Directed Graphs (Part II)

CSE 373
Data Structures

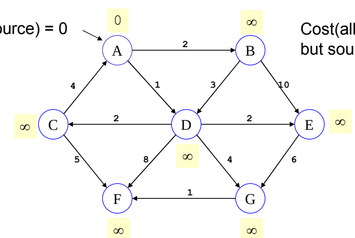
Dijkstra's Shortest Path Algorithm

- Initialize the cost of source to 0, and all the rest of the nodes to ∞
- Initialize set S to be ∅
  - › S is the set of nodes to which we have a shortest path
- While S is not all vertices
  - › Select the node A with the lowest cost that is not in S and identify the node as now being in S
  - › for each node B adjacent to A
    - if cost(A)+[A->B] < B's currently known cost
      - set cost(B) = cost(A)+[A->B]
      - set previous(B) = A so that we can remember the path

Example: Initialization

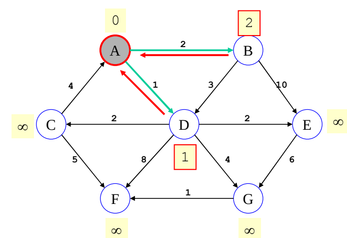
Cost(source) = 0

Cost(all vertices but source) = ∞



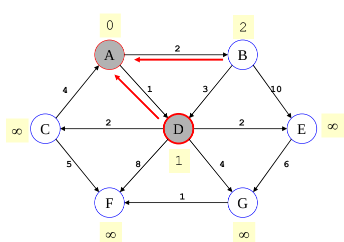
Pick vertex not in S with lowest cost.

Example: Update Cost neighbors



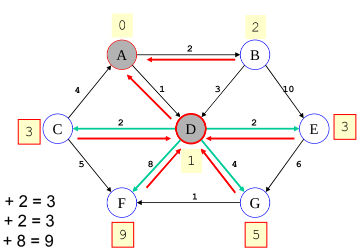
Cost(B) = 2
Cost(D) = 1

Example: pick vertex with lowest cost and add it to S



Pick vertex not in S with lowest cost, i.e., v_4

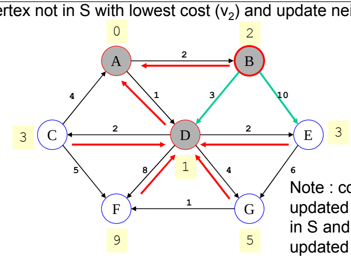
Example: update neighbors



Cost(C) = 1 + 2 = 3
Cost(E) = 1 + 2 = 3
Cost(F) = 1 + 8 = 9
Cost(G) = 1 + 4 = 5

Example (Ct'd)

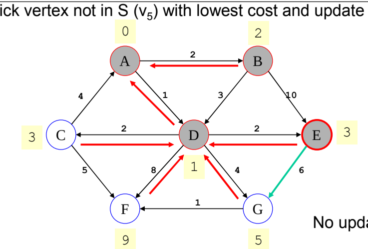
Pick vertex not in S with lowest cost (v_2) and update neighbors



Note : cost(v_4) not updated since already in S and cost(v_5) not updated since it is larger than previously computed

Example: (ct'd)

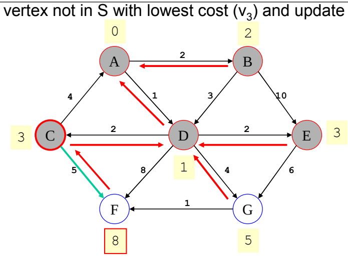
Pick vertex not in S (v_5) with lowest cost and update neighbors



No updating

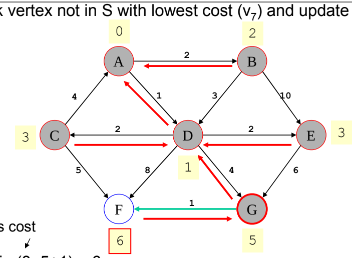
Example: (ct'd)

Pick vertex not in S with lowest cost (v_3) and update neighbors



Example: (ct'd)

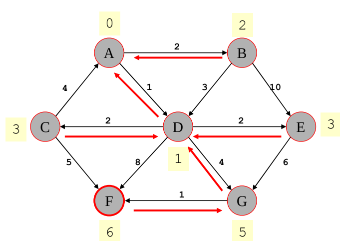
Pick vertex not in S with lowest cost (v_7) and update neighbors



Previous cost

Cost(v_6) = min (8, 5+1) = 6

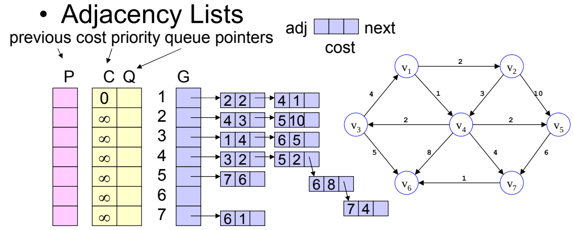
Example (end)



Pick vertex not in S with lowest cost (v_6) and update neighbors

Data Structures

- Adjacency Lists



Priority queue for finding and deleting lowest cost vertex and for decreasing costs (Binary Heap works)

Time Complexity

- n vertices and m edges
- Initialize data structures O(n+m)
- Find min cost vertices O(n log n)
  - n delete mins
- Update costs O(m log n)
  - Potentially m updates
- Update previous pointers O(m)
  - Potentially m updates
- Total time O((n + m) log n) - very fast.

(can be reduced to O(m log n) by fib or relaxed heap)

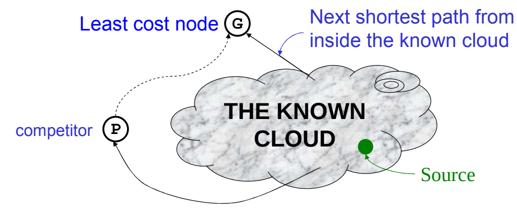
Or… using selection-sort pq

- n vertices and m edges
- Initialize data structures O(n+m)
- Find min cost vertices O(n^2)
  - n delete mins
- Update costs O(m)
  - Potentially m updates
- Update previous pointers O(m)
  - Potentially m updates
- Total time O(n^2+m) = O(n^2).

Correctness

- Dijkstra's algorithm is an example of a greedy algorithm
- Greedy algorithms always make choices that currently seem the best
  - Short-sighted – no consideration of long-term or global issues
  - Locally optimal does not always mean globally optimal
- In Dijkstra's case – choose the least cost node, but what if there is another path through other vertices that is cheaper?

"Cloudy" Proof: The Idea



- If the path to G is the next shortest path, the path to P must be at least as long. Therefore, any path through P to G cannot be shorter!

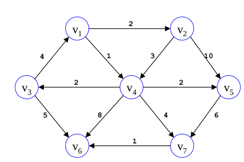
Inside the Cloud (Proof)

- Everything inside the cloud has the correct shortest path
- Proof is by induction on the number of nodes in the cloud:
  - Base case: Initial cloud is just the source s with shortest path 0.
  - Inductive hypothesis: Assume that a cloud of k-1 nodes all have shortest paths.
  - Inductive step: choose the least cost node G → has to be the shortest path to G (previous slide). Add k-th node G to the cloud.

All Pairs Shortest Path

- Given a edge weighted directed graph G = (V,E) find for all u,v in V the length of the shortest path from u to v. Use matrix representation.

C	1	2	3	4	5	6	7
1	0	2	:	1	:	:	:
2	:	0	:	3	10	:	:
3	4	:	0	:	:	5	:
4	:	:	2	0	2	8	4
5	:	:	:	:	0	:	6
6	:	:	:	:	:	0	:
7	:	:	:	:	:	1	0



: = infinity

A (simpler) Related Problem: Transitive Closure

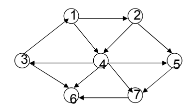
- Given a digraph G(V,E) the transitive closure is a digraph G'(V',E') such that
  - › V' = V (same set of vertices)
  - › If $(v_i, v_{i+1},\ldots,v_k)$ is a path in G, then (v_i, v_k) is an edge of E'

Unweighted Digraph Boolean Matrix Representation

- C is called the connectivity matrix

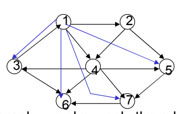
1 = connected
0 = not connected

C	1	2	3	4	5	6	7
1	0	1	0	1	0	0	0
2	0	0	0	1	1	0	0
3	1	0	0	0	0	1	0
4	0	0	1	0	1	1	1
5	0	0	0	0	0	0	1
6	0	0	0	0	0	0	0
7	0	0	0	0	0	1	0



Transitive Closure

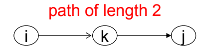
C	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1
4	1	1	1	1	1	1	1
5	0	0	0	0	0	1	1
6	0	0	0	0	0	0	0
7	0	0	0	0	0	1	0



On the graph, we show only the edges added with 1 as origin. The matrix represents the full transitive closure.

Finding Paths of Length 2

```
// First initialize C2 to all zero //
Length2 {
for k = 1 to n
 for i = 1 to n do
  for j = 1 to n do
     C2[i,j] := C2[i,j] ∪ (C[i,k] ∩ C[k,j]);
}
where ∩ is Boolean And (&&) and ∪ is Boolean OR (||)
This means if there is an edge from i to k
AND an edge from k to j, then there is a path
of length 2 between i and j.
Column k (C[i,k]) represents the predecessors of k
Row k (C[k,j]) represents the successors of k
```

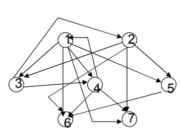


Paths of Length 2

C	1	2	3	4	5	6	7
1	0	1	0	1	0	0	0
2	0	0	0	1	1	0	0
3	1	0	0	0	0	1	0
4	0	0	1	0	1	1	1
5	0	0	0	0	0	0	1
6	0	0	0	0	0	0	0
7	0	0	0	0	0	1	0

Time $O(n^3)$

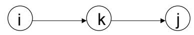
C2	1	2	3	4	5	6	7
1	0	0	1	1	1	1	1
2	0	0	1	0	1	1	1
3	0	1	0	1	0	0	0
4	1	0	0	0	0	1	1
5	0	0	0	0	0	1	0
6	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0



Transitive Closure

- Union of paths of length 0, length 1, length 2, …, length n-1.
  - › Time complexity n * $O(n^3)$ = $O(n^4)$
- There exists a better ($O(n^3)$) algorithm: Warshall's algorithm

Warshall Algorithm




```
TransitiveClosure {
for k = 1 to n do  // k is the step number //
  for i = 1 to n do
    for j = 1 to n do
      C [i,j] := C[i,j] ∪ (C[i,k] ∩ C[k,j]);
                        or          and
}
```

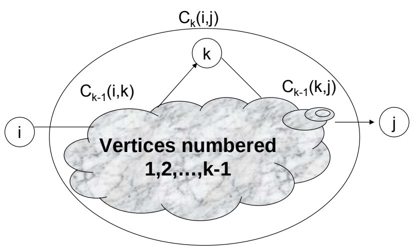
```
where C[i,j] starts as the original
connectivity matrix and C[i,j] is updated
after step k if a new path from i to j
through k is found.
```


Proof of Correctness

Prove: After the k-th time through the loop, C[i,j] =1 if there is a path from i to j that only passes through vertices numbered 1,2,…,k (except for the initial edges)

- Base case: k = 1. C [i,j] = 1 for the initial connectivity matrix (path of length 0) and C [i,j] = 1 if there is a path (i,1,j)

Cloud Argument



Inductive Step

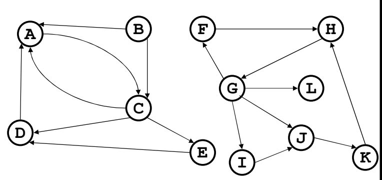
- Inductive Hypothesis: Suppose after step k-1 that C[i,j] contains a 1 if there is a path **from** i to j through vertices 1,…,k-1.
- Induction: Consider step k, which does

```
C[i,j] := C[i,j] ∪ (C[i,k] ∩ C[k,j]);
                 or          and
```

Either C[i,j] is already 1 or there is a new path through vertex k, which makes it 1.

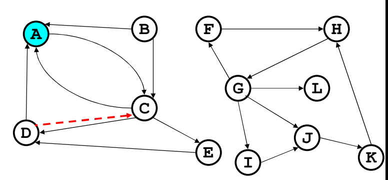
Warshall Algorithm

```
 ABCDEFGHIJKL
A001000000000
B101000000000
C100110000000
D100000000000
E000100000000
F000000010000
G000001001101
H000000100000
I000000000100
J000000000010
K000000010000
L000000000000
```



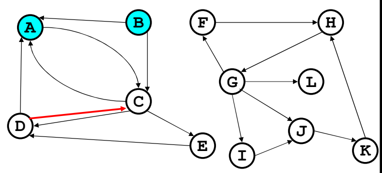
Warshall Algorithm

```
 ABCDEFGHIJKL
A001000000000
B101000000000
C100110000000
D101000000000
E000100000000
F000000010000
G000001001101
H000000100000
I000000000100
J000000000010
K000000010000
L000000000000
```



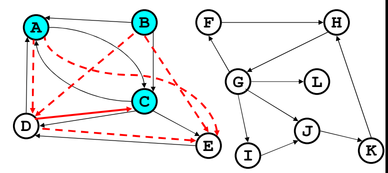
Warshall Algorithm

```
 ABCDEFGHIJKL
A001000000000
B101000000000
C100110000000
D101000000000
E000100000000
F000000010000
G000001001101
H000000100000
I000000000100
J000000000010
K000000010000
L000000000000
```



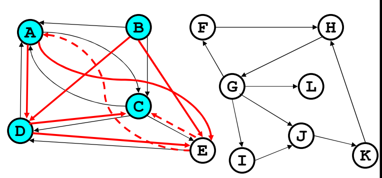
Warshall Algorithm

```
 ABCDEFGHIJKL
A001110000000
B101110000000
C100110000000
D101010000000
E000100000000
F000000010000
G000001001101
H000000100000
I000000000100
J000000000010
K000000010000
L000000000000
```



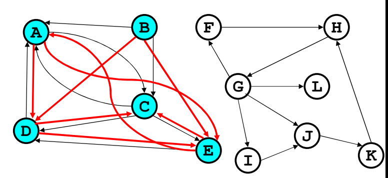
Warshall Algorithm

```
 ABCDEFGHIJKL
A001110000000
B101110000000
C100110000000
D101010000000
E101100000000
F000000010000
G000001001101
H000000100000
I000000000100
J000000000010
K000000010000
L000000000000
```



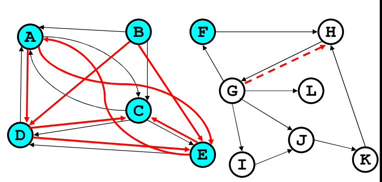
Warshall Algorithm

```
 ABCDEFGHIJKL
A001110000000
B101110000000
C100110000000
D101010000000
E101100000000
F000000010000
G000001001101
H000000100000
I000000000100
J000000000010
K000000010000
L000000000000
```



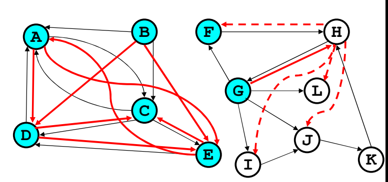
Warshall Algorithm

```
 ABCDEFGHIJKL
A001110000000
B101110000000
C100110000000
D101010000000
E101100000000
F000000010000
G000001011101
H000000100000
I000000000100
J000000000010
K000000010000
L000000000000
```



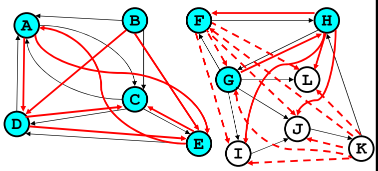
Warshall Algorithm

```
 ABCDEFGHIJKL
A001110000000
B101110000000
C100110000000
D101010000000
E101100000000
F000000010000
G000001011101
H000001101101
I000000000100
J000000000010
K000000010000
L000000000000
```



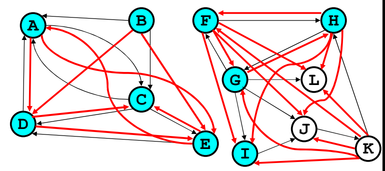
Warshall Algorithm

```
 ABCDEFGHIJKL
A001110000000
B101110000000
C100110000000
D101010000000
E101100000000
F000000111101
G000001011101
H000001101101
I000000000100
J000000000010
K000001111101
L000000000000
```



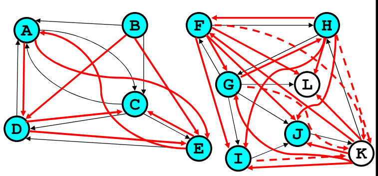
Warshall Algorithm

```
 ABCDEFGHIJKL
A001110000000
B101110000000
C100110000000
D101010000000
E101100000000
F000000111101
G000001011101
H000001101101
I000000000100
J000000000010
K000001111101
L000000000000
```



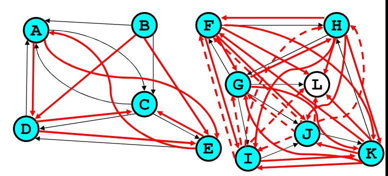
Warshall Algorithm

```
 ABCDEFGHIJKL
A001110000000
B101110000000
C100110000000
D101010000000
E101100000000
F000000111111
G000001011111
H000001101111
I000000000110
J000000000010
K000001111101
L000000000000
```



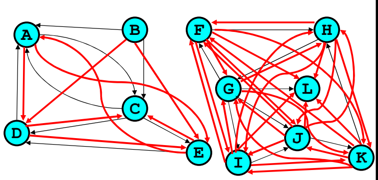
Warshall Algorithm

```
 ABCDEFGHIJKL
A001110000000
B101110000000
C100110000000
D101010000000
E101100000000
F000000111111
G000001011111
H000001101111
I000001110111
J000001111011
K000001111101
L000000000000
```



Warshall Algorithm

```
 ABCDEFGHIJKL
A001110000000
B101110000000
C100110000000
D101010000000
E101100000000
F000000111111
G000001011111
H000001101111
I000001110111
J000001111011
K000001111101
L000000000000
```



Back to Weighted graphs: Matrix Representation

- C[i,j] = the cost of the edge (i,j)
  - › C[i,i] = 0 because no cost to stay where you are
  - › C[i,j] = ∞ if no edge from i to j.

C	1	2	3	4	5	6	7
1	0	2	∞	1	∞	∞	∞
2	∞	0	∞	3	10	∞	∞
3	4	∞	0	∞	∞	5	∞
4	∞	∞	2	0	2	8	4
5	∞	∞	∞	∞	0	∞	6
6	∞	∞	∞	∞	∞	0	∞
7	∞	∞	∞	∞	∞	1	0

Floyd – Warshall Algorithm

// Start with the cost matrix C

```
All_Pairs_Shortest_Path {
  for k = 1 to n do
    for i = 1 to n do
      for j = 1 to n do
        C[i,j] := min(C[i,j], C[i,k] + C[k,j]);
}
```

old cost — updated new cost

```
Note x + ∞ = ∞ by definition
```

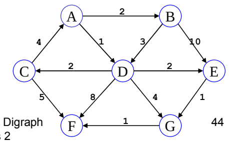
On termination C[i,j] is the length of the shortest path from i to j.

The Computation

	1	2	3	4	5	6	7
1	0	2	∞	1	∞	∞	∞
2	∞	0	∞	3	10	∞	∞
3	4	∞	0	∞	∞	5	∞
4	∞	∞	2	0	2	8	4
5	∞	∞	∞	∞	0	∞	1
6	∞	∞	∞	∞	∞	0	∞
7	∞	∞	∞	∞	∞	1	0

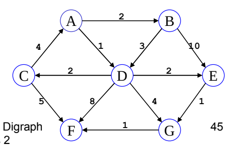
⇨

	1	2	3	4	5	6	7
1	0	2	3	1	3	6	5
2	9	0	5	3	5	8	7
3	4	6	0	5	4	5	9
4	6	8	2	0	2	5	4
5	∞	∞	∞	∞	0	7	1
6	∞	∞	∞	∞	∞	0	∞
7	∞	∞	∞	∞	∞	1	0



AB:	**DE:**
AC:	**DF:**
AD:	**DG:**
AE:	**EA:**
AF:	**EB:**
AG:	**EC:**
BA:	**ED:**
BC:	**EF:**
BD:	**EG:**
BE:	**FA:**
BF:	**FB:**
BG:	**FC:**
CA:	**FD:**
CB:	**FE:**
CD:	**FG:**
CE:	**GA:**
CF:	**GB:**
CG:	**GC:**
DA:	**GD:**
DB:	**GE:**
DC:	**GF:**

	A	B	C	D	E	F	G
A		2	∞	1	∞	∞	∞
B	∞		∞	3	10	∞	∞
C	4	∞		∞	∞	5	∞
D	∞	∞	2		2	8	4
E	∞	∞	∞	∞		∞	1
F	∞	∞	∞	∞	∞		∞
G	∞	∞	∞	∞	∞	1	



AB:	**DE:**
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B	∞		∞	3	10	∞	∞
C	4	6		5	∞	5	∞
D	∞	∞	2		2	8	4
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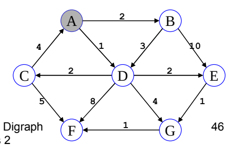

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DC:	**GF:**

	A	**B**	C	D	E	F	G
A		**2**	∞	1	12	∞	∞
B	∞		∞	3	10	∞	∞
C	4	6		5	16	5	∞
D	∞	∞	2		2	8	4
E	∞	∞	∞	∞		∞	1
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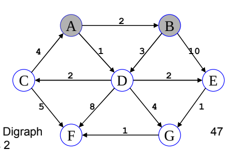

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	A	**B**	**C**	D	E	F	G
A		**2**	∞	1	12	∞	∞
B	∞		∞	3	10	∞	∞
C	**4**	**6**		5	16	5	∞
D	6	8	2		2	7	4
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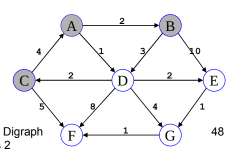

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DC:	GF:

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A		2	3	1	3	8	5
B	9		5	3	10	10	7
C	4	6		5	7	5	9
D	6	8	2		2	7	4
E	∞	∞	∞	∞		∞	1
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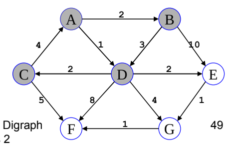

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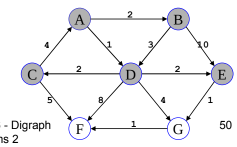

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DC:	GF:

	A	B	C	D	E	F	G
A		2	3	1	3	8	4
B	9		5	3	10	10	6
C	4	6		5	7	5	8
D	6	8	2		2	7	3
E	∞	∞	∞	∞		∞	1
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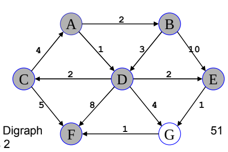

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DC:	GF:

	A	B	C	D	E	F	G
A		2	3	1	3	5	4
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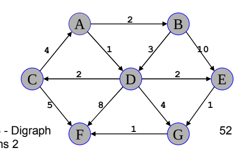

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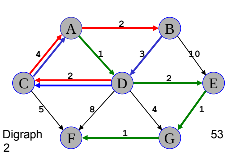

Time Complexity of All Pairs Shortest Path

- n is the number of vertices
- Three nested loops. $O(n^3)$
  - Shortest paths can be found too
- Repeated Dijkstra's algorithm
  - $O(n(n+m)\log n)$ (= $O(n^3 \log n)$ for dense graphs).
  - Run Dijkstra starting at each vertex.