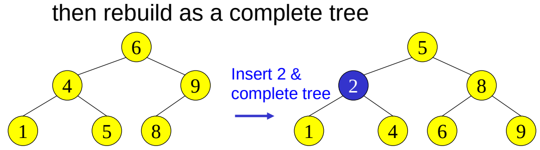
- Don't balance
  - › May end up with some nodes very deep
- Strict balance
  - › The tree must always be balanced perfectly
- Pretty good balance
  - › Only allow a little out of balance
- Adjust on access
  - › Self-adjusting

Balancing Binary Search Trees

- Many algorithms exist for keeping binary search trees balanced
  - › Adelson-Velskii and Landis (AVL) trees (height-balanced trees)
  - › Splay trees and other self-adjusting trees
  - › B-trees and other multiway search trees

Perfect Balance

- Want a complete tree after every operation
  - › tree is full except possibly in the lower right
- This is expensive
  - › For example, insert 2 in the tree on the left and then rebuild as a complete tree

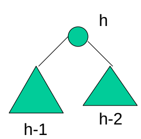


AVL - Good but not Perfect Balance

- AVL trees are height-balanced binary search trees
- Balance factor of a node
  - › height(left subtree) - height(right subtree)
- An AVL tree has balance factor calculated at every node
  - › For every node, heights of left and right subtree can differ by no more than 1
  - › Store current heights in each node

Height of an AVL Tree

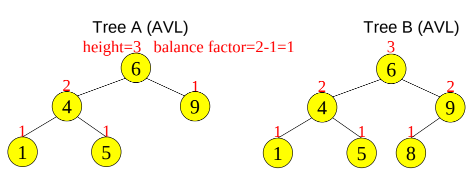
- N(h) = minimum number of nodes in an AVL tree of height h.
- Basis
  - › N(0) = 1, N(1) = 2
- Induction
  - › N(h) = N(h-1) + N(h-2) + 1
- Solution (Fibonacci)
  - › $N(h) \geq \phi^h$ ($\phi \approx 1.62$)



Height of an AVL Tree

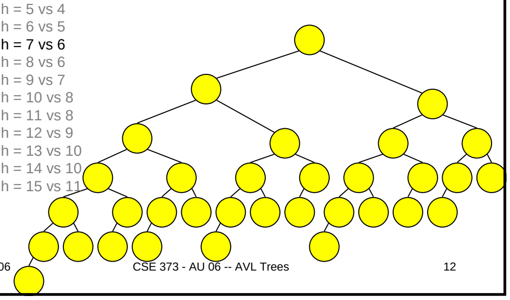
- $N(h) \geq \phi^h$ ($\phi \approx 1.62$)
- Suppose we have n nodes in an AVL tree of height h.
  - › $n \geq N(h)$ (because N(h) was the minimum)
  - › $n \geq \phi^h$ hence $\log_\phi n \geq h$ (relatively well balanced tree!!)
  - › $h \leq 1.44 \log_2 n$ (i.e., O(height) takes O(log n))

Node Heights

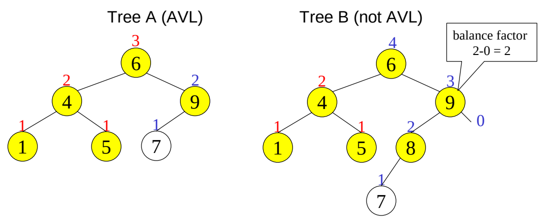


Worst-case AVL Trees

4 nodes: h = 3 vs 3
7 nodes: h = 4 vs 3
12 nodes: h = 5 vs 4
20 nodes: h = 6 vs 5
33 nodes: h = 7 vs 6
54 nodes: h = 8 vs 6
88 nodes: h = 9 vs 7
143 nodes: h = 10 vs 8
232 nodes: h = 11 vs 8
376 nodes: h = 12 vs 9
609 nodes: h = 13 vs 10
986 nodes: h = 14 vs 10
1596 nodes: h = 15 vs 11



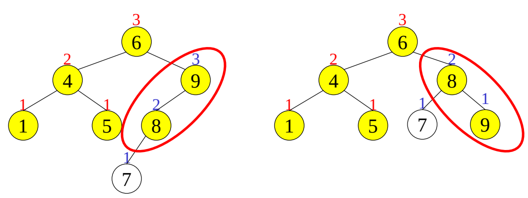
Node Heights after Insert 7



Insert and Rotation in AVL Trees

- Insert operation may cause balance factor to become 2 or –2 for some node
  - only nodes on the path from insertion point to root node have possibly changed in height
  - So after the Insert, go back up to the root node by node, updating heights
  - If a new balance factor (the difference h_{left}-h_{right}) is 2 or –2, adjust tree by *rotation* around the node

Single Rotation in an AVL Tree



Insertions in AVL Trees

Let the node that needs rebalancing be α.

There are 4 cases:

Outside Cases (require single rotation) :

1. Insertion into left subtree of left child of α.
2. Insertion into right subtree of right child of α.

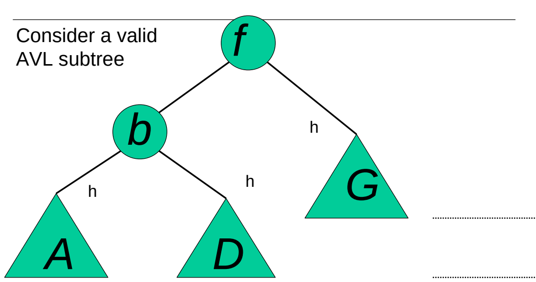
Inside Cases (require double rotation) :

3. Insertion into right subtree of left child of α.
4. Insertion into left subtree of right child of α.

The rebalancing is performed through four separate rotation algorithms.

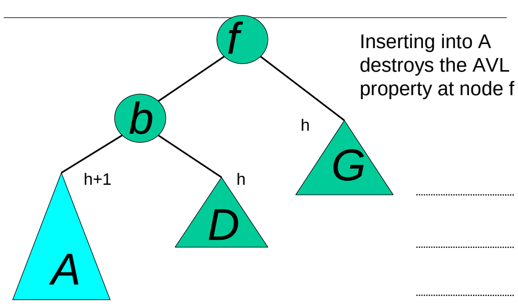
AVL Insertion: Outside Case

Consider a valid AVL subtree



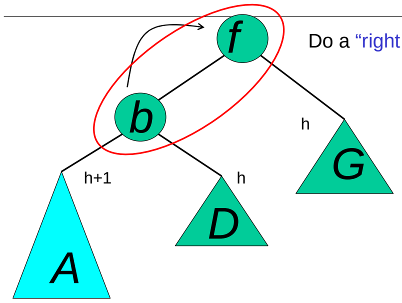
AVL Insertion: Outside Case

Inserting into A destroys the AVL property at node f



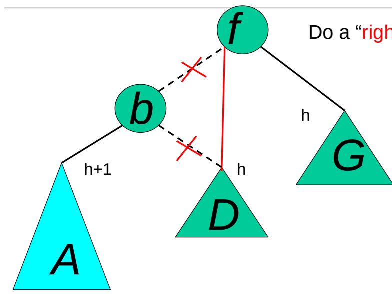
AVL Insertion: Outside Case

Do a “right rotation”



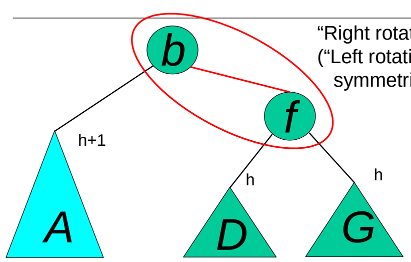
Single right rotation

Do a “right rotation”



Outside Case Completed

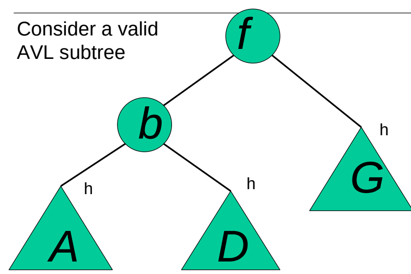
“Right rotation” done
(“Left rotation” is mirror symmetric)



AVL property has been restored

AVL Insertion: Inside Case

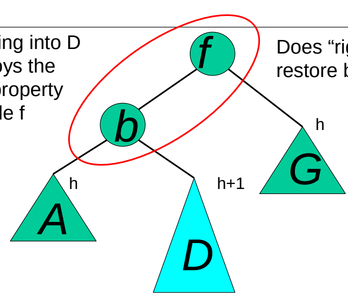
Consider a valid AVL subtree



AVL Insertion: Inside Case

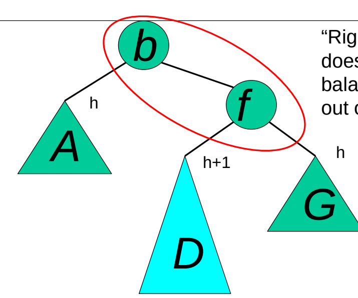
Inserting into D destroys the AVL property at node f

Does “right rotation” restore balance?



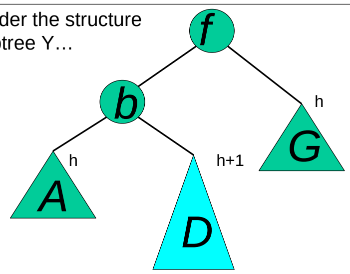
AVL Insertion: Inside Case

“Right rotation” does not restore balance… now b is out of balance



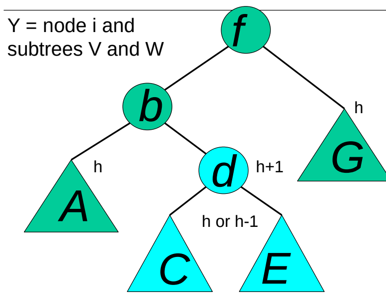
AVL Insertion: Inside Case

Consider the structure of subtree Y…



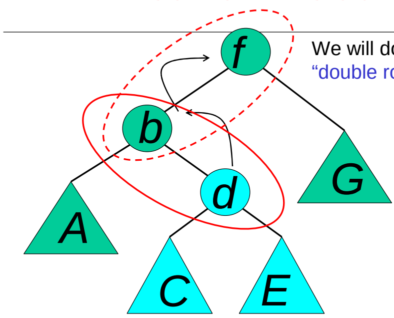
AVL Insertion: Inside Case

Y = node i and subtrees V and W



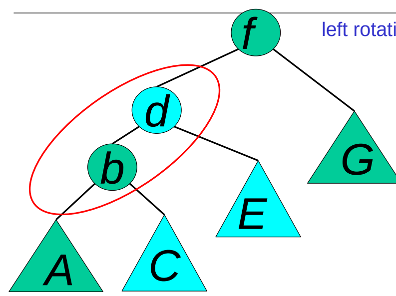
AVL Insertion: Inside Case

We will do a left-right "double rotation" . . .



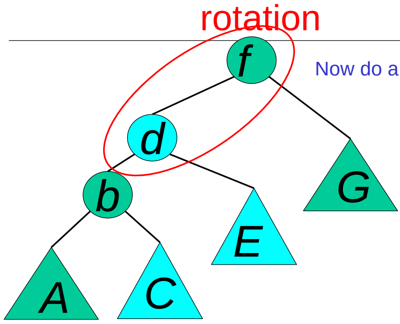
Double rotation : first rotation

left rotation complete



Double rotation : second rotation

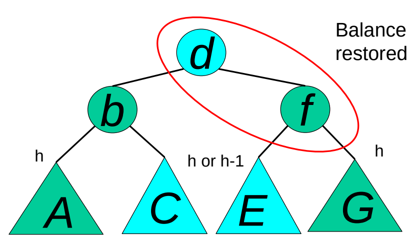
Now do a right rotation



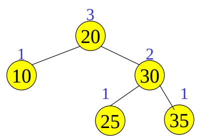
Double rotation : second rotation

right rotation complete

Balance has been restored

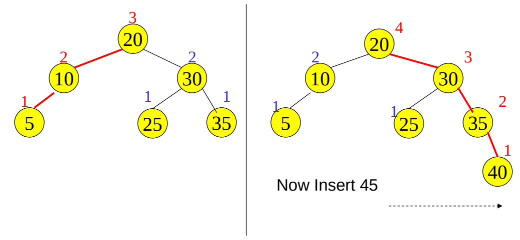


Example of Insertions in an AVL Tree

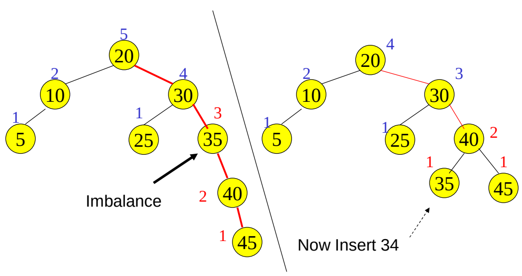


Insert 5, 40

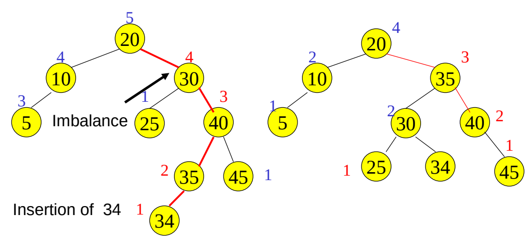
Example of Insertions in an AVL Tree



Single rotation (outside case)



Double rotation (inside case)



AVL Tree Deletion

- Similar but more complex than insertion
  - Rotations and double rotations needed to rebalance
  - Imbalance may propagate upward so that many rotations may be needed.

Pros and Cons of AVL Trees

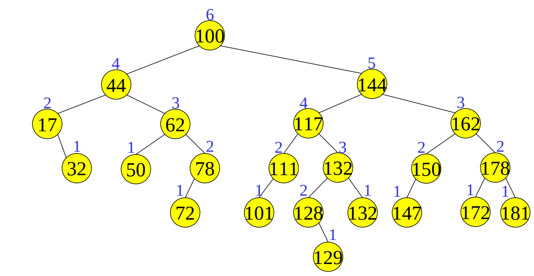
Arguments for AVL trees:

1. Search is O(log N) since AVL trees are always balanced.
2. Insertion and deletions are also O(logn)
3. The height balancing adds no more than a constant factor to the speed of insertion.

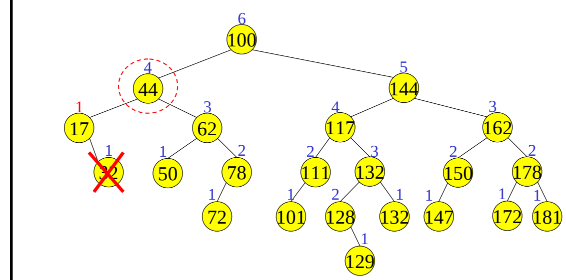
Arguments against using AVL trees:

1. Difficult to program & debug; more space for balance factor.
2. Asymptotically faster but rebalancing costs time.
3. Most large searches are done in database systems on disk and use other structures (e.g. B-trees).
4. May be OK to have O(N) for a single operation if total run time for many consecutive operations is fast (e.g. Splay trees).

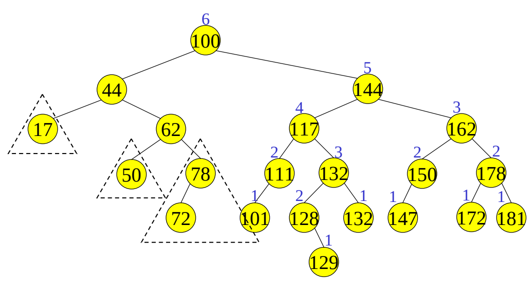
Deletion



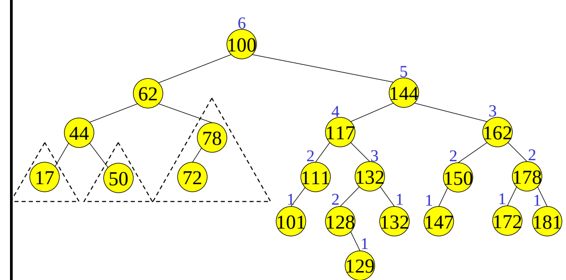
Deletion



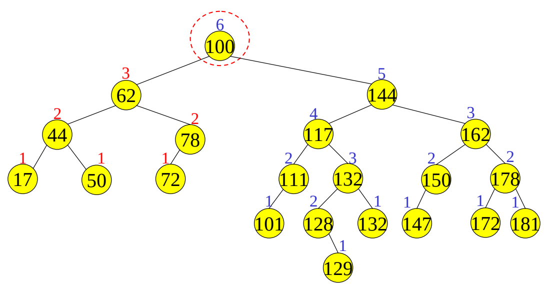
Deletion



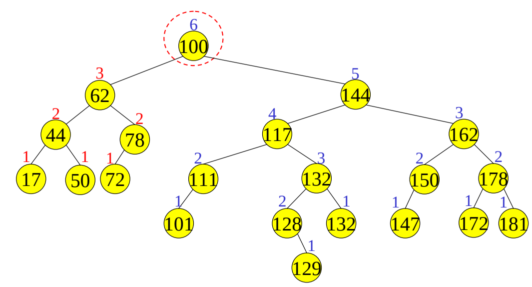
Deletion



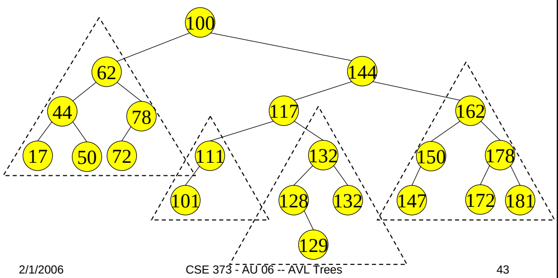
Deletion



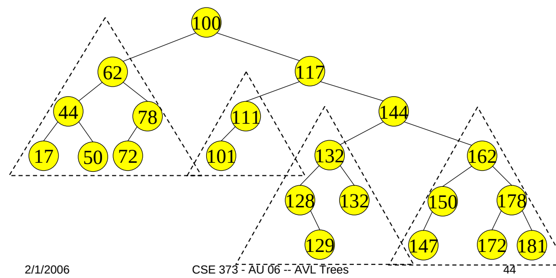
Deletion



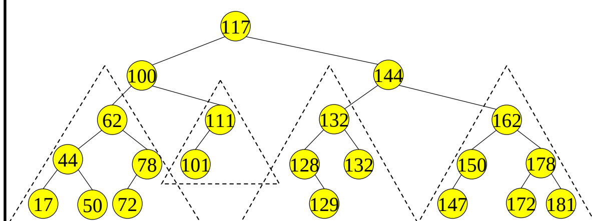
Deletion



Deletion



Deletion



Deletion