CSE 373
Data Structures & Algorithms

Lectures 19-20

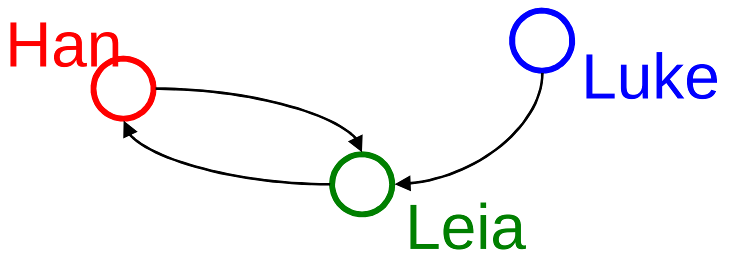
Graphs

Graph… ADT?

- Not quite an ADT…
  operations not clear

- A formalism for representing relationships between objects

  Graph **G = (V,E)**
  - *Set of vertices*:
    $\mathbf{V = \{v_1, v_2, \ldots, v_n\}}$
  - *Set of edges*:
    $\mathbf{E = \{e_1, e_2, \ldots, e_m\}}$
    where each $\mathbf{e_i}$ connects two vertices $\mathbf{(v_{i1}, v_{i2})}$



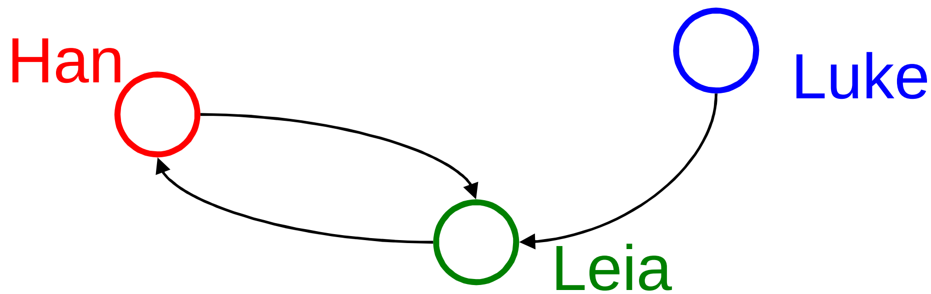

```
V = {Han, Leia, Luke}
E = {(Luke, Leia),
     (Han, Leia),
     (Leia, Han)}
```

Examples of Graphs

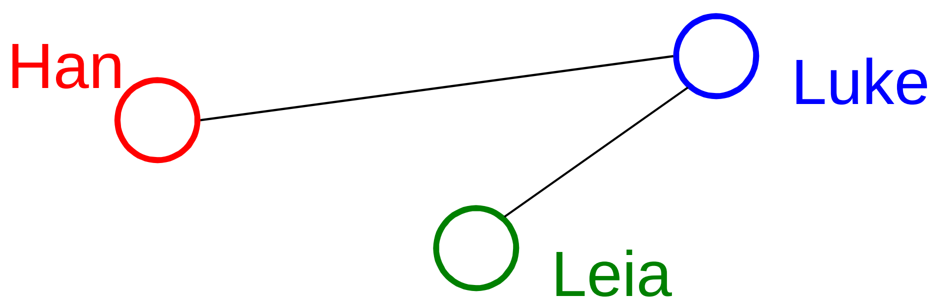
- The web
  - Vertices are webpages
  - Each edge is a link from one page to another
- Call graph of a program
  - Vertices are subroutines
  - Edges are calls and returns
- Social networks
  - Vertices are people
  - Edges connect friends

Graph Definitions

In *directed* graphs, edges have a direction:



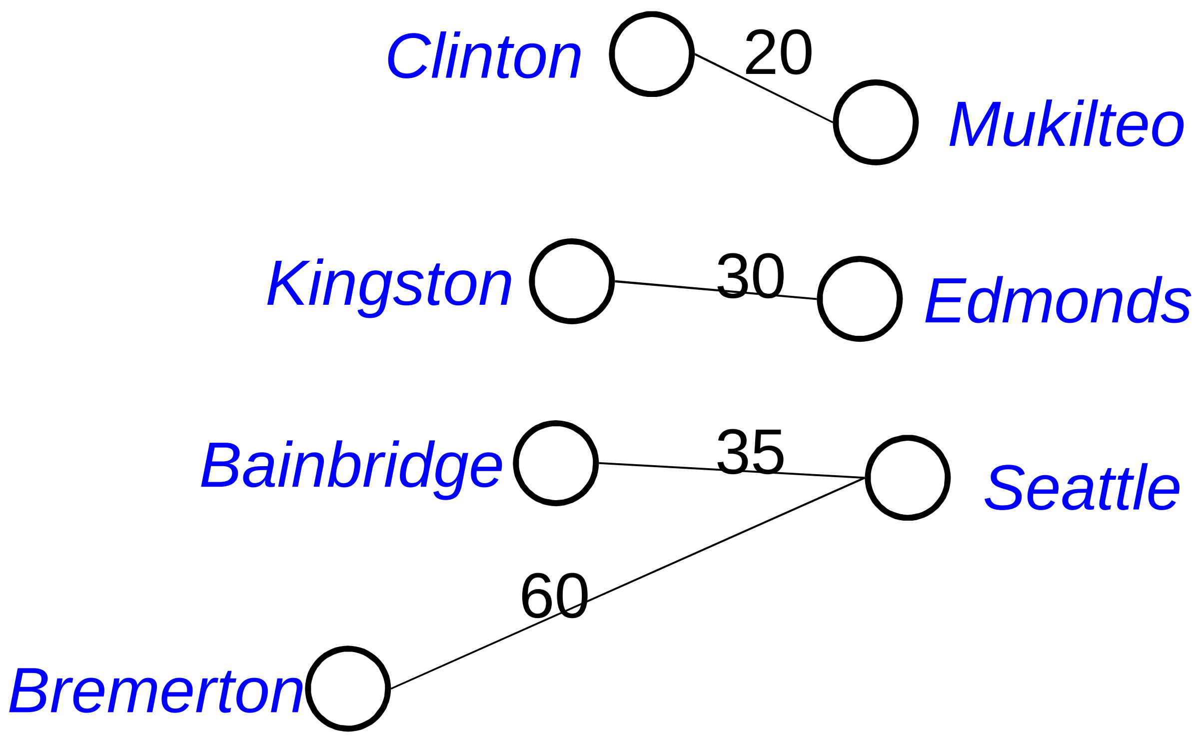
In *undirected* graphs, they don't (are two-way):



v is *adjacent* to **u** if $(\mathbf{u},\mathbf{v}) \in \mathbf{E}$

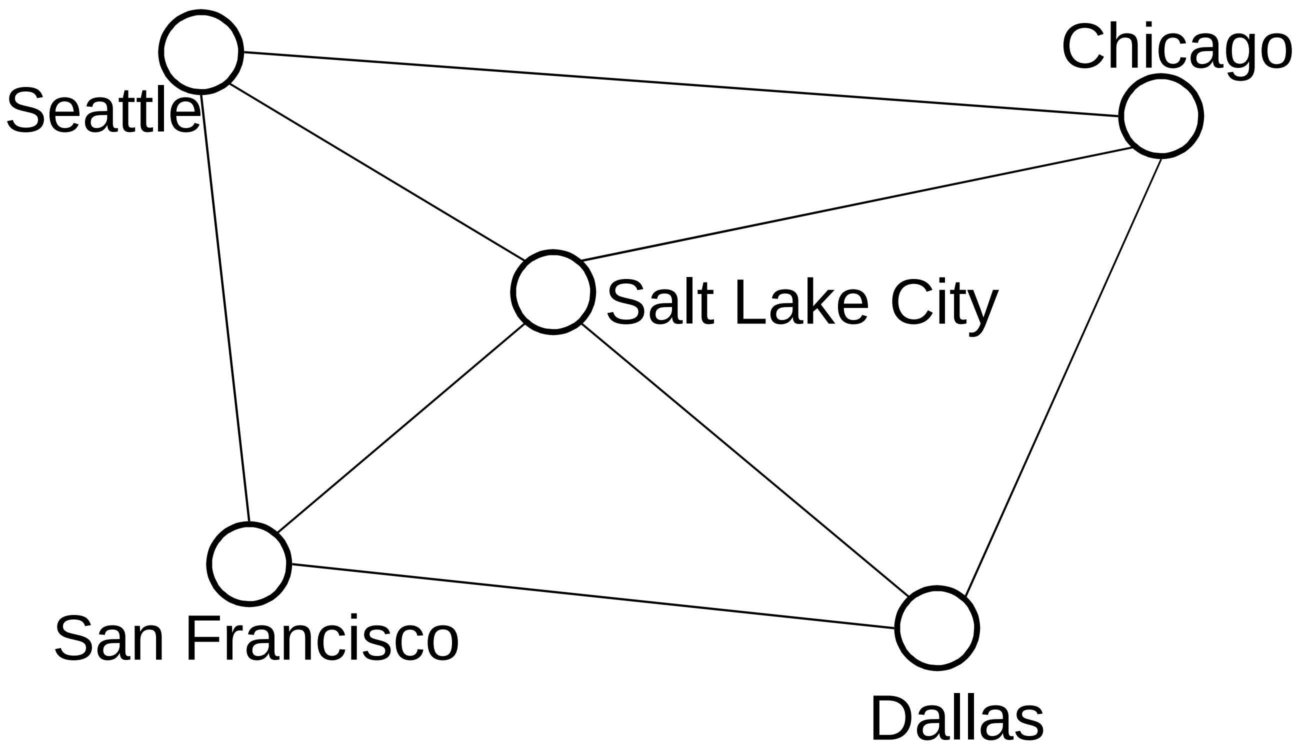
Weighted Graphs

Each edge has an associated weight or cost.



Paths and Cycles

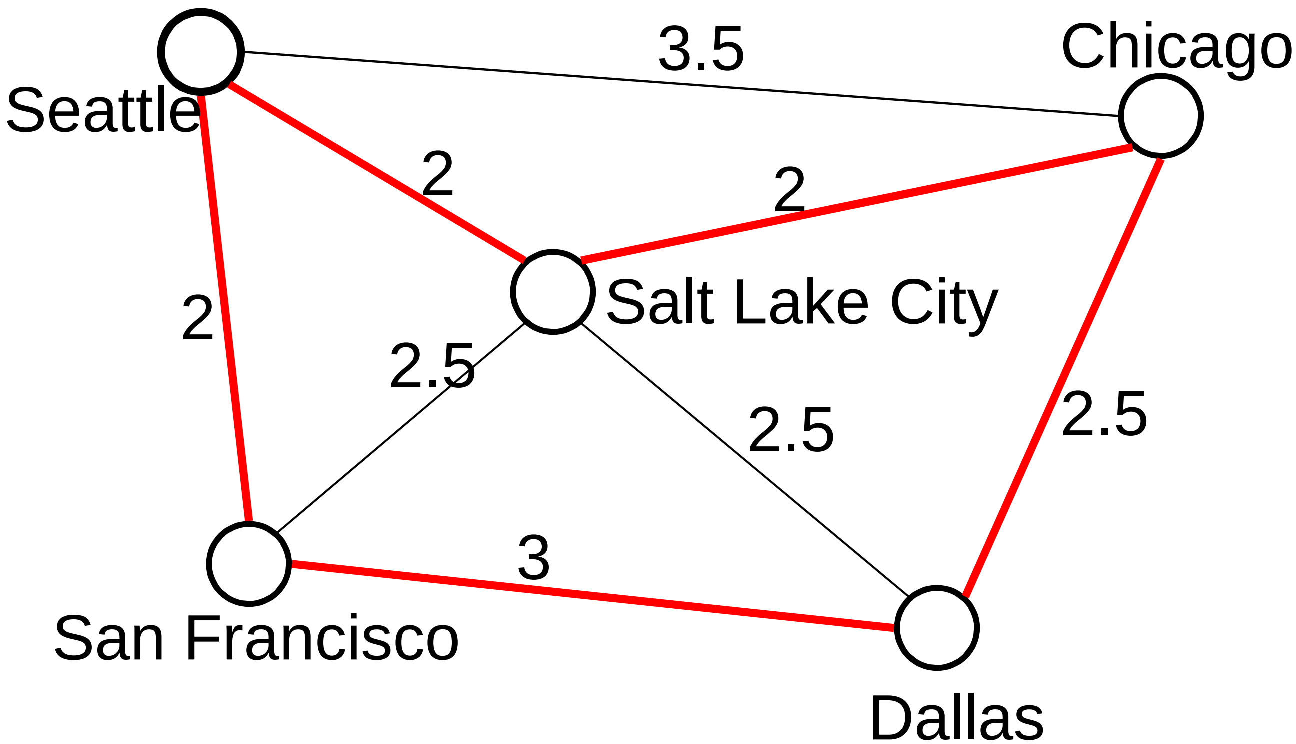
- A *path* is a list of vertices $\{v_1, v_2, \ldots, v_n\}$ such that $(v_i, v_{i+1}) \in E$ for all $0 \leq i < n$.
- A *cycle* is a path that begins and ends at the same node.



p ={Seattle, SaltLakeCity, Chicago, Dallas, SanFrancisco, Seattle}

Path Length and Cost

- *Path length*: the number of edges in the path
- *Path cost*: the sum of the costs of each edge



length(p) = 5

cost(p) = 11.5

More Definitions: Simple Paths and Cycles

A *simple path* repeats no vertices (except that the first can also be the last):

p = {Seattle, Salt Lake City, San Francisco, Dallas}

p = {Seattle, Salt Lake City, Dallas, San Francisco, Seattle}

A *cycle* is a path that starts and ends at the same node:

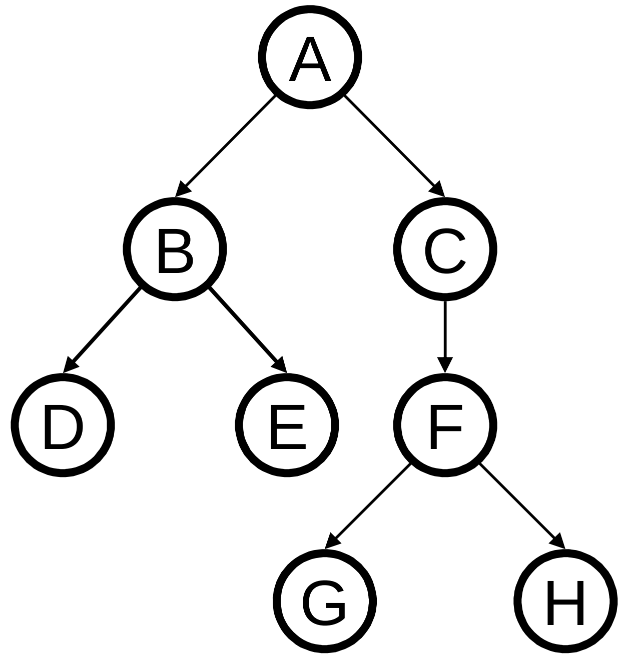
p = {Seattle, Salt Lake City, Dallas, San Francisco, Seattle}

p = {Seattle, Salt Lake City, Seattle, San Francisco, Seattle}

A *simple cycle* is a cycle that is also a simple path (in undirected graphs, no edge can be repeated)

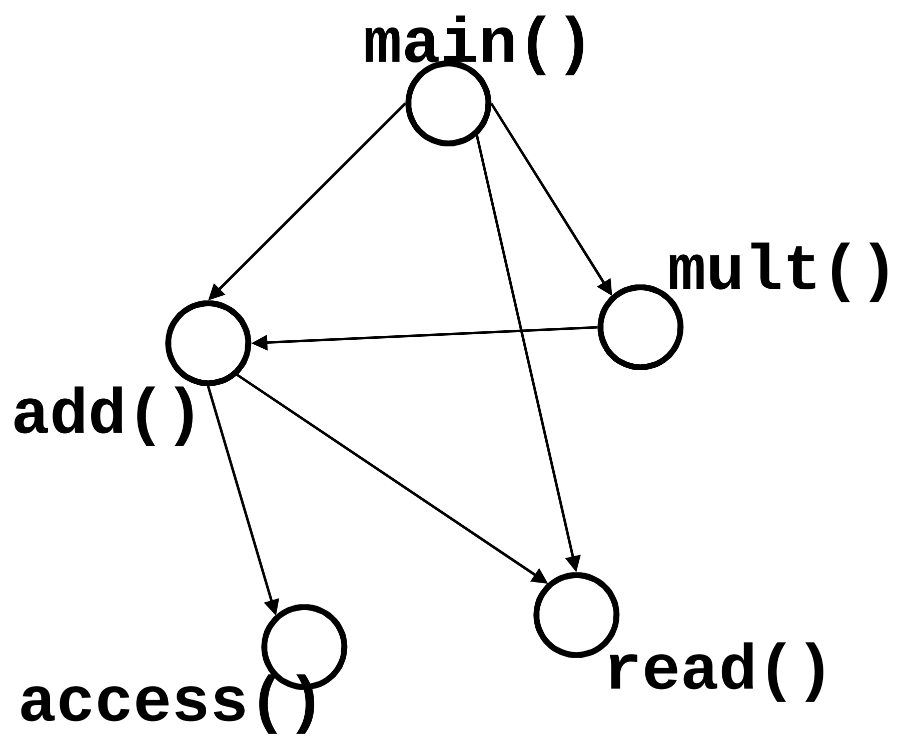
Trees as Graphs

- Every tree is a graph with some restrictions:
  - the tree is *directed*
  - there is *exactly one directed path from the root to every node*



Directed Acyclic Graphs (DAGs)

DAGs are directed graphs with no (directed) cycles.

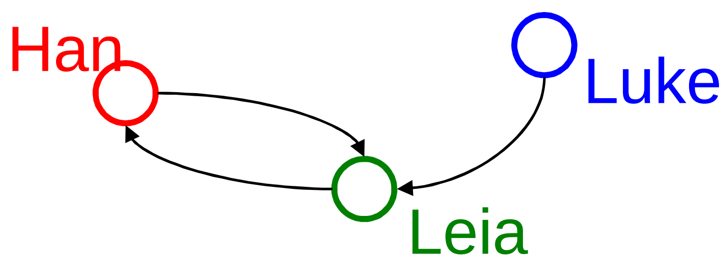


Aside: If program call-graph is a DAG, then all procedure calls can be in-lined

{Tree} ⊂ {DAG} ⊂ {Graph}

Rep 1: Adjacency Matrix

A **|V| x |V|** array in which an element **(u,v)** is true if and only if there is an edge from **u** to **v**



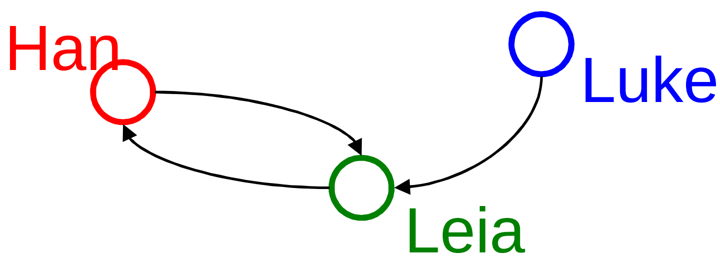
	Han	Luke	Leia
Han			
Luke			
Leia			

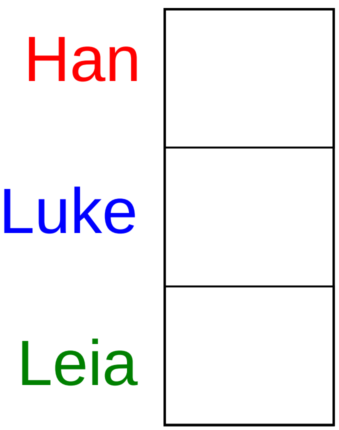
Runtimes:
Iterate over vertices?
Iterate over edges?
Iterate edges adj. to vertex?
Existence of edge?

Space requirements?

Rep 2: Adjacency List

A **|V|**-ary list (array) in which each entry stores a list (linked list) of all adjacent vertices

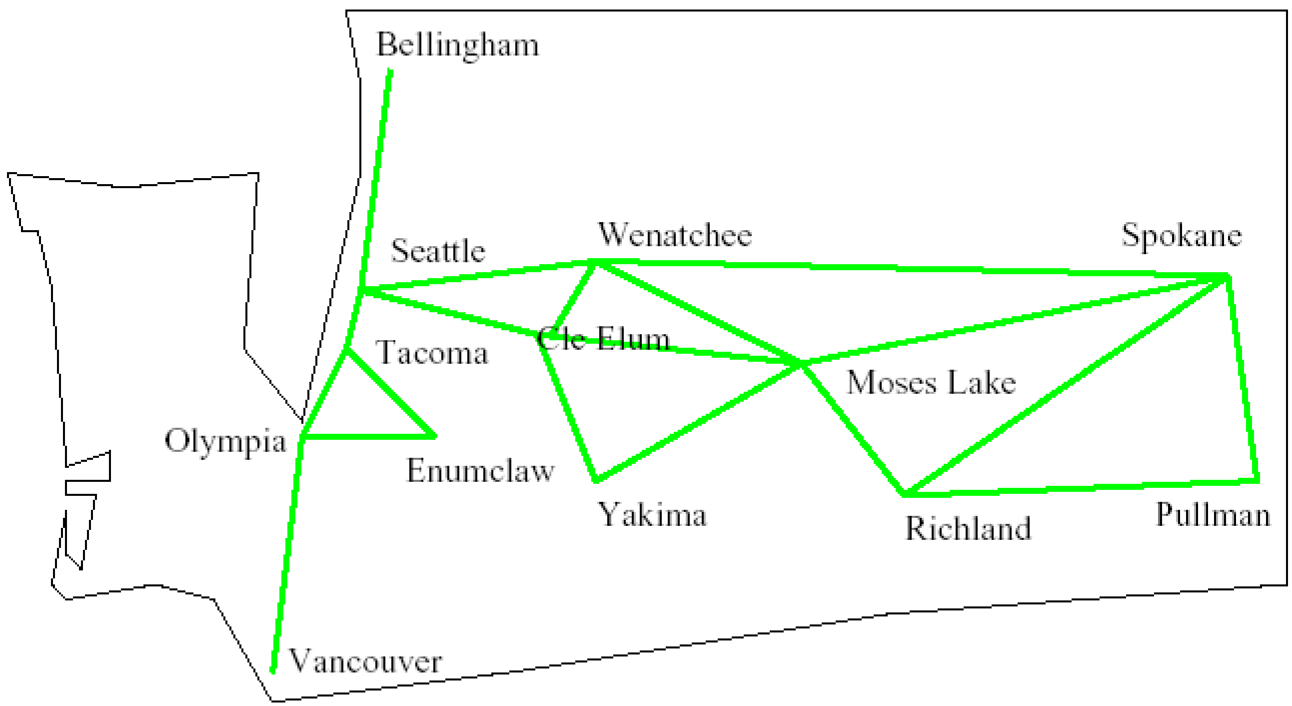




Runtimes:
Iterate over vertices?
Iterate over edges?
Iterate edges adj. to vertex?
Existence of edge?

Space requirements?

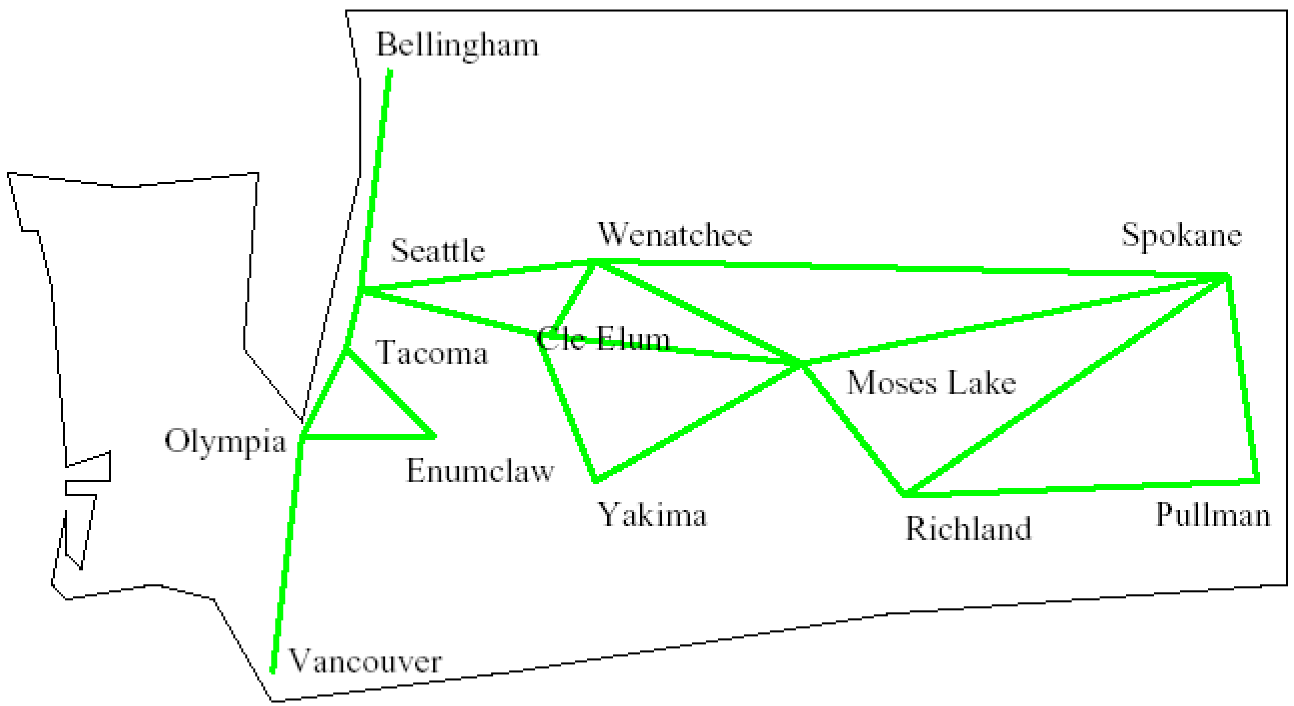
Some Applications: Moving Around Washington



What's the *shortest way* to get from Seattle to Pullman?

Edge labels: Distance

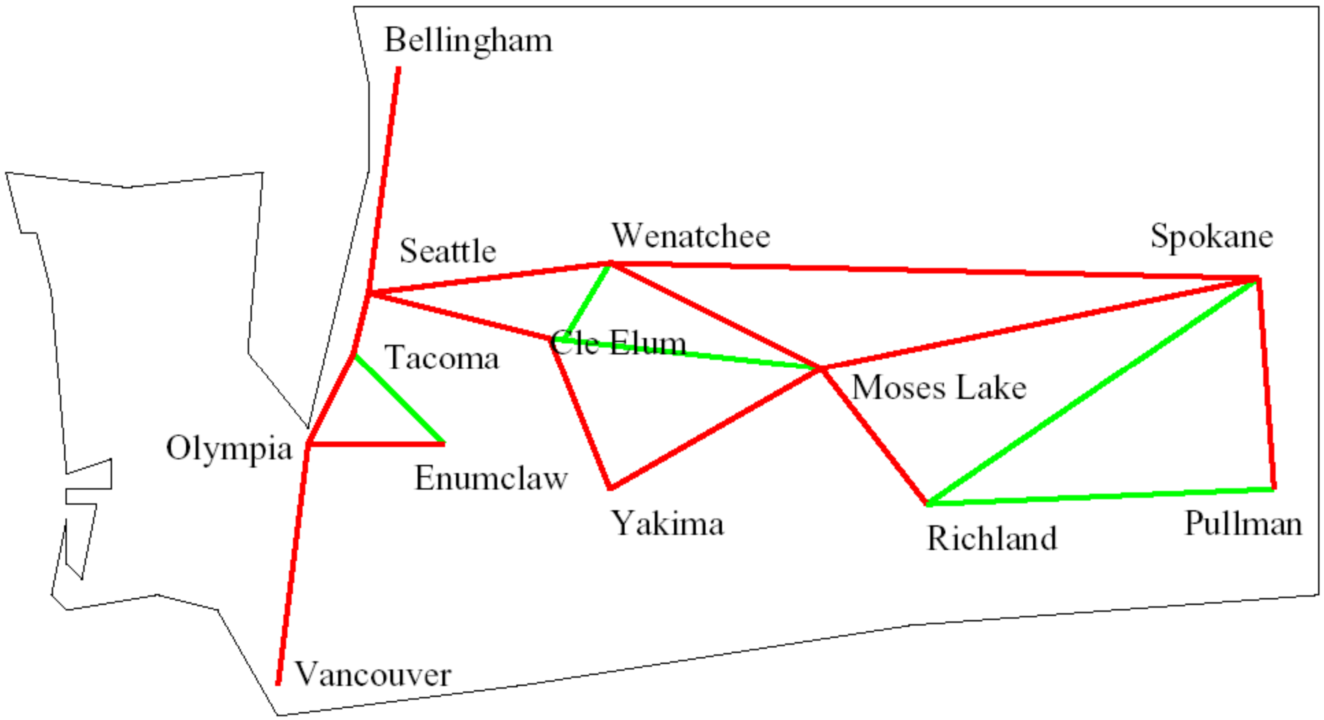
Some Applications: Moving Around Washington



What's the *fastest way* to get from Seattle to Pullman?

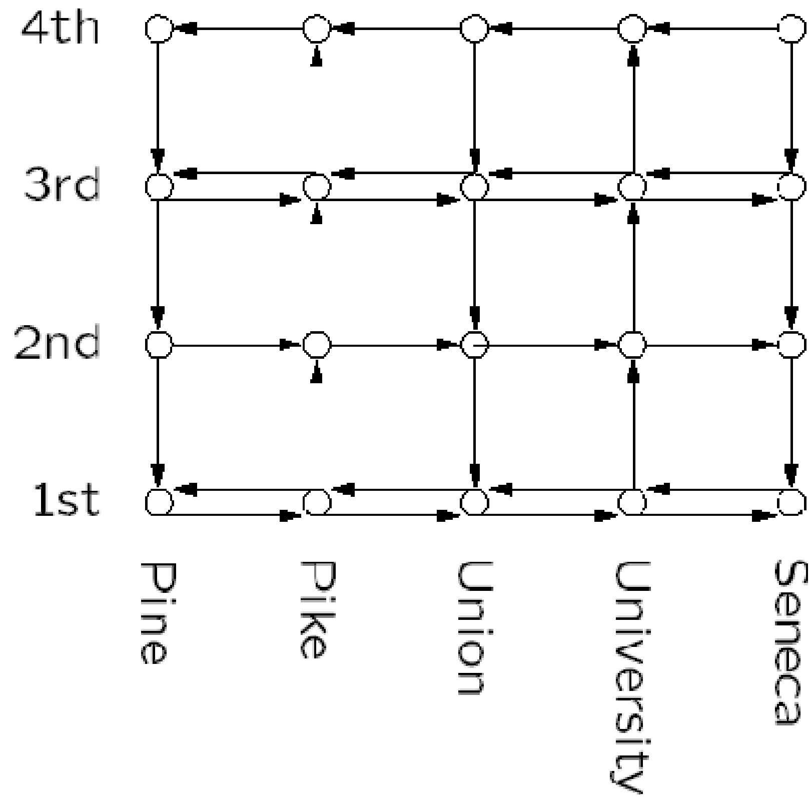
Edge labels: Distance, speed limit

Some Applications: Reliability of Communication



If Wenatchee's phone exchange *goes down*, can Seattle still talk to Pullman?

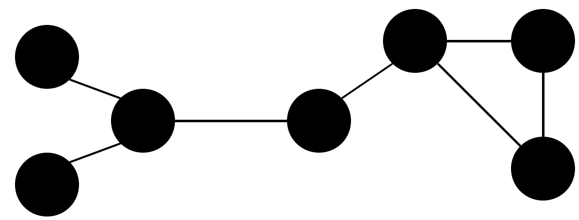
Some Applications:
Bus Routes in Downtown Seattle



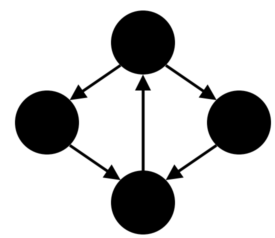
If we're at 3rd and Pine, how can we get to 1st and University using Metro?
How about 4th and Seneca?

Graph Connectivity

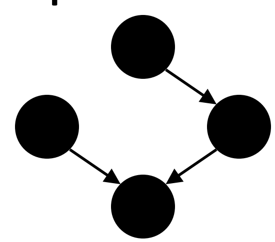
- Undirected graphs are *connected* if there is a path between any two vertices



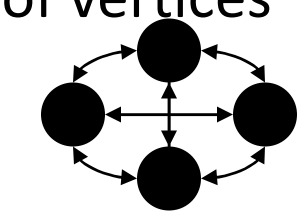
- Directed graphs are *strongly connected* if there is a path from any one vertex to any other



- Directed graphs are *weakly connected* if there is a path between any two vertices, *ignoring direction*



- A *complete* graph has an edge between every pair of vertices



Graph Traversals

- Breadth-first search (and depth-first search) work for arbitrary (directed or undirected) graphs - not just mazes!
  – Must mark visited vertices. Why?
  – So you do not go into an infinite loop! It's not a tree.
- Either can be used to determine connectivity:
  – Is there a path between two given vertices?
  – Is the graph (weakly/strongly) connected?
- Which one:
  – Uses a queue?
  – Uses a stack?
  – Always finds the shortest path (for unweighted graphs)?

The Shortest Path Problem

- Given a graph *G*, edge costs $c_{i,j}$, and vertices *s* and *t* in *G*, find the shortest path from *s* to *t*.

- For a path $p = v_0\ v_1\ v_2\ \ldots\ v_k$
  - *unweighted length* of path $p = k$ (a.k.a. *length*)
  - *weighted length* of path $p = \sum_{i=0..k-1} c_{i,i+1}$ (a.k.a *cost*)
  - Path length equals path cost when ?

Single Source Shortest Paths (SSSP)

- Given a graph *G*, edge costs $c_{i,j}$, and vertex *s*, find the shortest paths from *s* to all vertices in G.
  - Is this harder or easier than the previous problem?

All Pairs Shortest Paths (APSP)

- Given a graph *G* and edge costs $c_{i,j}$, find the shortest paths between all pairs of vertices in G.
  - Is this harder or easier than SSSP?
  - Could we use SSSP as a subroutine to solve this?

Breadth-First Graph Search

```
BFS( Start)
  for all nodes x do x.dist = ∞;
  Start.dist = 0;
  enqueue(Start, Open);
  repeat
    if (empty(Open)) then return;
    x:= dequeue(Open);
    for each y in children(x) do
      if (y.dist = ∞)
        then { y.dist = x.dist + 1;
               enqueue(y, Open);  }
  end-repeat
```

Depth-First Graph Search

```
DFS( Start)
   for all nodes x do x.dist = ∞;
   Start.dist = 0;
   push(Start, Open);
   repeat
      if (empty(Open)) then return;
      x:= pop(Open);
      for each y in children(x) do
         if (y.dist > x.dist + 1)
           then { y.dist = x.dist + 1;
                  push(y, Open);  }
    end-repeat
```

Comparison: DFS versus BFS

- Depth-first search
  - –Does not find shortest paths naturally
    - Had to do the extra test y.dist > x.dist + 1
  - –Must be careful to mark visited vertices (using x.dist, or some other means), or you could go into an infinite loop if there is a cycle
- Breadth-first search
  - –Always finds shortest paths – optimal solutions
  - –Marking visited nodes can improve efficiency, but even without doing so search is guaranteed to terminate

  - –Is BFS always preferable?

DFS Space Requirements

- Assume:
  - Longest path in graph is length *d*
  - Highest number of out-edges is *k*
- DFS stack grows at most to size *dk*
  - For *k*=10, *d*=15, size is 150

BFS Space Requirements

- Assume
  - Distance from start to a goal is *d*
  - Highest number of out edges is *k* BFS
- Queue could grow to size k^d
  - For *k*=10, *d*=15, size is 1,000,000,000,000,000

Conclusion

- For large graphs, DFS is more memory efficient, *if we can limit the maximum path length to some fixed d.*
  - If we *knew* the distance from the start to the goal in advance, we can just *not add any children to stack after level d*
  - But what if we don't know *d* in advance?

Edsger Wybe Dijkstra
(1930-2002)

- Invented concepts of structured programming, synchronization, weakest precondition, and "semaphores" for controlling computer processes. The Oxford English Dictionary cites his use of the words "vector" and "stack" in a computing context.
- Believed programming should be taught without computers
- 1972 Turing Award
- “In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind.”

Shortest Path for Weighted Graphs

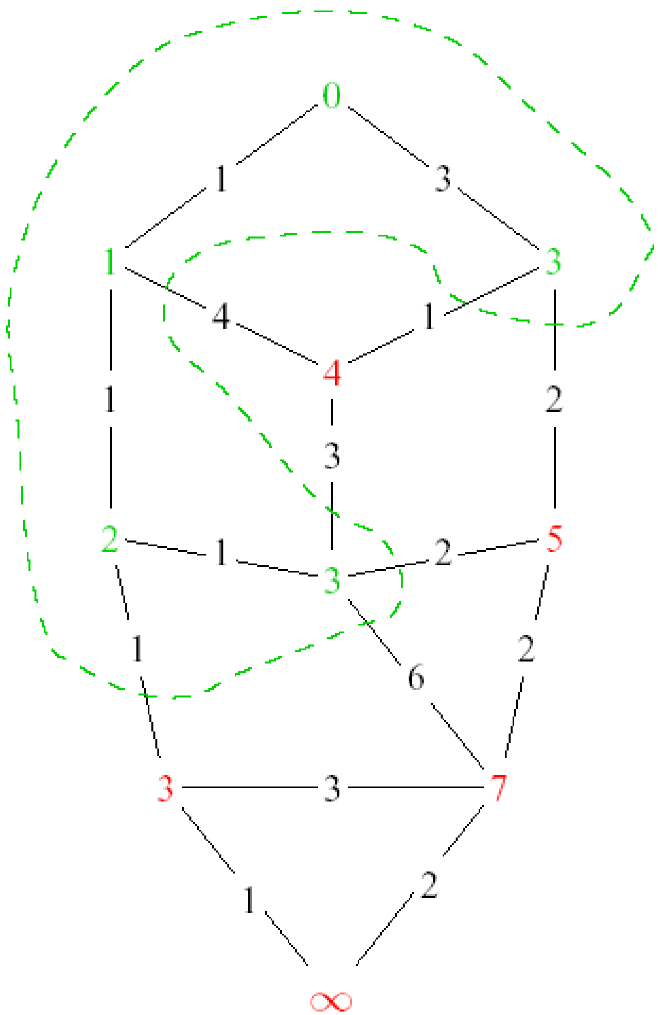
- Given a graph **`G = (V, E)`** with edge costs c(e), and a vertex **`s`** $\in$ **`V`**, find the shortest (lowest cost) path from s to every vertex in **`V`**

- Assume: only *positive* edge costs

Dijkstra's Algorithm for Single Source Shortest Path

- Similar to breadth-first search, but uses a heap instead of a queue:
  - Always select (expand) the vertex that has a lowest-cost path to the start vertex
- Correctly handles the case where the lowest-cost (shortest) path to a vertex is not the one with fewest edges

Dijkstra's Algorithm: Idea

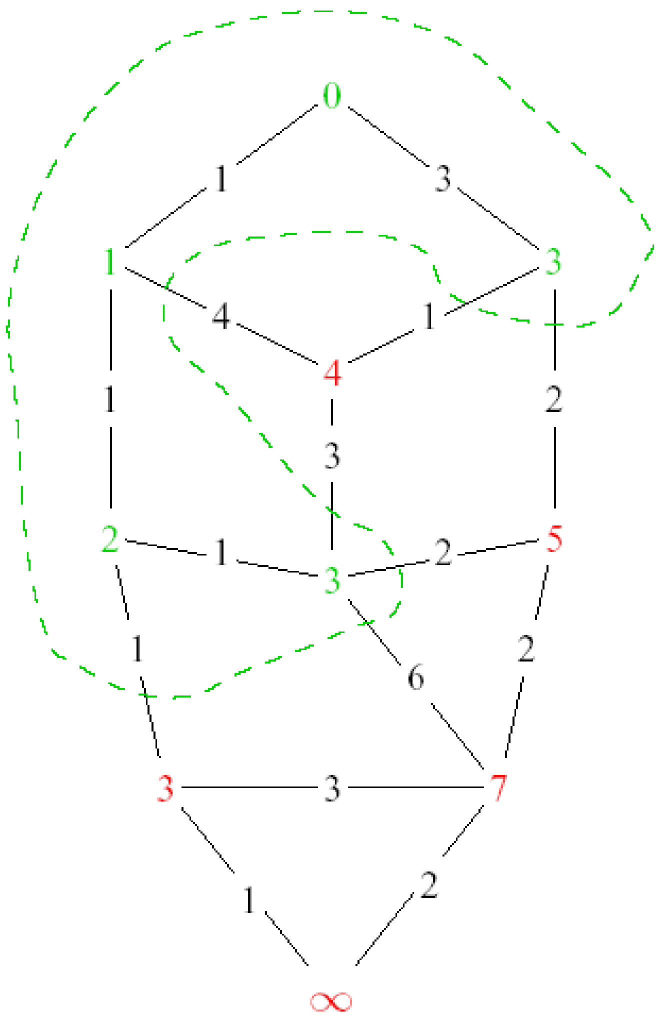


Adapt BFS to handle weighted graphs

Two kinds of vertices:

- Finished or known vertices
  - Shortest distance has been computed
- Unknown vertices
  - Have tentative distance

Dijkstra’s Algorithm: Idea



At each step:

1) Pick closest unknown vertex
2) Add it to known vertices
3) Update distances

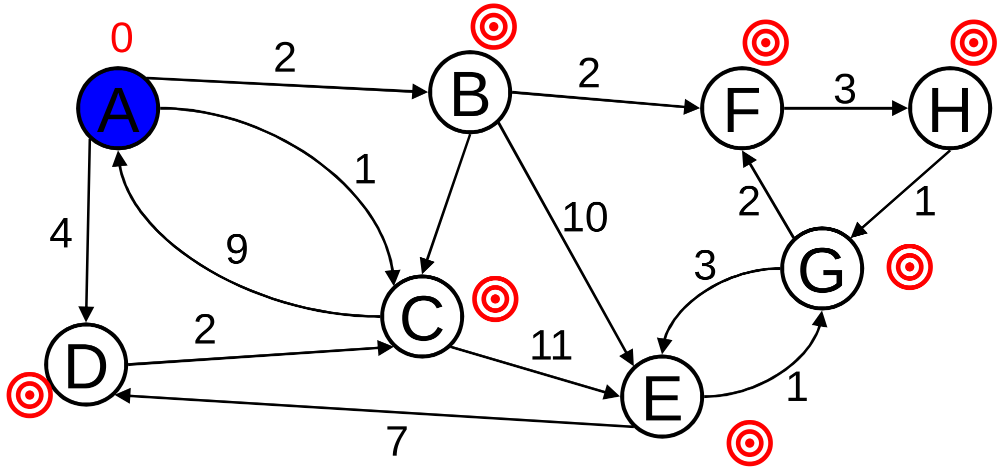
Dijkstra's Algorithm: Pseudocode

```
Initialize the cost of each node to ∞

Initialize the cost of the source to 0

While there are unknown nodes left in the graph
    Select an unknown node b with the lowest cost
    Mark b as known
    For each node a adjacent to b
        if b's cost + cost of (b, a) < a's old cost
                    a's cost = b's cost + cost of (b, a)
                    a's prev path node = b
```

Important Features

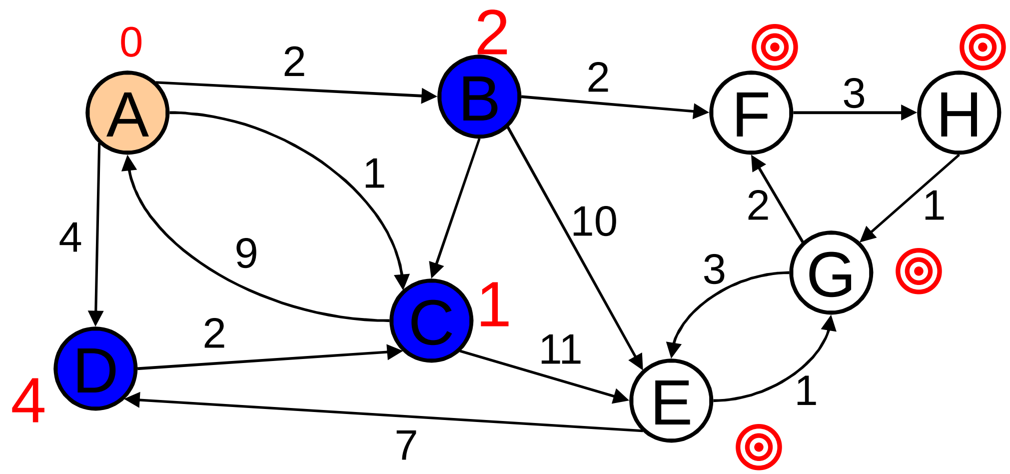
- Once a vertex is made **known**, the cost of the shortest path to that node is known
- While a vertex is still not **known**, another shorter path to it might still be found
- The shortest path itself can found by following the backward pointers stored in **`node.path`**

Dijkstra’s Algorithm in action



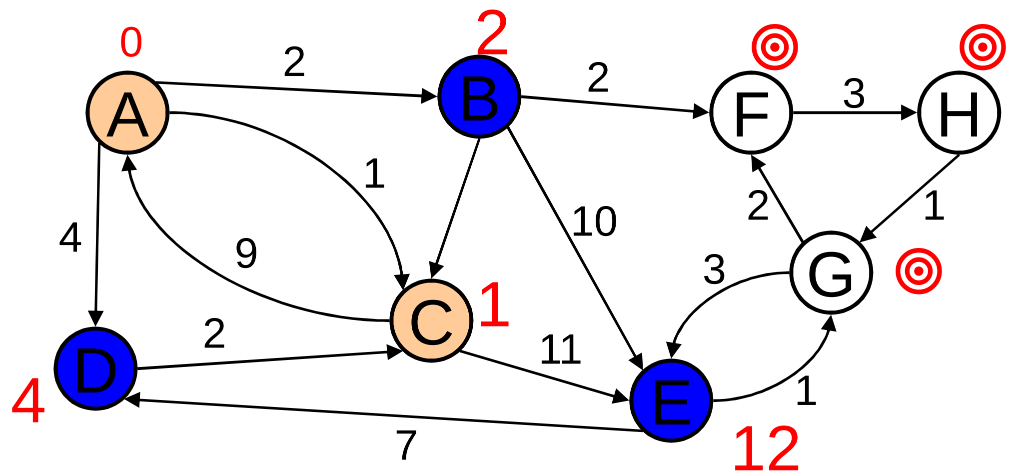
Vertex	Visited?	Cost	Found by
A		0	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	
H		??	

Dijkstra’s Algorithm in action



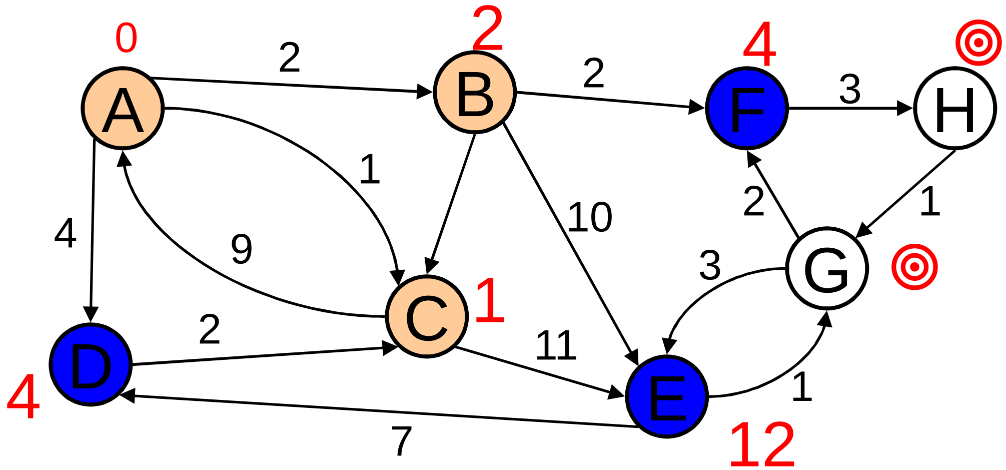
Vertex	Visited?	Cost	Found by
A	Y	0	
B		<=2	A
C		<=1	A
D		<=4	A
E		??	
F		??	
G		??	
H		??	

Dijkstra’s Algorithm in action



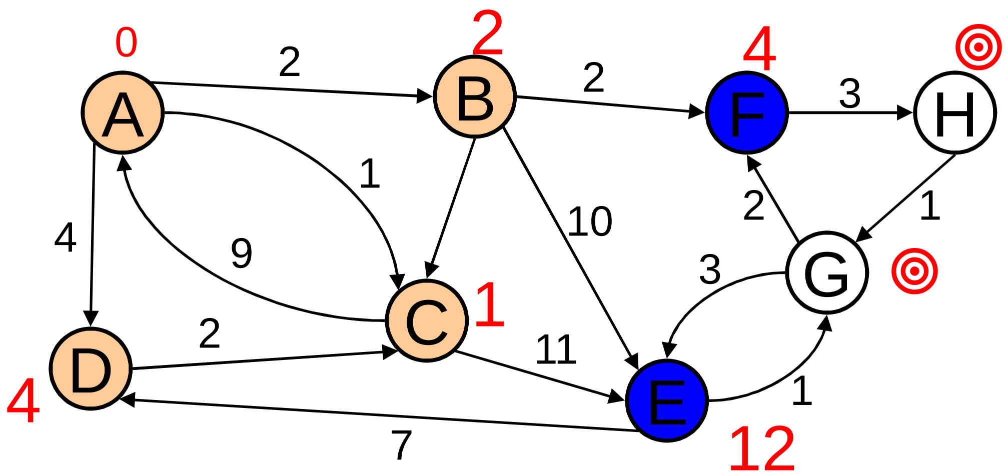
Vertex	Visited?	Cost	Found by
A	Y	0	
B		<=2	A
C	Y	1	A
D		<=4	A
E		<=12	C
F		??	
G		??	
H		??	

Dijkstra's Algorithm in action



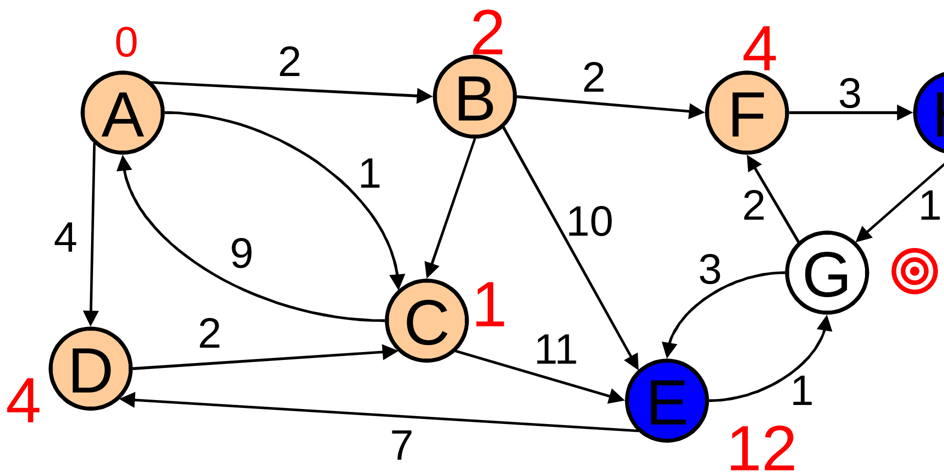
Vertex	Visited?	Cost	Found by
A	Y	0	
B	Y	2	A
C	Y	1	A
D		<=4	A
E		<=12	C
F		<=4	B
G		??	
H		??	

Dijkstra's Algorithm in action



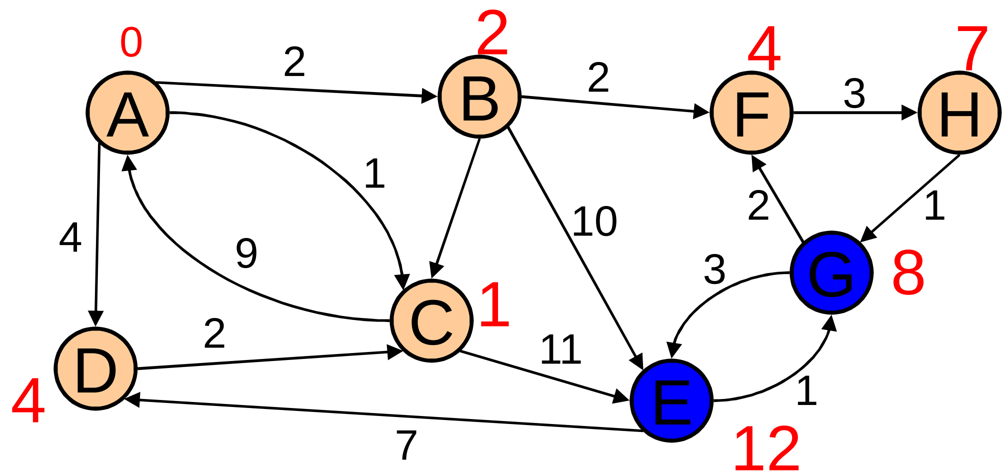
Vertex	Visited?	Cost	Found by
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		<=12	C
F		<=4	B
G		??	
H		??	

Dijkstra’s Algorithm in action



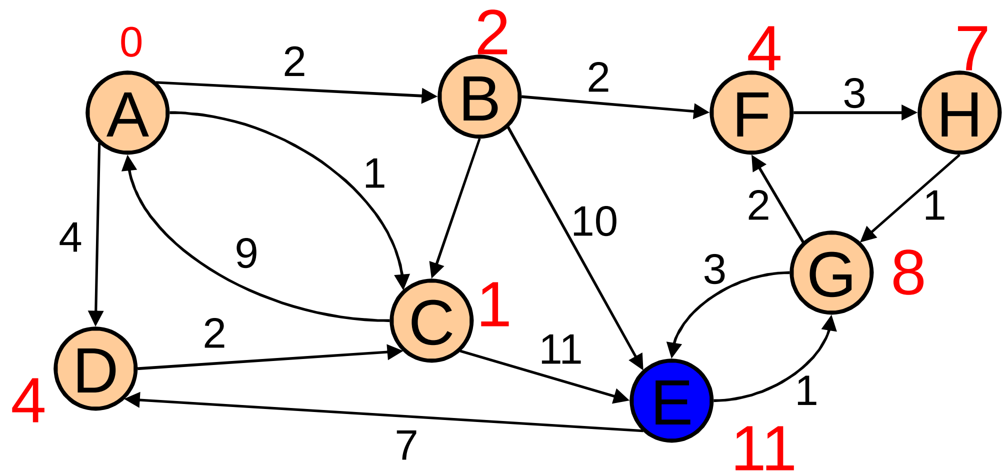
Vertex	Visited?	Cost	Found by
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		<=12	C
F	Y	4	B
G		??	
H		<=7	F

Dijkstra's Algorithm in action



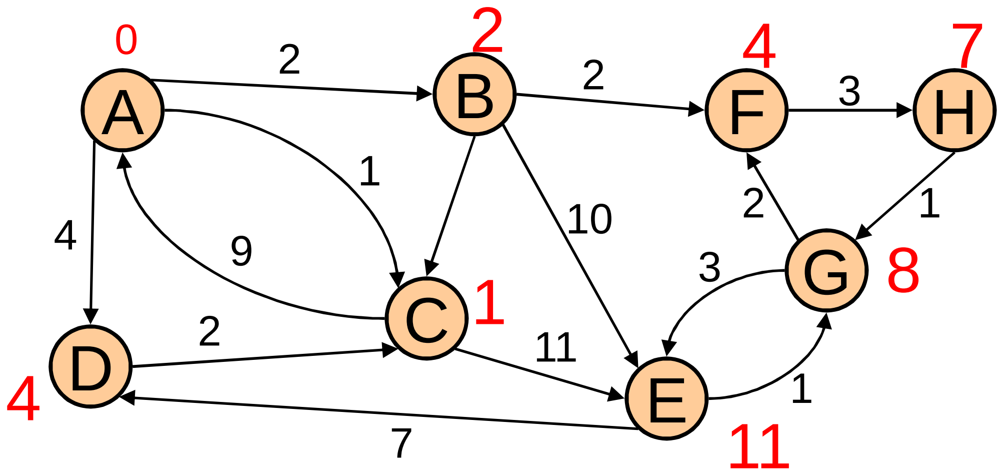
Vertex	Visited?	Cost	Found by
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		<=12	C
F	Y	4	B
G		<=8	H
H	Y	7	F

Dijkstra's Algorithm in action



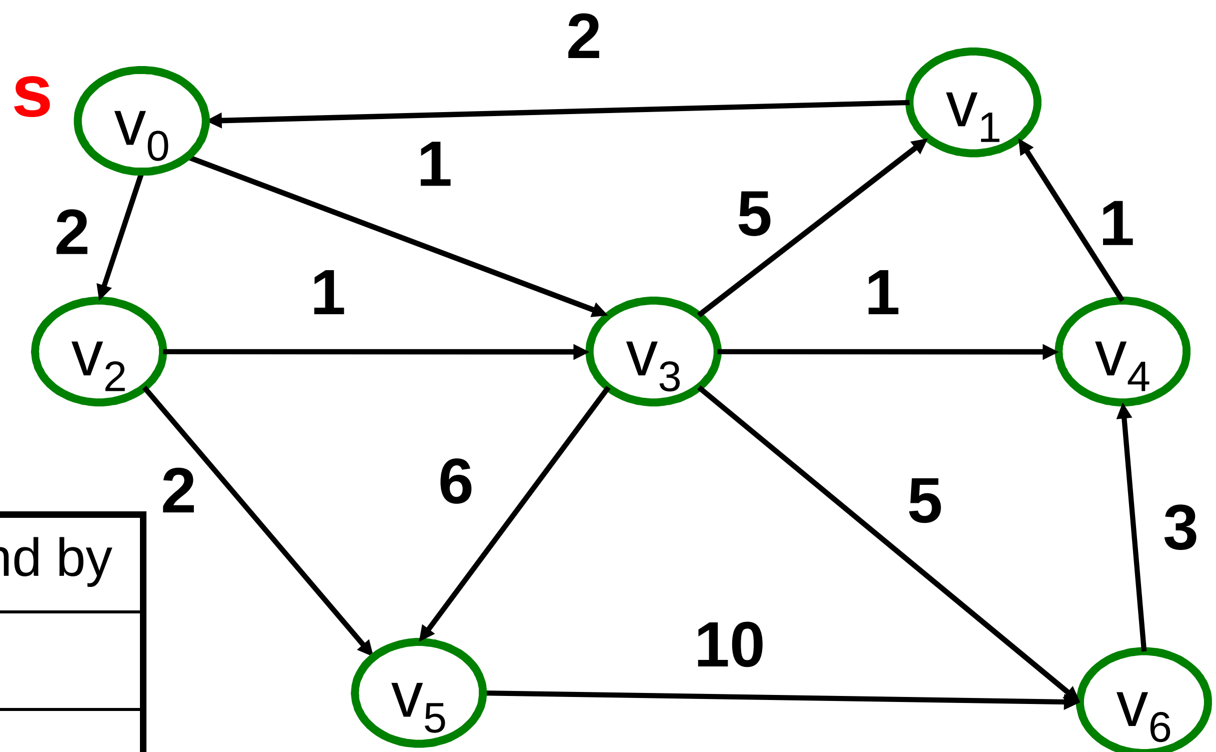
Vertex	Visited?	Cost	Found by
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		<=11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Dijkstra's Algorithm in action



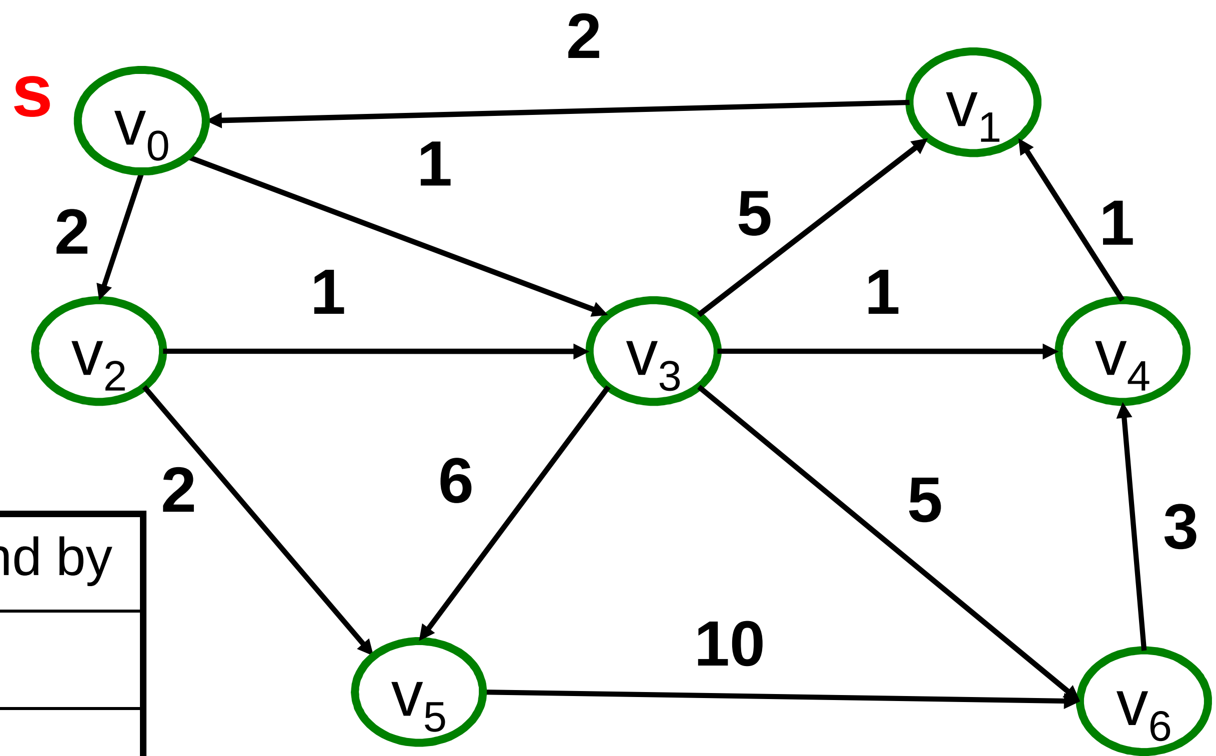
Vertex	Visited?	Cost	Found by
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Another



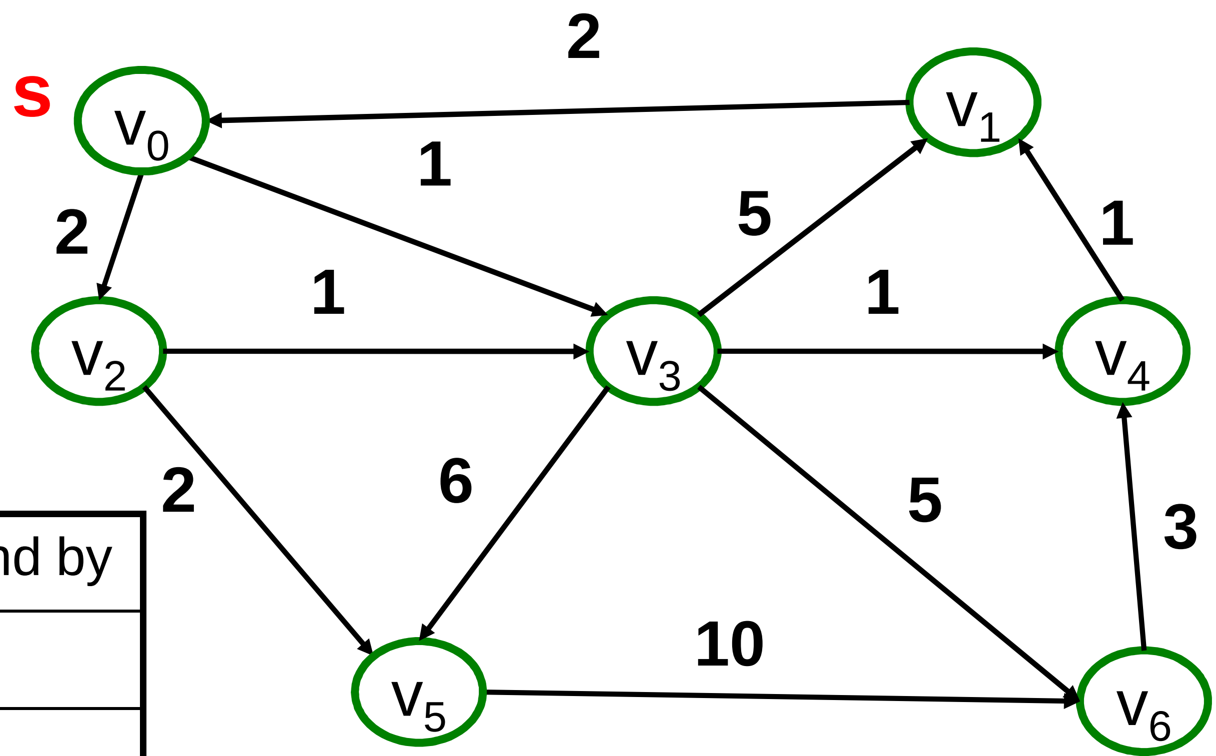
V	Visited?	Cost	Found by
v0			
v1			
v2			
v3			
v4			
v5			
v6			

Another



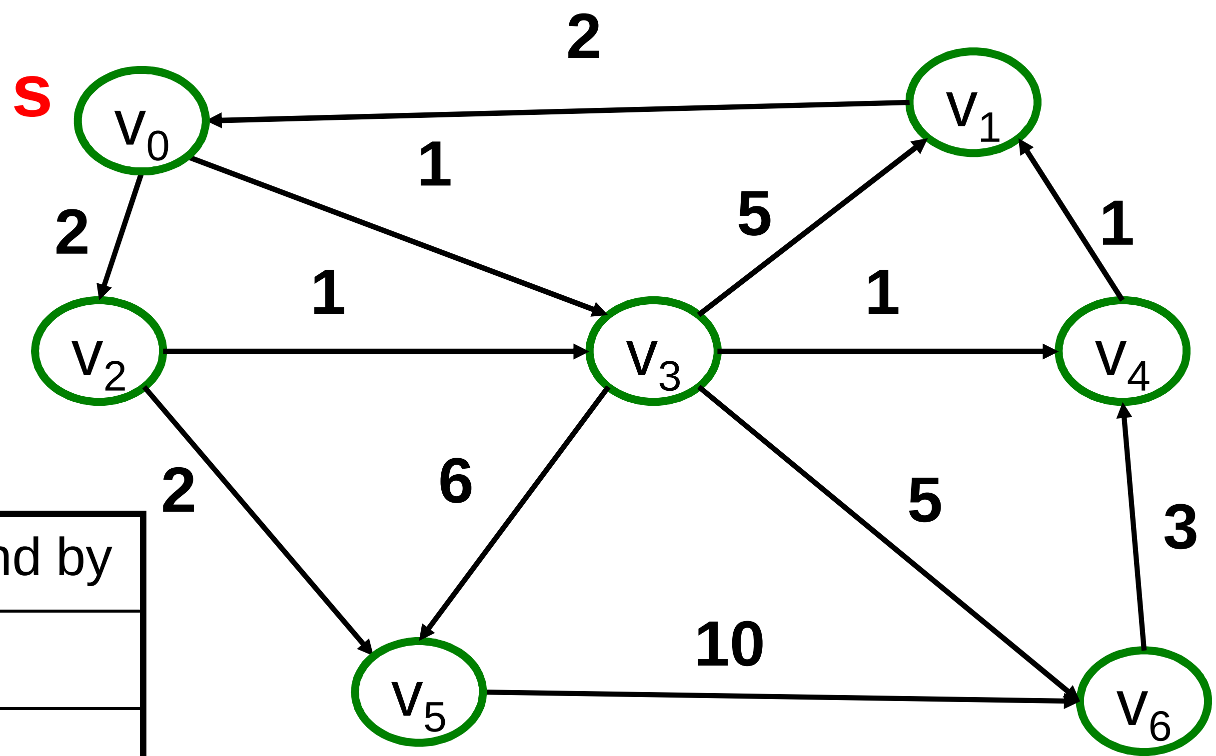
V	Visited?	Cost	Found by
v0	Y	0	
v1			
v2		<= 2	V0
v3		<= 1	V0
v4			
v5			
v6			

Another



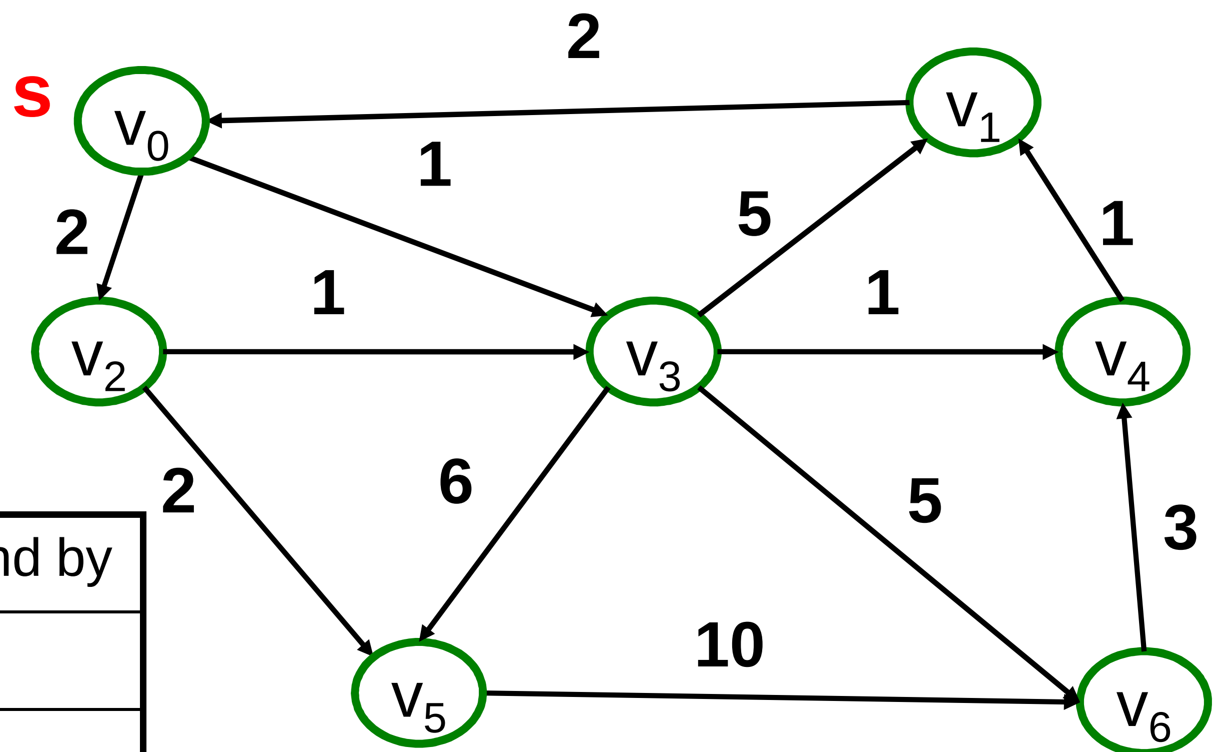
V	Visited?	Cost	Found by
v0	Y	0	
v1		<= 6	V3
v2		<= 2	V0
v3	Y	1	V0
v4		<= 2	V3
v5		<= 7	V3
v6		<= 6	V3

Another



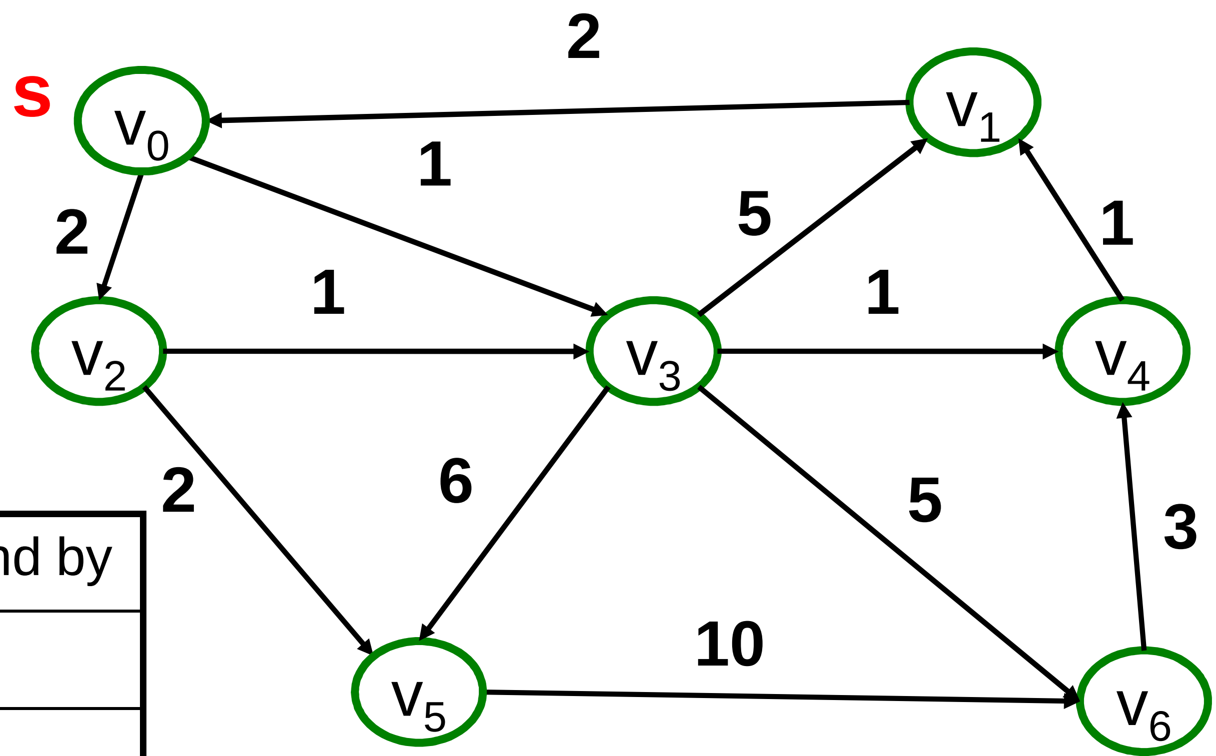
V	Visited?	Cost	Found by
v0	Y	0	
v1		<= 6	V3
v2	Y	2	V0
v3	Y	1	V0
v4		<= 2	V3
v5		<= 4	V2
v6		<= 6	V3

Another



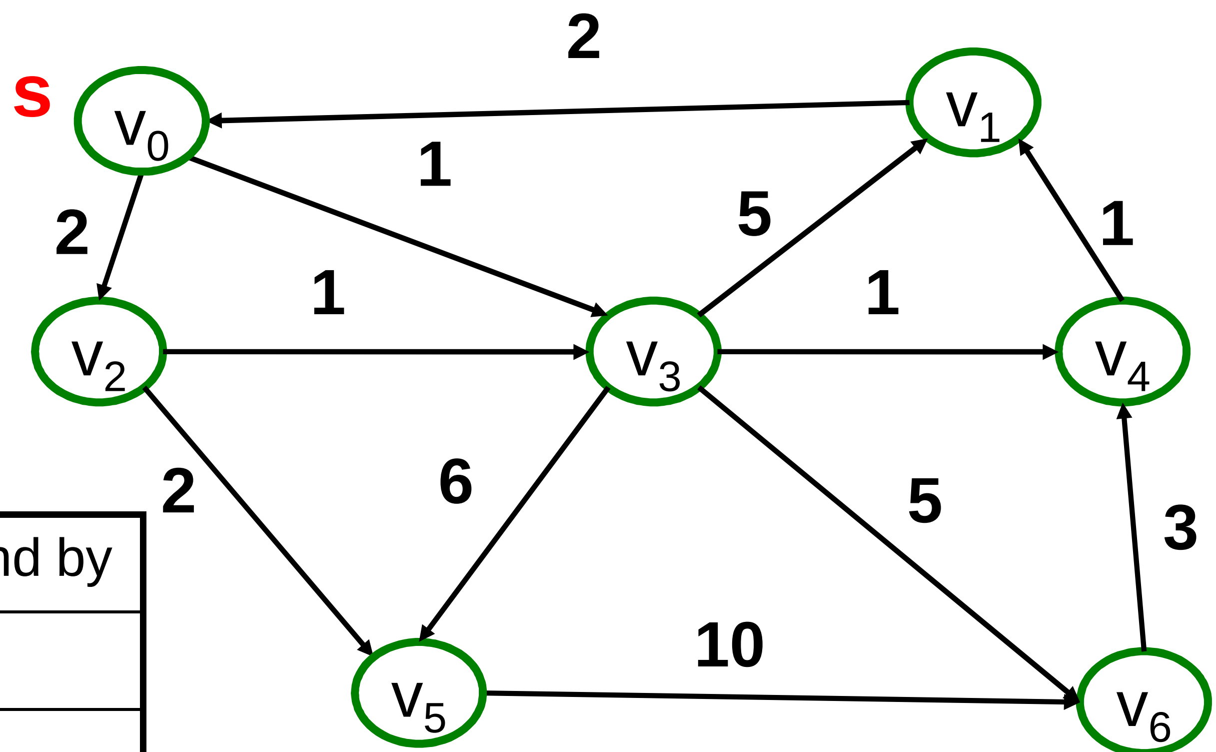
V	Visited?	Cost	Found by
v0	Y	0	
v1		<= 3	V4
v2	Y	2	V0
v3	Y	1	V0
v4	Y	2	V3
v5		<= 4	V2
v6		<= 6	V3

Another



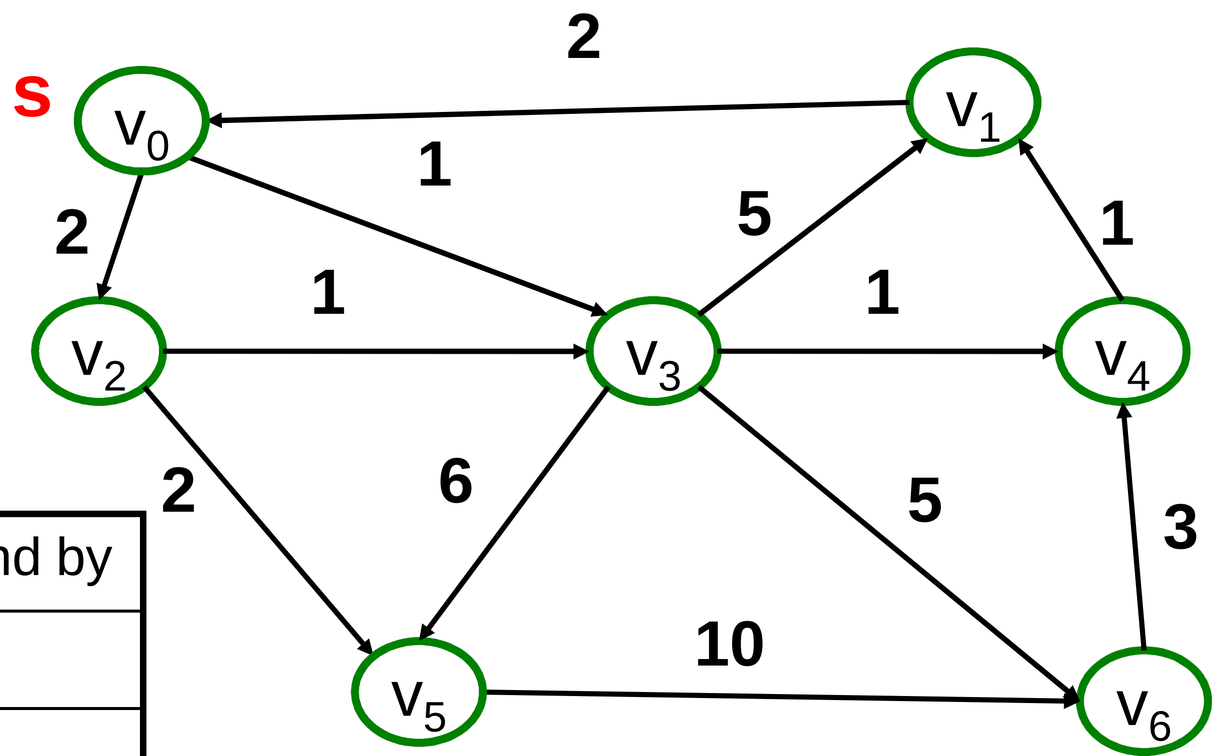
V	Visited?	Cost	Found by
v0	Y	0	
v1	Y	3	V4
v2	Y	2	V0
v3	Y	1	V0
v4	Y	2	V3
v5		<= 4	V2
v6		<= 6	V3

Another



V	Visited?	Cost	Found by
v0	Y	0	
v1	Y	3	V4
v2	Y	2	V0
v3	Y	1	V0
v4	Y	2	V3
v5	Y	4	V2
v6		<= 6	V3

Another



V	Visited?	Cost	Found by
v0	Y	0	
v1	Y	3	V4
v2	Y	2	V0
v3	Y	1	V0
v4	Y	2	V3
v5	Y	4	V2
v6	Y	6	V3

```
void Graph::dijkstra(Vertex s){
  Vertex v,w;

  Initialize s.dist = 0 and set dist of
  all other vertices to infinity

  while (there exist unknown vertices,
  find the one b with the smallest distance)
    b.known = true;

    for each a adjacent to b
      if (!a.known)
        if (b.dist + weight(b,a) < a.dist){
       a.dist = (b.dist + weight(b,a));
          a.path = b;
        }
  }
}
```

deleteMin on a heap…

adjacency lists

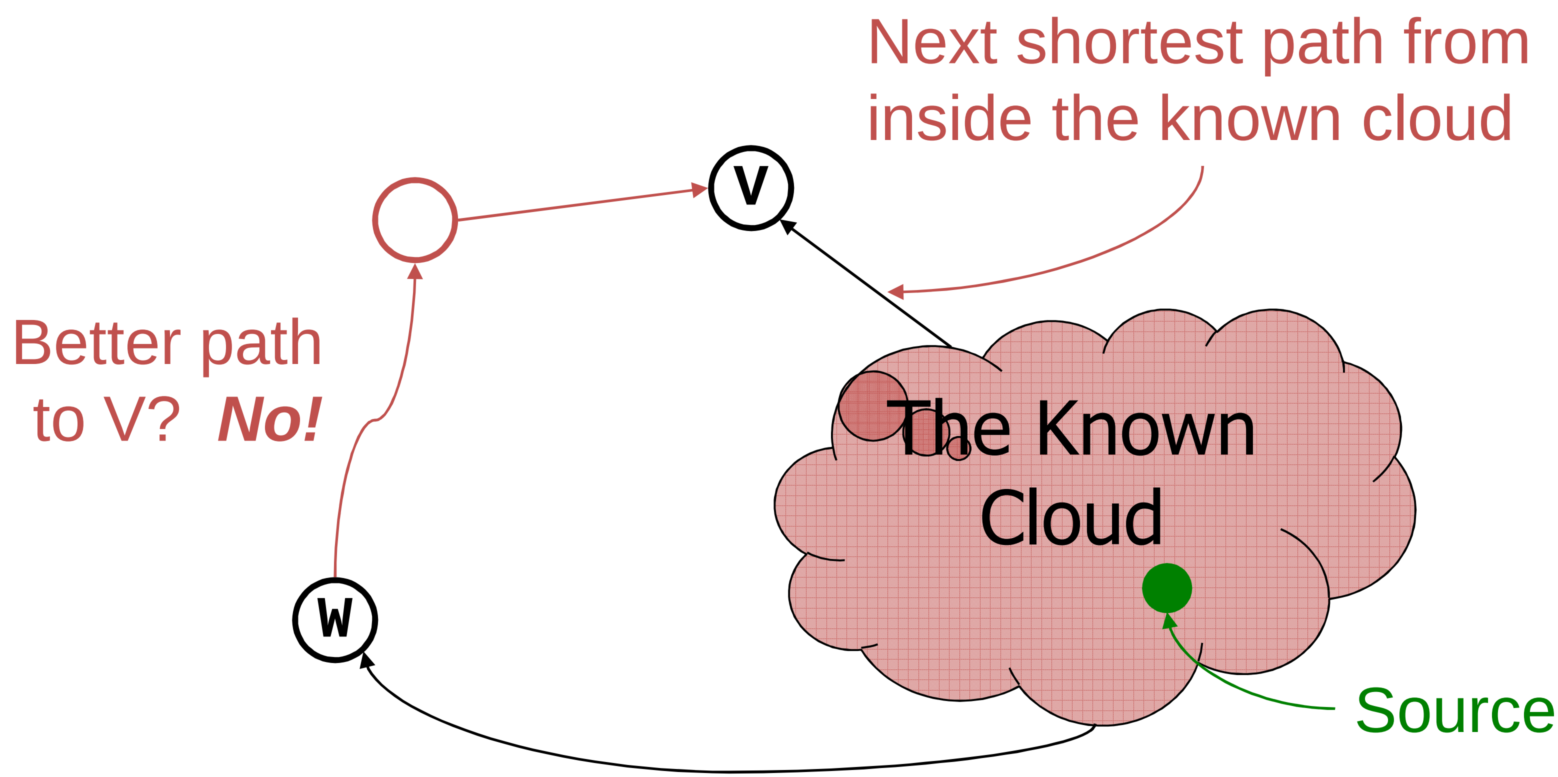
decreaseKey

Running time: O(|E| log |V|) – there are |E| edges to examine, and each one causes a heap operation of time O(log |V|)

Dijkstra's Algorithm: Summary

- Classic algorithm for solving SSSP in weighted graphs *without negative weights*
- A *greedy* algorithm (irrevocably makes decisions without considering future consequences)
- Intuition for correctness:
  - shortest path from source vertex to itself is 0
  - cost of going to adjacent nodes is at most edge weights
  - cheapest of these must be shortest path to that node
  - update paths for new node and continue picking cheapest path

Correctness: The Cloud Proof



How does Dijkstra's decide which vertex to add to the Known set next?

- If path to **V** is shortest, path to **W** must be *at least as long*
  (or else we would have picked **W** *as the next vertex)*
- So the path through **W** to **V** cannot be any shorter!

Correctness: Inside the Cloud

Prove by induction on # of nodes in the cloud:

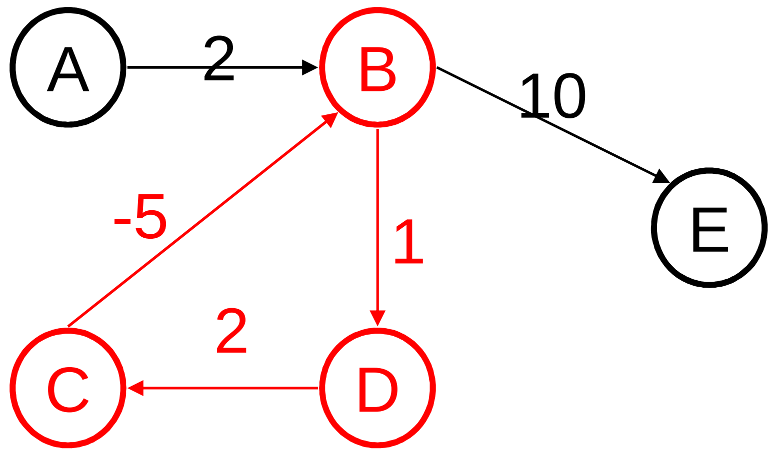
Initial cloud is just the source with shortest path 0

Assume: Everything inside the cloud has the correct shortest path

Inductive step: Only when we prove the shortest path to some node ***v*** (which is *not* in the cloud) is correct, we add it to the cloud

When does Dijkstra's algorithm not work?

The Trouble with Negative Weight Cycles



What's the shortest path from A to E?

Problem?

Dijkstra's vs BFS

At each step:

1) Pick closest unknown vertex
2) Add it to finished vertices
3) Update distances

Dijkstra's Algorithm

At each step:

1) Pick vertex from queue
2) Add it to visited vertices
3) Update queue with neighbors

Breadth-first Search

Two Questions

- What if I had multiple potential start points, and need to know the minimum cost of reaching each node from any start point?
- What if I want to know the minimum cost between every pair of nodes in the graph?

Single-Source Shortest Path

- Given a graph **G** = (V, E) and a single distinguished vertex **s**, find the shortest weighted path from **s** to every other vertex in **G**.

All-Pairs Shortest Path:

- Find the shortest paths between all pairs of vertices in the graph.
- How?

Analysis

- Total running time for Dijkstra's:

  $O(|V| \log |V| + |E| \log |V|)$ (heaps)

  What if we want to find the shortest path from each point to ALL other points?

Dynamic Programming

Algorithmic technique that systematically records the answers to sub-problems in a table and re-uses those recorded results (rather than re-computing them).

Simple Example: Calculating the Nth Fibonacci number.

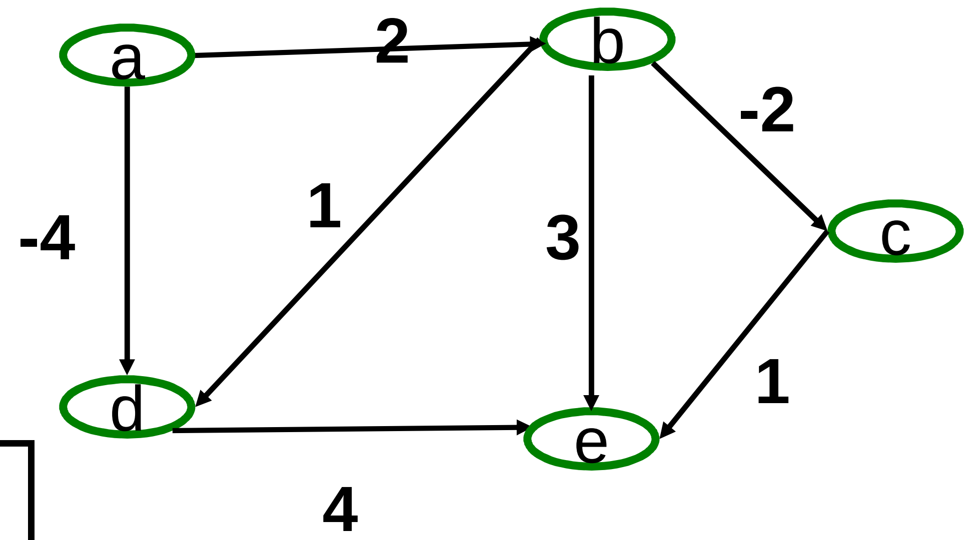
Fib(N) = Fib(N-1) + Fib(N-2)

Floyd-Warshall

```
for (int k = 1; k =< V; k++)
 for (int i = 1; i =< V; i++)
  for (int j = 1; j =< V; j++)
   if ( ( M[i][k]+ M[k][j] ) < M[i][j] )
   M[i][j] =  M[i][k]+ M[k][j]
```

Invariant: After the kth iteration, the matri
includes the shortest pathsfor all
pairs of vertices (i,j) containing only
vertices 1..k as intermediate vertices

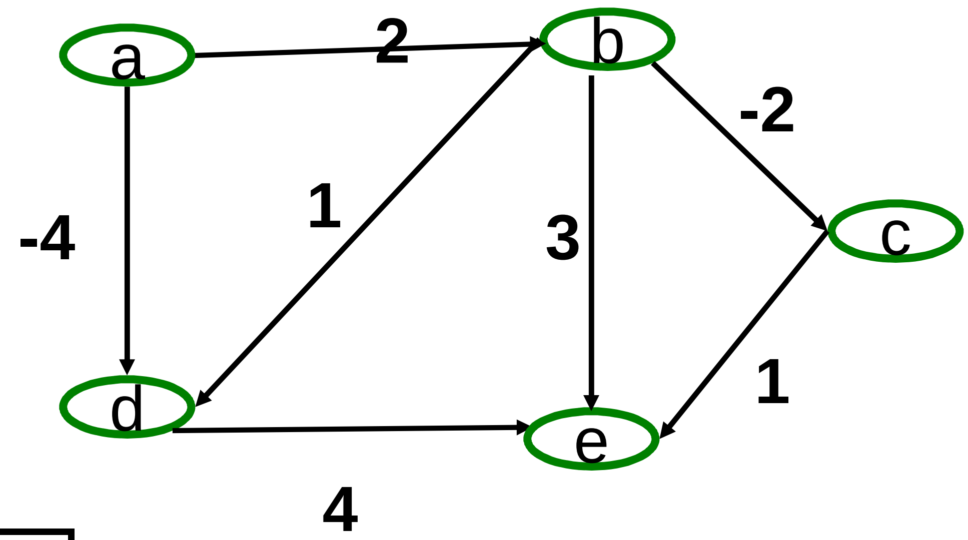
Initial state of the matrix:



	a	b	c	d	e
a	0	2	-	-4	-
b	-	0	-2	1	3
c	-	-	0	-	1
d	-	-	-	0	4
e	-	-	-	-	0

M[i][j] = min(M[i][j], M[i][k]+ M[k][j])

Floyd-Warshall - for All-pairs shortest path



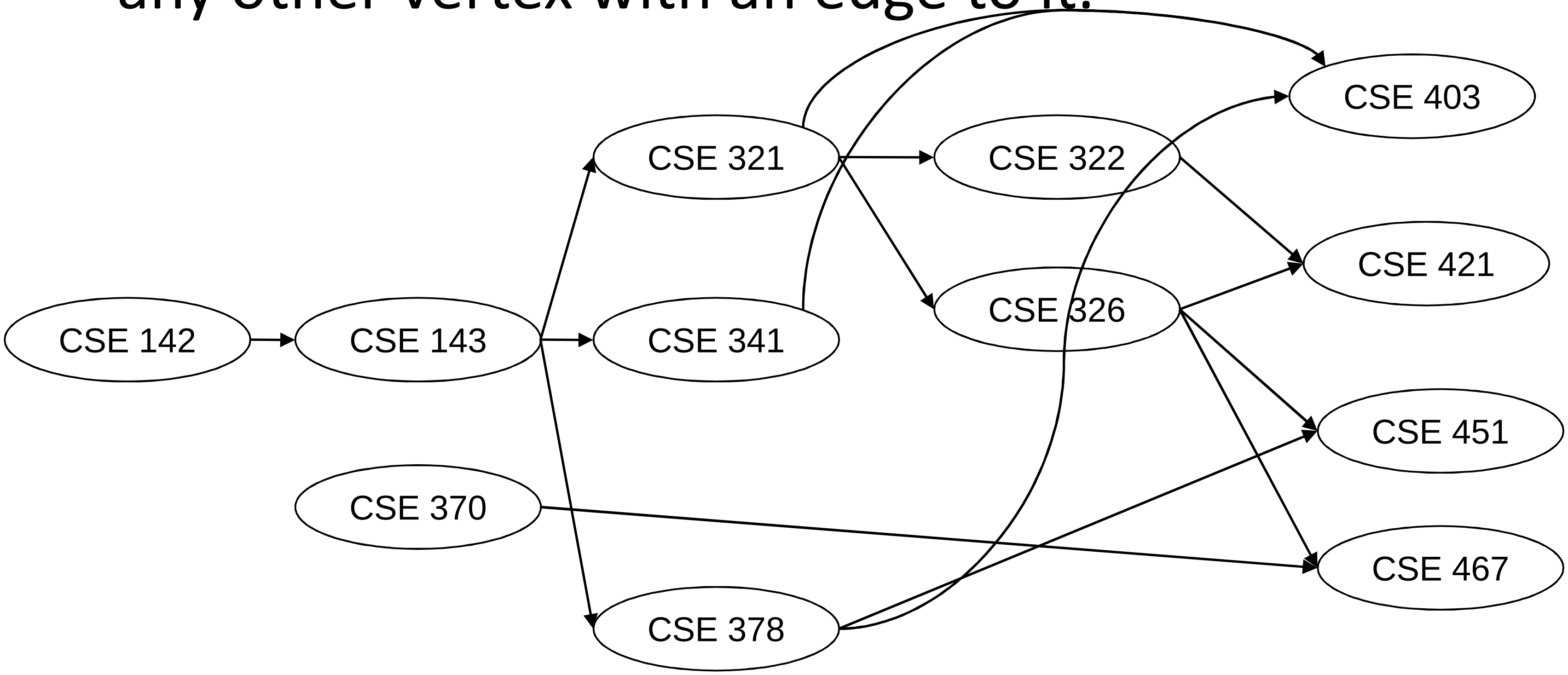
	a	b	c	d	e
a	0	2	0	-4	0
b	-	0	-2	1	-1
c	-	-	0	-	1
d	-	-	-	0	4
e	-	-	-	-	0

Final Matrix Contents

This is a partial ordering, for sorting we had a total ordering

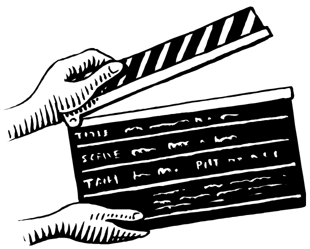
Application: Topological Sort

Given a directed graph, **`G = (V,E)`**, output all the vertices in **V** such that no vertex is output before any other vertex with an edge to it.



Is the output unique?

Minimize and DO a topo sort



Topological Sort: Take One

1. Label each vertex with its *in-degree* (# of inbound edges)
2. **While** there are vertices remaining:
    a. Choose a vertex *v* of *in-degree zero*; output *v*
    b. Reduce the in-degree of all vertices adjacent to *v*
    c. Remove *v* from the list of vertices

Runtime:

```
void Graph::topsort(){
  Vertex v, w;

  labelEachVertexWithItsIn-degree();

  for (int counter=0; counter < NUM_VERTICES;
                         counter++){
    v = findNewVertexOfDegreeZero();

    v.topologicalNum = counter;
    for each w adjacent to v
      w.indegree--;
  }
}
```

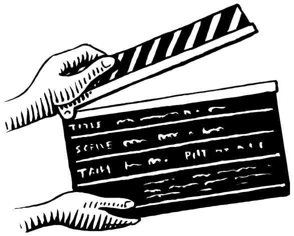
Time?

Time?

Time?

What's the bottleneck?

O(depends)



Topological Sort: Take Two

1. Label each vertex with its in-degree
2. Initialize a queue *Q* to contain all in-degree zero vertices
3. While *Q* not empty
   a. *v* = *Q*.dequeue; output *v*
   b. Reduce the in-degree of all vertices adjacent to *v*
   c. If new in-degree of any such vertex *u* is zero *Q*.enqueue(*u*)

<u>Note</u>: could use a stack, list, set, box, … instead of a queue

Runtime:

```
void Graph::topsort(){
  Queue q(NUM_VERTICES);  int counter = 0; Vertex v, w;
   labelEachVertexWithItsIn-degree();

  q.makeEmpty();
  for each vertex v
    if (v.indegree == 0)
      q.enqueue(v);

  while (!q.isEmpty()){
    v = q.dequeue();
    v.topologicalNum = ++counter;
    for each w adjacent to v
      if (--w.indegree == 0)
        q.enqueue(w);
  }
}
```

intialize the queue

get a vertex with indegree 0

insert new eligible vertices

Runtime: O(|V| + |E|)