CSE373: Data Structure & Algorithms

Lecture 18: Comparison Sorting

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Introduction to Sorting

- Stacks, queues, priority queues, and dictionaries all focused on providing one element at a time

- But often we know we want "all the things" in some order
  - Humans can sort, but computers can sort fast
  - Very common to need data sorted somehow
    - Alphabetical list of people
    - List of countries ordered by population
    - Search engine results by relevance
    - …

- Algorithms have different asymptotic and constant-factor trade-offs
  - No single "best" sort for all scenarios
  - Knowing one way to sort just isn't enough

More Reasons to Sort

General technique in computing:

Preprocess data to make subsequent operations faster

Example: Sort the data so that you can

- Find the **`k`**th largest in constant time for any **`k`**
- Perform binary search to find elements in logarithmic time

Whether the performance of the preprocessing matters depends on

- How often the data will change (and how much it will change)
- How much data there is

The main problem, stated carefully

For now, assume we have *n* comparable elements in an array and we want to rearrange them to be in increasing order

Input:

- An array **`A`** of data records
- A key value in each data record
- A comparison function (consistent and total)

Effect:

- Reorganize the elements of **`A`** such that for any **`i`** and **`j`**, if **`i < j`** then **`A[i]`** ≤ **`A[j]`**
- (Also, **`A`** must have exactly the same data it started with)
- Could also sort in reverse order, of course

An algorithm doing this is a comparison sort

Variations on the Basic Problem

1. Maybe elements are in a linked list (could convert to array and back in linear time, but some algorithms needn't do so)

2. Maybe ties need to be resolved by "original array position"
   – Sorts that do this naturally are called stable sorts
   – Others could tag each item with its original position and adjust comparisons accordingly (non-trivial constant factors)

3. Maybe we must not use more than $O(1)$ "auxiliary space"
   – Sorts meeting this requirement are called in-place sorts

4. Maybe we can do more with elements than just compare
   – Sometimes leads to faster algorithms

5. Maybe we have too much data to fit in memory
   – Use an "external sorting" algorithm

Sorting: The Big Picture

Surprising amount of neat stuff to say about sorting:

Simple algorithms: $O(n^2)$

- **Insertion sort**
- **Selection sort**
- **Shell sort**
- **…**

Fancier algorithms: $O(n \log n)$

- **Heap sort**
- **Merge sort**
- **Quick sort (avg)**
- **…**

Comparison lower bound: $\Omega(n \log n)$

Specialized algorithms: $O(n)$

- **Bucket sort**
- **Radix sort**

Handling huge data sets

- **External sorting**

Insertion Sort

- Idea: At step **`k`**, put the **`k`**th element in the correct position among the first **`k`** elements

- Alternate way of saying this:
  - Sort first two elements
  - Now insert 3rd element in order
  - Now insert 4th element in order
  - …

- "Loop invariant": when loop index is **`i`**, first **`i`** elements are sorted

- Time?

  Best-case _____ Worst-case _____ "Average" case ____

Insertion Sort

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- Alternate way of saying this:
  - Sort first two elements
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  - Now insert 4th element in order
  - …

- "Loop invariant": when loop index is **`i`**, first **`i`** elements are sorted

- Time?

  Best-case O(n) Worst-case O(n^2) "Average" case O(n^2)
  start sorted start reverse sorted (see text)

Selection sort

- Idea: At step **`k`**, find the smallest element among the not-yet-sorted elements and put it at position k

- Alternate way of saying this:
  - Find smallest element, put it 1st
  - Find next smallest element, put it 2nd
  - Find next smallest element, put it 3rd
  - …

- "Loop invariant": when loop index is **`i`**, first **`i`** elements are the **`i`** smallest elements in sorted order

- Time?

  Best-case _____ Worst-case _____ "Average" case ____

Selection sort

- Idea: At step **`k`**, find the smallest element among the not-yet-sorted elements and put it at position k

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  - …

- "Loop invariant": when loop index is **`i`**, first **`i`** elements are the **`i`** smallest elements in sorted order

- Time?

  Best-case $O(n^2)$ Worst-case $O(n^2)$ "Average" case $O(n^2)$

  Always $T(1) = 1$ and $T(n) = n + T(n-1)$

Mystery

This is one implementation of which sorting algorithm (for ints)?

```
void mystery(int[] arr) {
  for(int i = 1; i < arr.length; i++) {
     int tmp = arr[i];
     int j;
     for(j=i; j > 0 && tmp < arr[j-1]; j--)
        arr[j] = arr[j-1];
     arr[j] = tmp;
  }
}
```

Note: Like with heaps, “moving the hole” is faster than unnecessary swapping (constant-factor issue)

Insertion Sort vs. Selection Sort

- Different algorithms
- Solve the same problem
- Have the same worst-case and average-case asymptotic complexity
  - Insertion-sort has better best-case complexity; preferable when input is “mostly sorted”
- Other algorithms are more efficient *for non-small arrays that are not already almost sorted*
  - Insertion sort may do well on small arrays

Aside: We Will Not Cover Bubble Sort

- It is not, in my opinion, what a “normal person” would think of
- It doesn’t have good asymptotic complexity: $O(n^2)$
- It’s not particularly efficient with respect to common factors

Basically, almost everything it is good at some other algorithm is at least as good at

- Perhaps people teach it just because someone taught it to them?

Fun, short, optional read:

Bubble Sort: An Archaeological Algorithmic Analysis, Owen Astrachan, SIGCSE 2003

http://www.cs.duke.edu/~ola/bubble/bubble.pdf

The Big Picture

Surprising amount of juicy computer science: 2-3 lectures…

Simple algorithms: $O(n^2)$	Fancier algorithms: $O(n \log n)$	Comparison lower bound: $\Omega(n \log n)$	Specialized algorithms: $O(n)$	Handling huge data sets
Insertion sort Selection sort Shell sort …	Heap sort Merge sort Quick sort (avg) …		Bucket sort Radix sort	External sorting

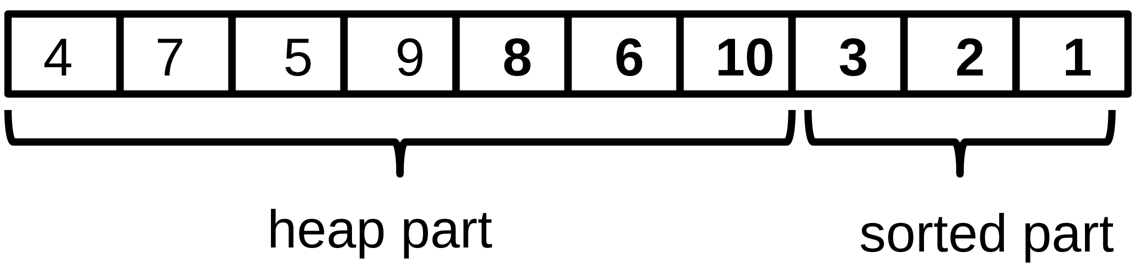
Heap sort

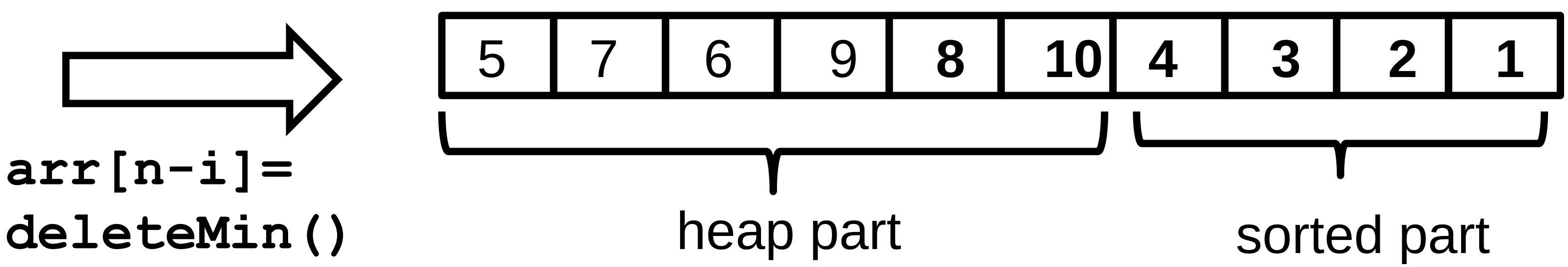
- Sorting with a heap is easy:
  - **`insert`** each **`arr[i]`**, or better yet use **`buildHeap`**
  - ```
    for(i=0; i < arr.length; i++)
        arr[i] = deleteMin();
    ```
- Worst-case running time: $O(n \log n)$
- We have the array-to-sort and the heap
  - So this is not an in-place sort
  - There's a trick to make it in-place…
    ```

In-place heap sort

But this reverse sorts – how would you fix that?

– Treat the initial array as a heap (via **`buildHeap`**)
– When you delete the **`i`**th element, put it at **`arr[n-i]`**
  • That array location isn't needed for the heap anymore!





"AVL sort"

- We can also use a balanced tree to:
  - **`insert`** each element: total time $O(n \log n)$
  - Repeatedly **`deleteMin`**: total time $O(n \log n)$
    - Better: in-order traversal $O(n)$, but still $O(n \log n)$ overall

- But this cannot be made in-place and has worse constant factors than heap sort
  - both are $O(n \log n)$ in worst, best, and average case
  - neither parallelizes well
  - heap sort is better

“Hash sort”???

- Don’t even think about trying to sort with a hash table!

- Finding min item in a hashtable is O(n), so this would be a slower, more complicated selection sort

Divide and conquer

Very important technique in algorithm design

1. Divide problem into smaller parts

2. Independently solve the simpler parts
   - Think recursion
   - Or potential parallelism

3. Combine solution of parts to produce overall solution

(The name “divide and conquer” is rather clever.)

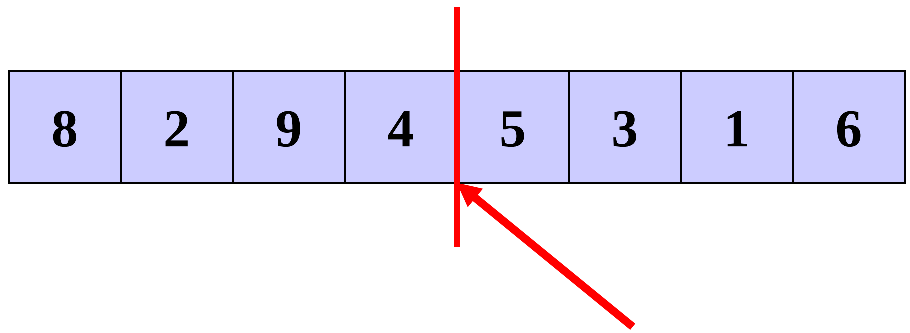
Divide-and-Conquer Sorting

Two great sorting methods are fundamentally divide-and-conquer

1. Mergesort: Sort the left half of the elements (recursively)
   Sort the right half of the elements (recursively)
   Merge the two sorted halves into a sorted whole

2. Quicksort: Pick a “pivot” element
   Divide elements into less-than pivot
   and greater-than pivot
   Sort the two divisions (recursively on each)
   Answer is sorted-less-than then pivot then
   sorted-greater-than

Mergesort



- To sort array from position **`lo`** to position **`hi`**:
  - If range is 1 element long, it is already sorted! (Base case)
  - Else:
    - Sort from **`lo`** to **`(hi+lo)/2`**
    - Sort from **`(hi+lo)/2`** to **`hi`**
    - Merge the two halves together

- Merging takes two sorted parts and sorts everything
  - $O(n)$ but requires auxiliary space…

Example, Focus on Merging

Start with:

8	**2**	**9**	**4**	**5**	**3**	**1**	**6**

After recursion:

(not magic ☺)

2	**4**	**8**	**9**	**1**	**3**	**5**	**6**

Merge:

Use 3 “fingers”

and 1 more array

(After merge,
copy back to
original array)

Example, focus on merging

Start with:

8	**2**	**9**	**4**	**5**	**3**	**1**	**6**

After recursion:
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1							

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Example, focus on merging

Start with:

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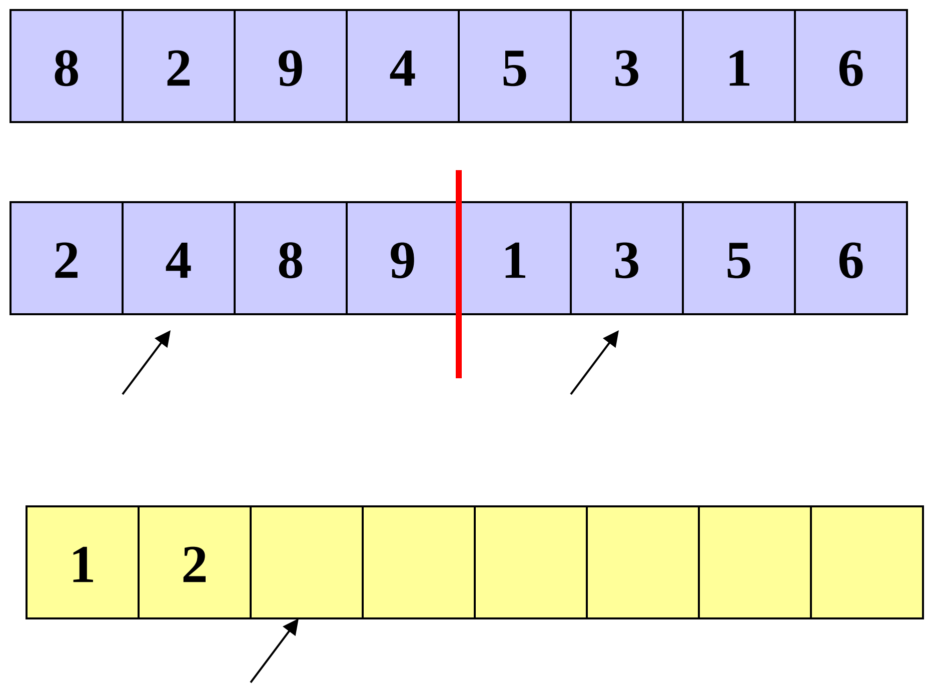

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Example, focus on merging

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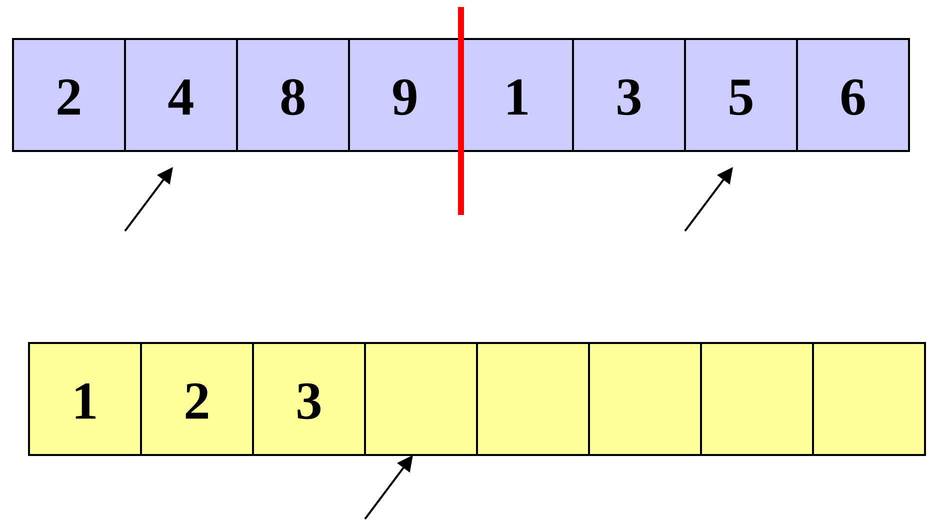

(After merge,
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Example, focus on merging

Start with:

8	2	9	4	5	3	1	6

After recursion:
(not magic ☺)

2	4	8	9	1	3	5	6

Merge:
Use 3 “fingers”
and 1 more array

1	2	3	4				

(After merge,
copy back to
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Example, focus on merging

Start with:

8	**2**	**9**	**4**	**5**	**3**	**1**	**6**

After recursion:

(not magic ☺)

2	**4**	**8**	**9**	**1**	**3**	**5**	**6**

Merge:

Use 3 “fingers”

and 1 more array

1	**2**	**3**	**4**	**5**			

(After merge,
copy back to
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Example, focus on merging

Start with:

8	**2**	**9**	**4**	**5**	**3**	**1**	**6**

After recursion:

(not magic ☺)

2	**4**	**8**	**9**	**1**	**3**	**5**	**6**

Merge:

Use 3 “fingers”

and 1 more array

1	**2**	**3**	**4**	**5**	**6**		

(After merge, copy back to original array)

Example, focus on merging

Start with:

After recursion:
(not magic ☺)

Merge:
Use 3 “fingers”
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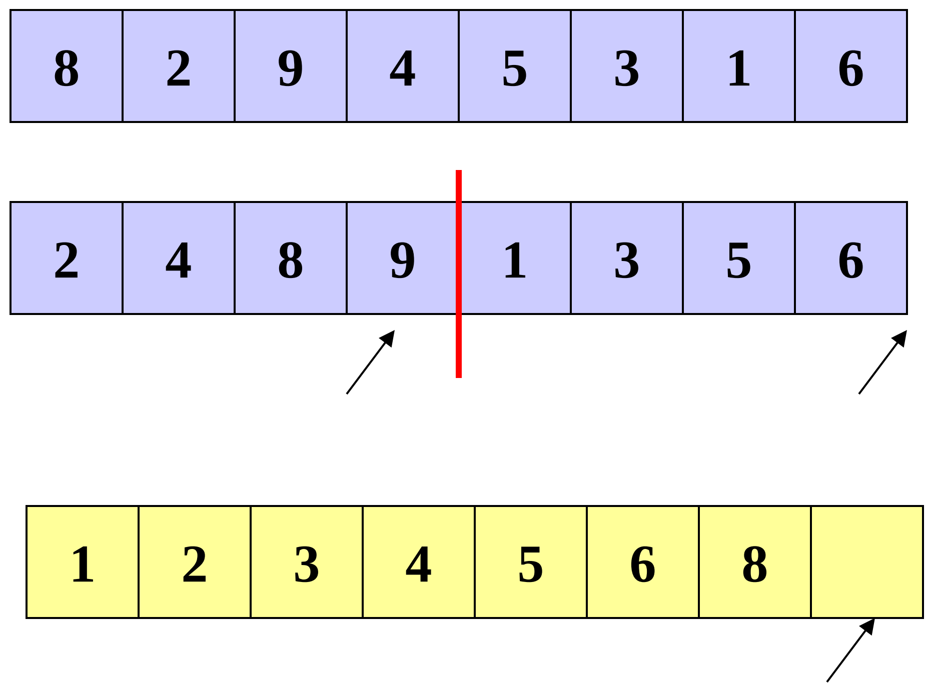

(After merge,
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Example, focus on merging

Start with:

8	**2**	**9**	**4**	**5**	**3**	**1**	**6**

After recursion:

(not magic ☺)

2	**4**	**8**	**9**	**1**	**3**	**5**	**6**

Merge:

Use 3 “fingers”

and 1 more array

1	**2**	**3**	**4**	**5**	**6**	**8**	**9**

(After merge,
copy back to
original array)

Example, focus on merging

Start with:

8	2	9	4	5	3	1	6

After recursion:
(not magic ☺)

2	4	8	9	1	3	5	6

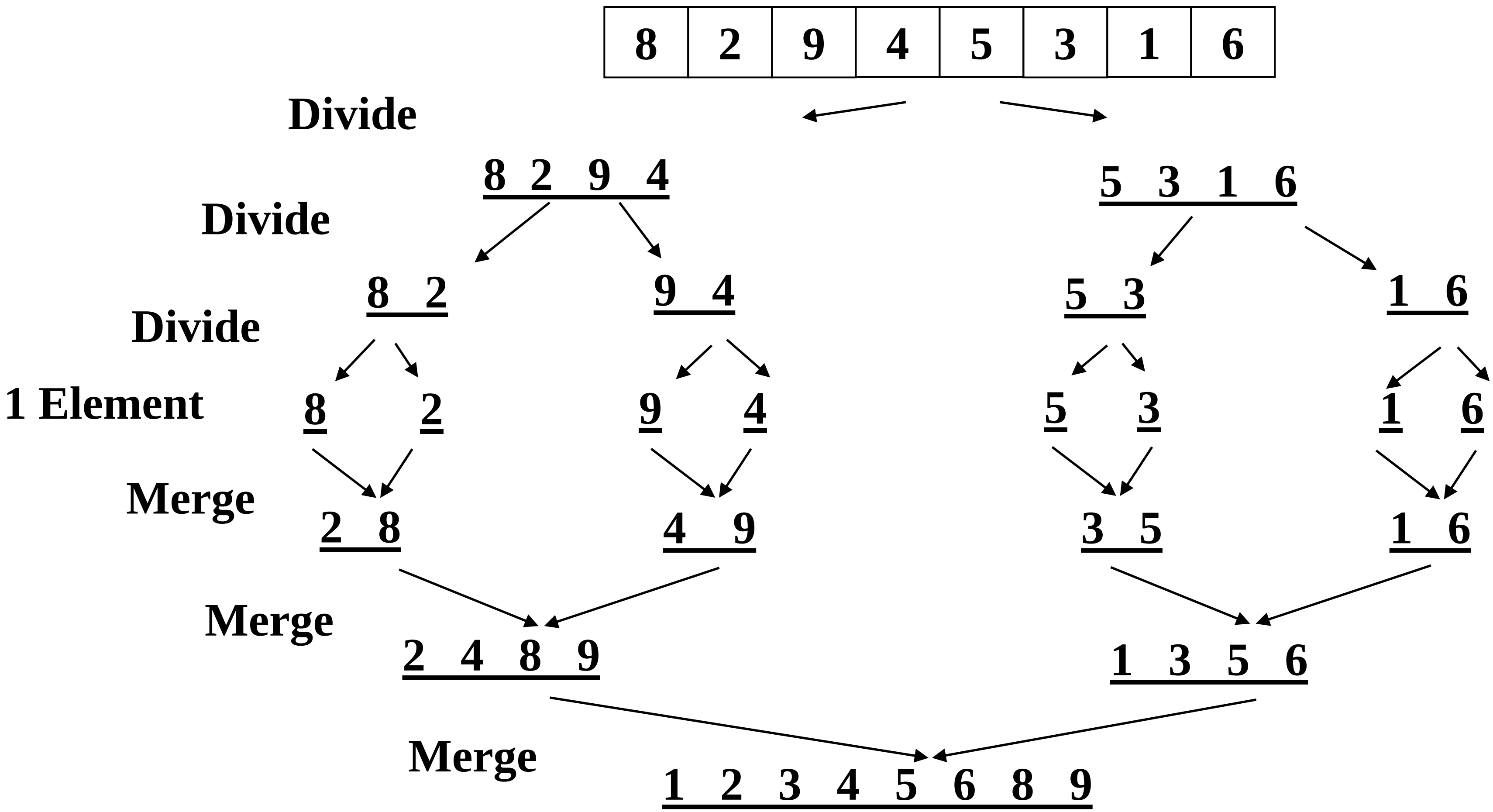
Merge:
Use 3 “fingers”
and 1 more array

1	2	3	4	5	6	8	9

(After merge,
copy back to
original array)

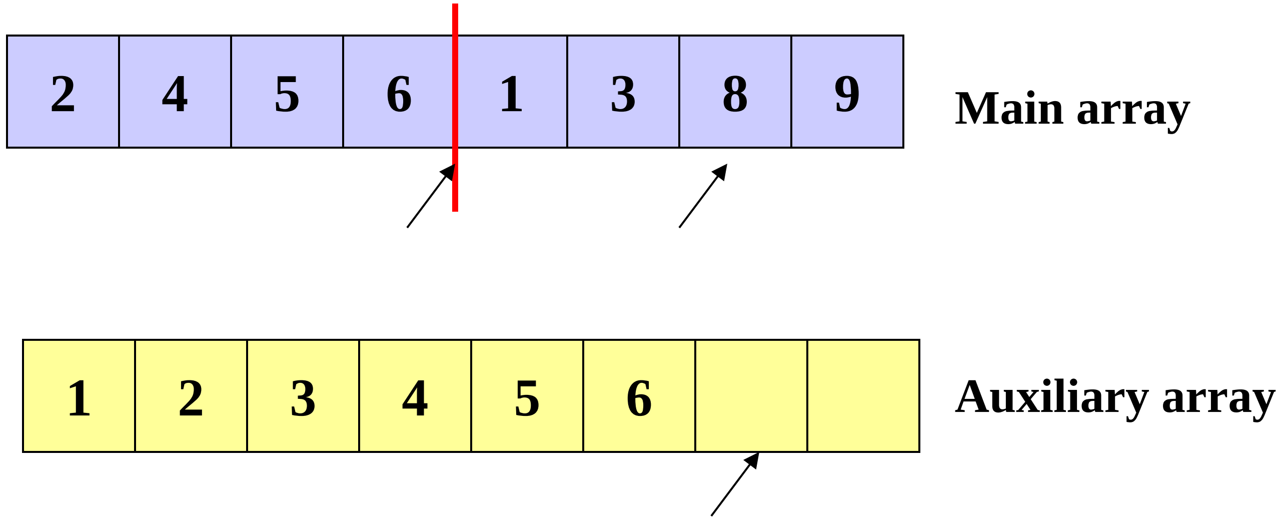
1	2	3	4	5	6	8	9

Example, Showing Recursion



Some details: saving a little time

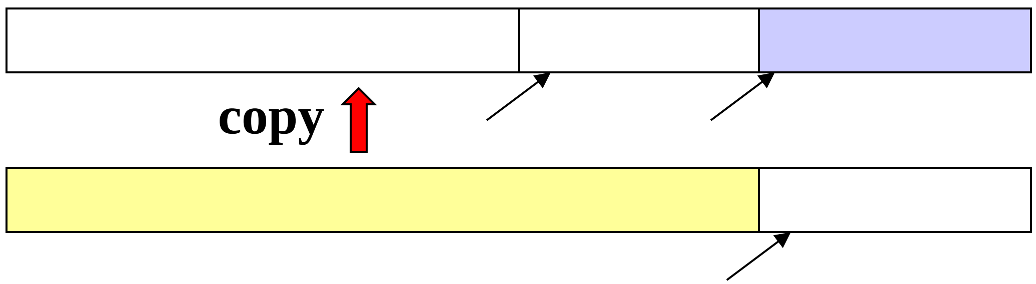
- What if the final steps of our merge looked like this:



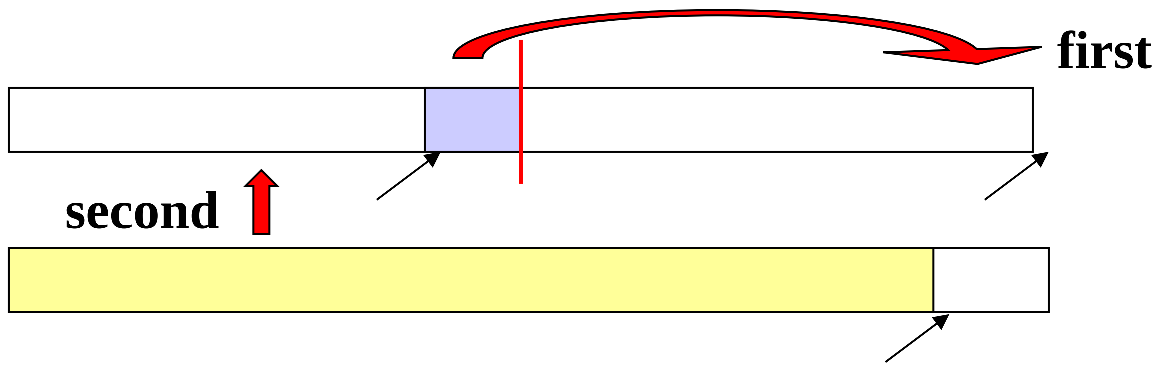
- Wasteful to copy to the auxiliary array just to copy back…

Some details: saving a little time

- If left-side finishes first, just stop the merge and copy back:



- If right-side finishes first, copy dregs into right then copy back



Some details: Saving Space and Copying

Simplest / Worst:

Use a new auxiliary array of size **`(hi-lo)`** for every merge

Better:

Use a new auxiliary array of size **`n`** for every merging stage

Better:

Reuse same auxiliary array of size **`n`** for every merging stage

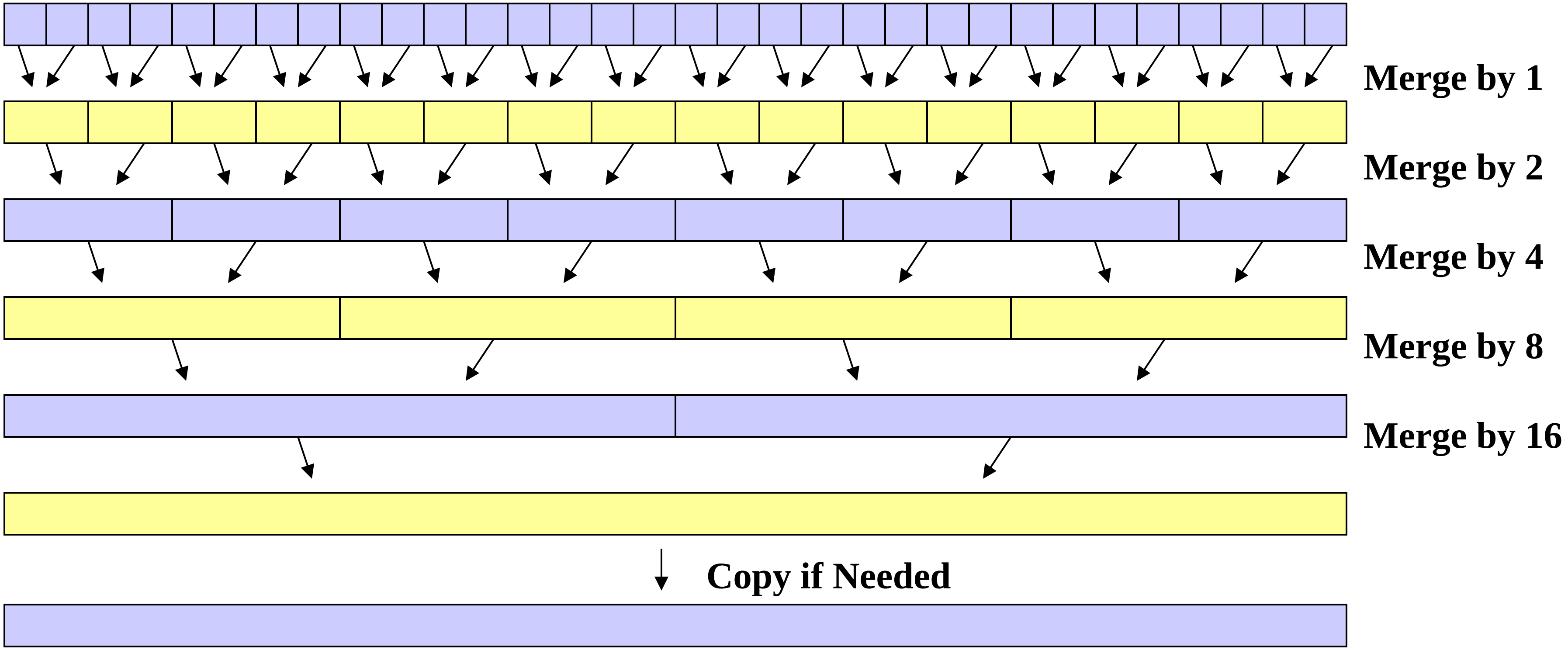
Best (but a little tricky):

Don't copy back – at 2^{nd}, 4^{th}, 6^{th}, … merging stages, use the original array as the auxiliary array and vice-versa

- Need one copy at end if number of stages is odd

Swapping Original / Auxiliary Array ("best")

- First recurse down to lists of size 1
- As we return from the recursion, swap between arrays



(Arguably easier to code up without recursion at all)

Linked lists and big data

We defined sorting over an array, but sometimes you want to sort linked lists

One approach:

- Convert to array: $O(n)$
- Sort: $O(n \texttt{ log } n)$
- Convert back to list: $O(n)$

Or: merge sort works very nicely on linked lists directly

- Heapsort and quicksort do not
- Insertion sort and selection sort do but they’re slower

Merge sort is also the sort of choice for external sorting

- Linear merges minimize disk accesses
- And can leverage multiple disks to get streaming accesses

Analysis

Having defined an algorithm and argued it is correct, we should analyze its running time and space:

To sort n elements, we:

- Return immediately if n=1
- Else do 2 subproblems of size n/2 and then an O(n) merge

Recurrence relation:

$$T(1) = c_1$$

$$T(n) = 2T(n/2) + c_2 n$$

One of the recurrence classics…

For simplicity let constants be 1 – no effect on asymptotic answer

$T(1) = 1$

$T(n) = 2T(n/2) + n$

$= 2(2T(n/4) + n/2) + n$

$= 4T(n/4) + 2n$

$= 4(2T(n/8) + n/4) + 2n$

$= 8T(n/8) + 3n$

….

$= 2^{\mathbf{k}}T(n/2^{\mathbf{k}}) + kn$

So total is $2^{\mathbf{k}}T(n/2^{\mathbf{k}}) + kn$ where

$n/2^{\mathbf{k}} = 1$, i.e., $\log n = k$

That is, $2^{\mathbf{log\ n}}\, T(1) + n \log n$

$= n + n \log n$

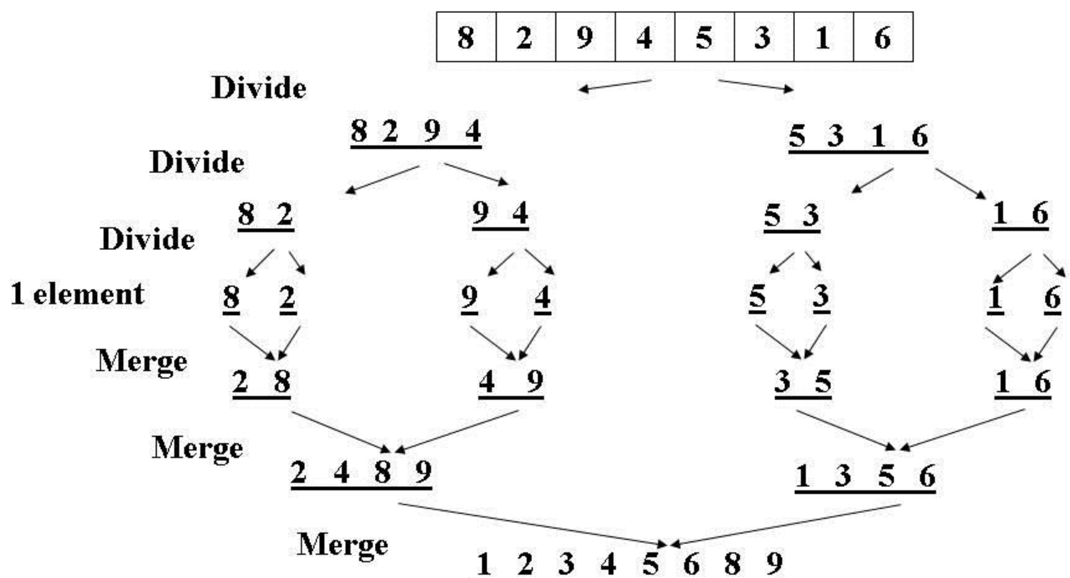
$= O(n \log n)$

Or more intuitively…

This recurrence is common you just "know" it's $O(n \texttt{ log } n)$

Merge sort is relatively easy to intuit (best, worst, and average):

- The recursion "tree" will have **`log`** *n* height
- At each level we do a *total* amount of merging equal to *n*



Quicksort

- Also uses divide-and-conquer
  - Recursively chop into two pieces
  - Instead of doing all the work as we merge together,
    we will do all the work as we recursively split into halves
  - Unlike merge sort, does not need auxiliary space
- $O(n \texttt{ log } n)$ on average ☺, but $O(n^2)$ worst-case ☹
- Faster than merge sort in practice?
  - Often believed so
  - Does fewer copies and more comparisons, so it depends on the relative cost of these two operations!

Quicksort Overview

1. Pick a pivot element

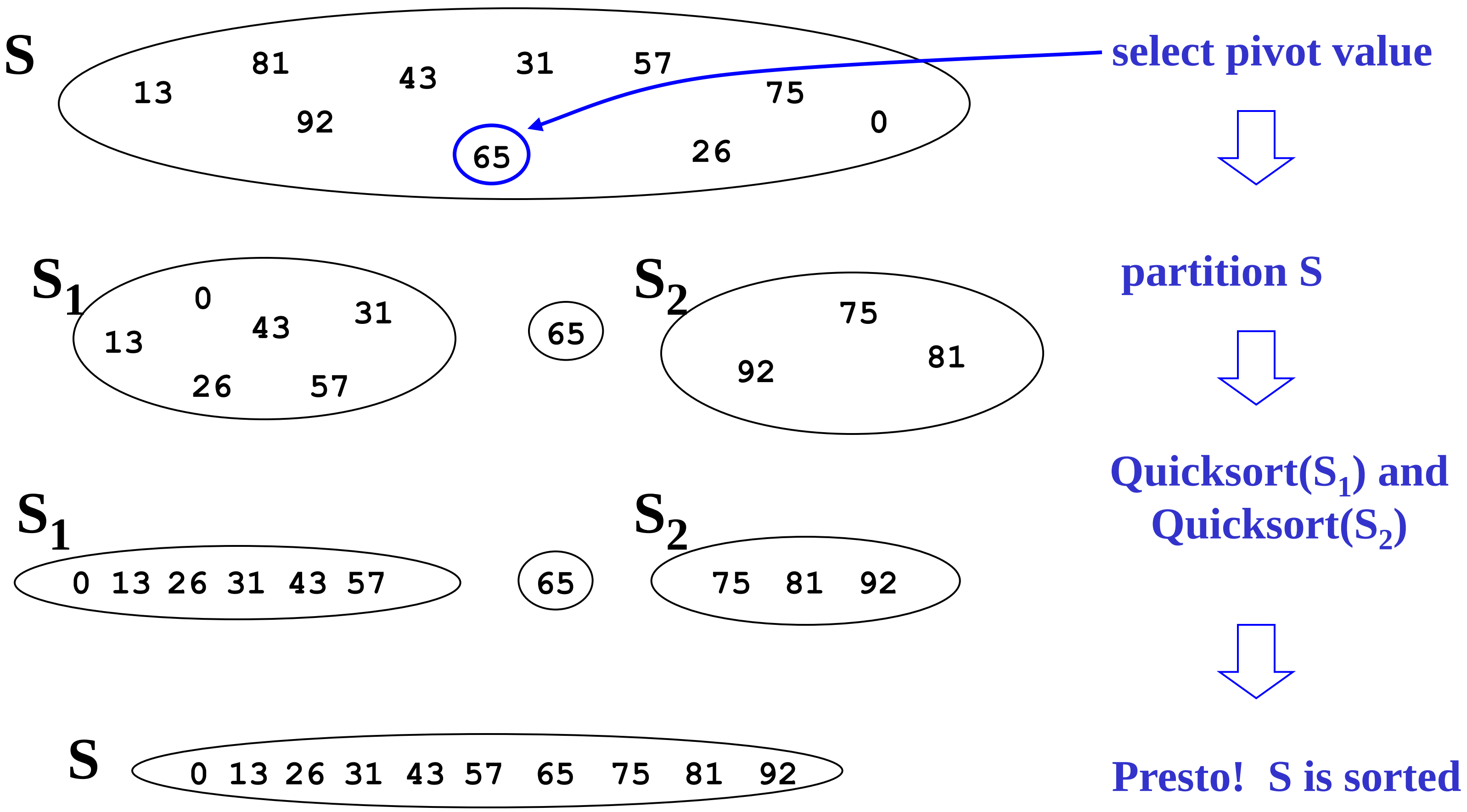
2. Partition all the data into:
   A. The elements less than the pivot
   B. The pivot
   C. The elements greater than the pivot

3. Recursively sort A and C

4. The answer is, “as simple as A, B, C”

(Alas, there are some details lurking in this algorithm)

Think in Terms of Sets



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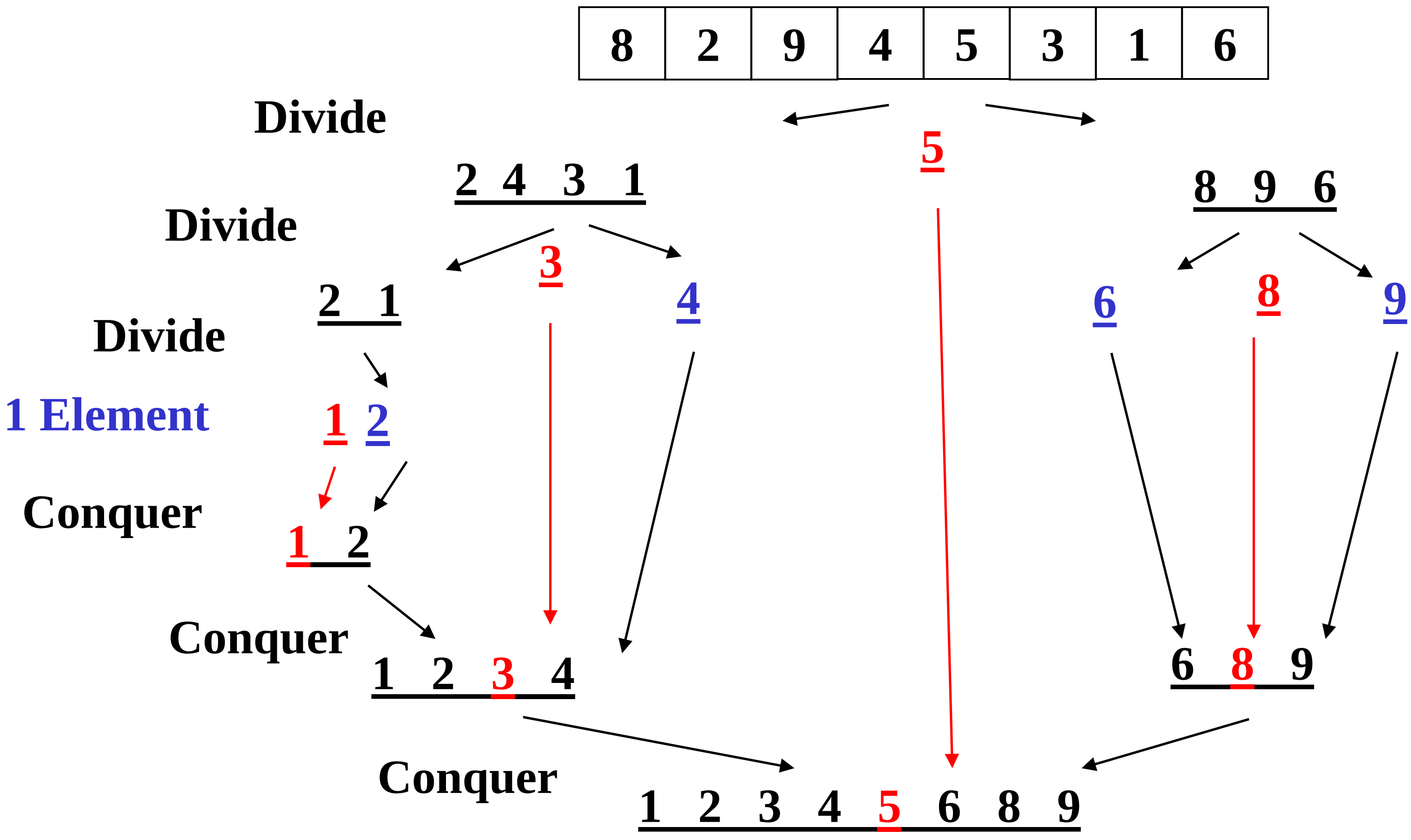
Example, Showing Recursion



Details

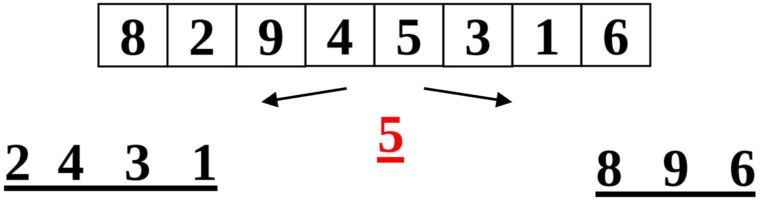
Have not yet explained:

- How to pick the pivot element
  - Any choice is correct: data will end up sorted
  - But as analysis will show, want the two partitions to be about equal in size

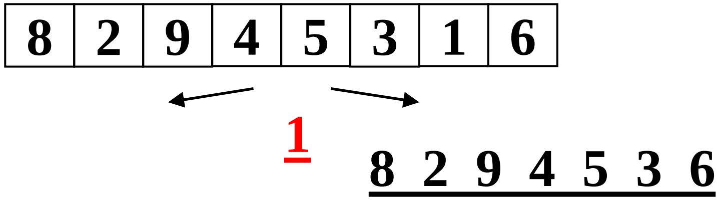
- How to implement partitioning
  - In linear time
  - In place

Pivots

- Best pivot?
  - Median
  - Halve each time



- Worst pivot?
  - Greatest/least element
  - Problem of size n - 1
  - $O(n^2)$



Potential pivot rules

While sorting **`arr`** from **`lo`** (inclusive) to **`hi`** (exclusive)…

- Pick **`arr[lo]`** or **`arr[hi-1]`**
  - Fast, but worst-case occurs with mostly sorted input

- Pick random element in the range
  - Does as well as any technique, but (pseudo)random number generation can be slow
  - Still probably the most elegant approach

- Median of 3, e.g., **`arr[lo], arr[hi-1], arr[(hi+lo)/2]`**
  - Common heuristic that tends to work well

Partitioning

- Conceptually simple, but hardest part to code up correctly
  - After picking pivot, need to partition in linear time in place

- One approach (there are slightly fancier ones):
  1. Swap pivot with **`arr[lo]`**
  2. Use two fingers **`i`** and **`j`**, starting at **`lo+1`** and **`hi-1`**
  3. **`while (i < j)`**

```
        if (arr[j] > pivot) j--
        else if (arr[i] < pivot) i++
        else swap arr[i] with arr[j]
```

  4. Swap pivot with **`arr[i]`** *

*skip step 4 if pivot ends up being least element

Example

- Step one: pick pivot as median of 3
  - **`lo`** = 0, **`hi`** = 10

0	1	2	3	4	5	6	7	8	9
8	1	4	9	0	3	5	2	7	6

- Step two: move pivot to the **`lo`** position

0	1	2	3	4	5	6	7	8	9
6	1	4	9	0	3	5	2	7	8

Example

Often have more than one swap during partition – this is a short example

Now partition in place

6	1	4	9	0	3	5	2	7	8

Move fingers

6	1	4	9	0	3	5	2	7	8

Swap

6	1	4	2	0	3	5	9	7	8

Move fingers

6	1	4	2	0	3	5	9	7	8

Move pivot

5	1	4	2	0	3	6	9	7	8

Analysis

- Best-case: Pivot is always the median

  T(0)=T(1)=1

  T(n)=2T(n/2) + n -- linear-time partition

  Same recurrence as mergesort: $O(n \log n)$

- Worst-case: Pivot is always smallest or largest element

  T(0)=T(1)=1

  T(n) = 1T(n-1) + n

  Basically same recurrence as selection sort: $O(n^2)$

- Average-case (e.g., with random pivot)
  - O(n `log` n), not responsible for proof (in text)

Cutoffs

- For small *n*, all that recursion tends to cost more than doing a quadratic sort
  - Remember asymptotic complexity is for large *n*

- Common engineering technique: switch algorithm below a cutoff
  - Reasonable rule of thumb: use insertion sort for $n < 10$

- Notes:
  - Could also use a cutoff for merge sort
  - Cutoffs are also the norm with parallel algorithms
    - Switch to sequential algorithm
  - None of this affects asymptotic complexity

Cutoff skeleton

```
void quicksort(int[] arr, int lo, int hi) {
  if(hi – lo < CUTOFF)
     insertionSort(arr,lo,hi);
  else
     …
}
```

Notice how this cuts out the vast majority of the recursive calls

- Think of the recursive calls to quicksort as a tree
- Trims out the bottom layers of the tree