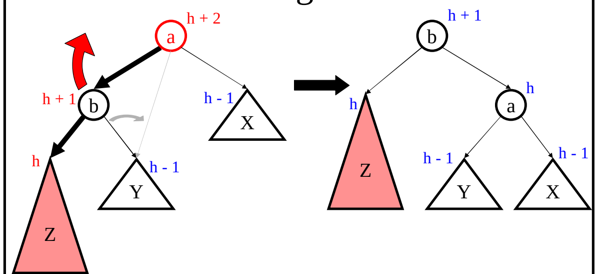
Extra AVL Tree Slides

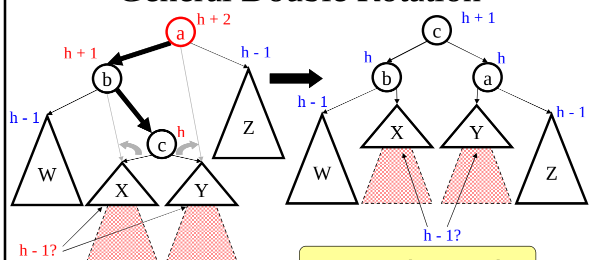
Insert into Z, increasing height

General Single Rotation



- Height of subtree same as it was before insert!
- Height of all ancestors unchanged.

General Double Rotation



X,Y: one is h-1, one is h-2

- Height of subtree **still** the same as it was before insert!
- Height of all ancestors unchanged.

Height of an AVL tree

Theorem: Any AVL tree with *n* nodes has height less than 1.441 log *n.*

Proof: Given an *n*-node AVL tree, we want to find an upper bound on the height of the tree.

Fix *h*. What is the smallest *n* such that there is an AVL tree of height *h* with *n* nodes?

Let W_h be the set of all AVL trees of height *h* that have as few nodes as possible.

Let S(h) be the number of nodes in any one of these trees.

S(0) = 1, S(1) = 2

Suppose $T \in W_h$, where $h \geq 2$. Let T_L and T_R be T's left and right subtrees. Since T has height *h*, either T_L or T_R has height *h*-1. Suppose it's T_R.

By definition, both T_L and T_R are AVL trees. In fact, $T_R \in W_{h-1}$ or else it could be replaced by a smaller AVL tree of height *h*-1 to give an AVL tree of height *h* that is smaller than T.

Similarly, $T_L \in W_{h-2}$.

Therefore, S(h) = 1 + S(h-2) + S(h-1) .

Claim: For $h \geq 0$, $S(h) \geq \varphi^h$, where $\varphi = (1 + \sqrt{5}) / 2 \approx 1.6$.

Note: from Fibonacci #s, Golden Ratio

Proof: The proof is by induction on *h*.

Basis step: *h*=0. $S(0) = 1 = \varphi^0$.

h=1. $S(1) = 2 > \varphi^1$.

Induction step: Suppose the claim is true for $0 \leq m \leq h$, where $h \geq 1$.

Then:

$S(h+1) = 1 + S(h-1) + S(h)$

$\geq 1 + \varphi^{h-1} + \varphi^{h}$ (by the i.h.)

$= 1 + \varphi^{h-1}(1 + \varphi)$ (by math)

$= 1 + \varphi^{h+1}$ (using $1+\varphi = \varphi^2$)

$> \varphi^{h+1}$ Thus, the claim is true.

From the claim, in an n-node AVL tree of height h,

$n \geq S(h) \geq \varphi^h$ (from the Claim)

$h \leq \log_\varphi n$ (by math – $\log_\varphi$ of both sides)

$= (\log n) / (\log \varphi)$

$< 1.441 \log n$

AVL tree: Running times

- find takes O(log n) time, because height of the tree is always O(log n).
- insert: O(log n) time because we do a find (O(log n) time), and then we may have to visit every node on the path back to the root, performing up to 2 single rotations (O(1) time each) to fix the tree.
- remove: O(log n) time. Left as an exercise.

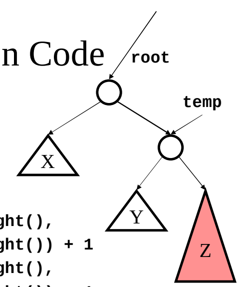
AVL Insert Algorithm

- Recursive
```
1. Search downward for
   spot
2. Insert node
3. Unwind stack,
   correcting heights
   a. If imbalance #1,
      single rotate
   b. If imbalance #2,
      double rotate
```
- Iterative
```
1. Search downward for
   spot, stacking
   parent nodes
2. Insert node
3. Unwind stack,
   correcting heights
   a. If imbalance #1,
      single rotate and
      exit
   b. If imbalance #2,
      double rotate and
      exit
```

Why use a stack?

RotateRight brings up the right child

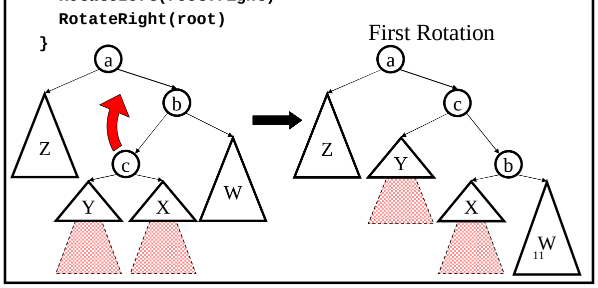
Single Rotation Code




```
void RotateRight(Node root) {
  Node temp = root.right
  root.right = temp.left
  temp.left = root
  root.height = max(root.right.height(),
                    root.left.height()) + 1
  temp.height = max(temp.right.height(),
                    temp.left.height()) + 1
  root = temp
}
```


Double Rotation Code

```
void DoubleRotateRight(Node root) {
  RotateLeft(root.right)
  RotateRight(root)
}
```



Double Rotation Completed