CSE 373: Data Structures and Algorithms

Lecture 5: Math Review/Asymptotic Analysis III

Efficiency examples 6

```
int sum = 0;
for (int i = 1; i <= n; i++) {
    for (int j = 1; j <= i / 2; j += 2) {
        sum++;
    }
}
```

Math background: Arithmetic series

- Series

$$\sum_{i=j}^{k} Expr$$

  - for some expression *Expr* (possibly containing *i*), means the sum of all values of *Expr* with each value of *i* between *j* and *k* inclusive

Example:

$$\sum_{i=0}^{4} 2i+1$$

= (2(0) + 1) + (2(1) + 1) + (2(2) + 1)
+ (2(3) + 1) + (2(4) + 1)
= 1 + 3 + 5 + 7 + 9
= 25

Series identities

- sum from 1 through N inclusive

$$\sum_{i=1}^{N} i = \frac{N(N+1)}{2}$$

- is there an intuition for this identity?
  - sum of all numbers from 1 to N

    1 + 2 + 3 + ... + (N-2) + (N-1) + N

  - how many terms are in this sum? Can we rearrange them?

More series identities

- sum from *a* through N inclusive
(when the series doesn't start at 1)

$$\sum_{i=a}^{N} i = \sum_{i=1}^{N} i - \sum_{i=1}^{a-1} i$$

- is there an intuition for this identity?

Series of constants

- sum of constants
  (when the body of the series doesn't contain the counter variable such as *i*)

$$\sum_{i=a}^{b} k = k\sum_{i=a}^{b} 1 = k(b-a+1)$$

- example:

$$\sum_{i=4}^{10} 5 = 5\sum_{i=4}^{10} 1 = 5(10-4+1) = 35$$

Splitting series

for any constant *k*,

- splitting a sum with addition

$$\sum_{i=a}^{b}(i+k)=\sum_{i=a}^{b}i+\sum_{i=a}^{b}k$$

- moving out a constant multiple

$$\sum_{i=a}^{b}ki=k\sum_{i=a}^{b}i$$

Series of powers

- sum of powers of 2

$$\sum_{i=0}^{N} 2^i = 2^{N+1} - 1$$

  – 1 + 2 + 4 + 8 + 16 + 32 = 64 - 1 = 63
  – think about binary representation of numbers...

```
  111111 (63)
 +     1 (1)
 -------
 1000000 (64)
```

- when the series doesn't start at 0:

$$\sum_{i=a}^{N} 2^i = \sum_{i=0}^{N} 2^i - \sum_{i=0}^{a-1} 2^i$$

Series practice problems

- Give a closed form expression for the following summation.
  - A closed form expression is one without the Σ or "…".

$$\sum_{i=0}^{N-2} 2i$$

- Give a closed form expression for the following summation.

$$\sum_{i=10}^{N-1} (i-5)$$

Efficiency examples 6 (revisited)

```
int sum = 0;
for (int i = 1; i <= n; i++) {
    for (int j = 1; j <= i / 2; j += 2) {
        sum++;
    }
}
```

- Compute the value of the variable sum after the following code fragment, as a closed-form expression in terms of input size n.
  - Ignore small errors caused by i not being evenly divisible by 2 and 4.

Efficiency examples 6 (revisited)

```
int sum = 0;
for (int i = 1; i <= n; i++) {
    for (int j = 1; j <= i / 2; j += 2) {
        sum++;
    }
}
```

Growth rate terminology (recap)

- f(n) = O(g(N))
  - g(n) is an **upper bound** on f(n)
  - f(n) **grows no faster** than g(n)

- f(n) = Ω(g(N))
  - g(N) is a **lower bound** on f(n)
  - f(n) grows at least as fast as g(N)

- f(n) = Θ(g(N))
  - f(n) grows at the same rate as g(N)

Facts about big-Oh

- If $T_1(N) = O(f(N))$ and $T_2(N) = O(g(N))$, then
  - $T_1(N) + T_2(N) = O(f(N) + g(N))$
  - $T_1(N) * T_2(N) = O(f(N) * g(N))$
- If T(N) is a polynomial of degree *k*, then: $T(N) = \Theta(N^k)$
  - example: $17n^3 + 2n^2 + 4n + 1 = \Theta(n^3)$
- $\log^k N = O(N)$, for any constant *k*

Complexity classes

- **complexity class**: A category of algorithm efficiency based on the algorithm's relationship to the input size N.

Class	Big-Oh	If you double N, ...	Example
constant	O(1)	unchanged	10ms
logarithmic	$O(\log_2 N)$	increases slightly	175ms
linear	O(N)	doubles	3.2 sec
log-linear	$O(N \log_2 N)$	slightly more than doubles	6 sec
quadratic	$O(N^2)$	quadruples	1 min 42 sec
cubic	$O(N^3)$	multiplies by 8	55 min
...	...	...	...
exponential	$O(2^N)$	multiplies drastically	$5 * 10^{61}$ years

Complexity cases

- **Worst-case complexity**: “most challenging” input of size n
- **Best-case complexity:** “easiest” input of size n
- **Average-case complexity**: random inputs of size n
- **Amortized complexity**: m “most challenging” *consecutive* inputs of size n, divided by m

Bounds vs. Cases

Two orthogonal axes:

- Bound
  - Upper bound (O)
  - Lower bound (Ω)
  - Asymptotically tight (Θ)
- Analysis Case
  - Worst Case (Adversary), $T_{worst}(n)$
  - Average Case, $T_{avg}(n)$
  - Best Case, $T_{best}(n)$
  - Amortized, $T_{amort}(n)$

One can estimate the bounds for any given case.

Example

`List.contains(Object o)`

- returns `true` if the list contains `o`; `false` otherwise
- Input size: *n* (the length of the `List`)
- *f*(*n*) = "running time for size *n*"
- But *f*(*n*) needs clarification:
  - Worst case *f*(*n*): it runs in at most *f*(*n*) time
  - Best case *f*(*n*): it takes at least *f*(*n*) time
  - Average case *f*(*n*): average time

Recursive programming

- A method in Java can call itself; if written that way, it is called a *recursive method*
- The code of a recursive method should be written to handle the problem in one of two ways:
  - **base case**: a simple case of the problem that can be answered directly; does not use recursion.
  - **recursive case**: a more complicated case of the problem, that isn't easy to answer directly, but can be expressed elegantly with recursion; makes a recursive call to help compute the overall answer

Recursive power function

- Defining powers recursively:

```
pow(x, 0) = 1
pow(x, y) = x * pow(x, y-1),   y > 0
```

```
// recursive implementation
public static int pow(int x, int y) {
    if (y == 0) {
        return 1;
    } else {
        return x * pow(x, y - 1);
    }
}
```