CSE373: Data Structures & Algorithms
Lecture 20: Beyond Comparison Sorting

Dan Grossman

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The Big Picture

Surprising amount of juicy computer science: 2-3 lectures…

Simple algorithms: $O(n^2)$

- **Insertion sort**
- **Selection sort**
- **Shell sort**
- **…**

Fancier algorithms: $O(n \log n)$

- **Heap sort**
- **Merge sort**
- **Quick sort (avg)**
- **…**

Comparison lower bound: $\Omega(n \log n)$

Specialized algorithms: $O(n)$

- **Bucket sort**
- **Radix sort**

Handling huge data sets

- **External sorting**

How Fast Can We Sort?

- Heapsort & mergesort have $O(n \log n)$ worst-case running time

- Quicksort has $O(n \log n)$ average-case running time

- These bounds are all tight, actually $\Theta(n \log n)$

- So maybe we need to dream up another algorithm with a lower asymptotic complexity, such as $O(n)$ or $O(n \log \log n)$
  - Instead: we *know* that this is *impossible*
    - *Assuming* our comparison *model*: The only operation an algorithm can perform on data items is a 2-element comparison

A General View of Sorting

- Assume we have *n* elements to sort
  - For simplicity, assume none are equal (no duplicates)

- How many *permutations* of the elements (possible orderings)?

- Example, n=3

  a[0]<a[1]<a[2]   a[0]<a[2]<a[1]   a[1]<a[0]<a[2]

  a[1]<a[2]<a[0]   a[2]<a[0]<a[1]   a[2]<a[1]<a[0]

- In general, *n* choices for least element, *n*-1 for next, *n*-2 for next, …
  - $n(n-1)(n-2)\ldots(2)(1) = n!$ possible orderings

Counting Comparisons

- So *every* sorting algorithm has to "find" the right answer among the n! possible answers
  - Starts "knowing nothing", "anything is possible"
  - Gains information with each comparison
  - Intuition: Each comparison can *at best* eliminate *half* the remaining possibilities
  - Must narrow answer down to a single possibility

- What we can show:

  Any sorting algorithm must do *at least* $(1/2)n \log n - (1/2)n$

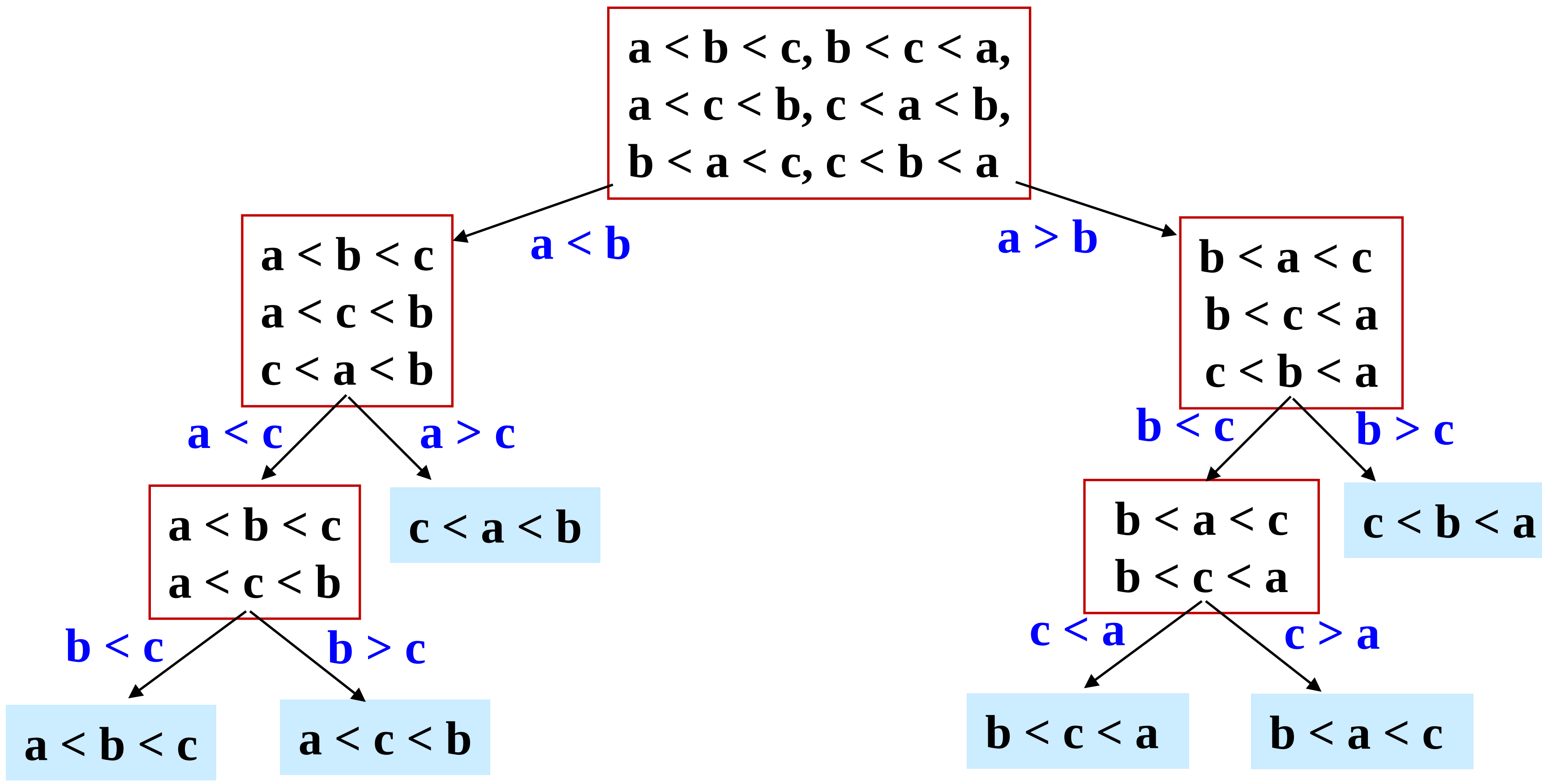
  (which is $\Omega(n \log n)$) comparisons
  - Otherwise there are at least two permutations among the n! possible that cannot yet be distinguished, so the algorithm would have to guess and could be wrong [incorrect algorithm]

Optional: *Counting Comparisons*

- Don't know what the algorithm is, but it cannot make progress without doing comparisons
  - Eventually does a first comparison "is $a < b$?"
  - Can use the result to decide what second comparison to do
  - Etc.: comparison k can be chosen based on first k-1 results

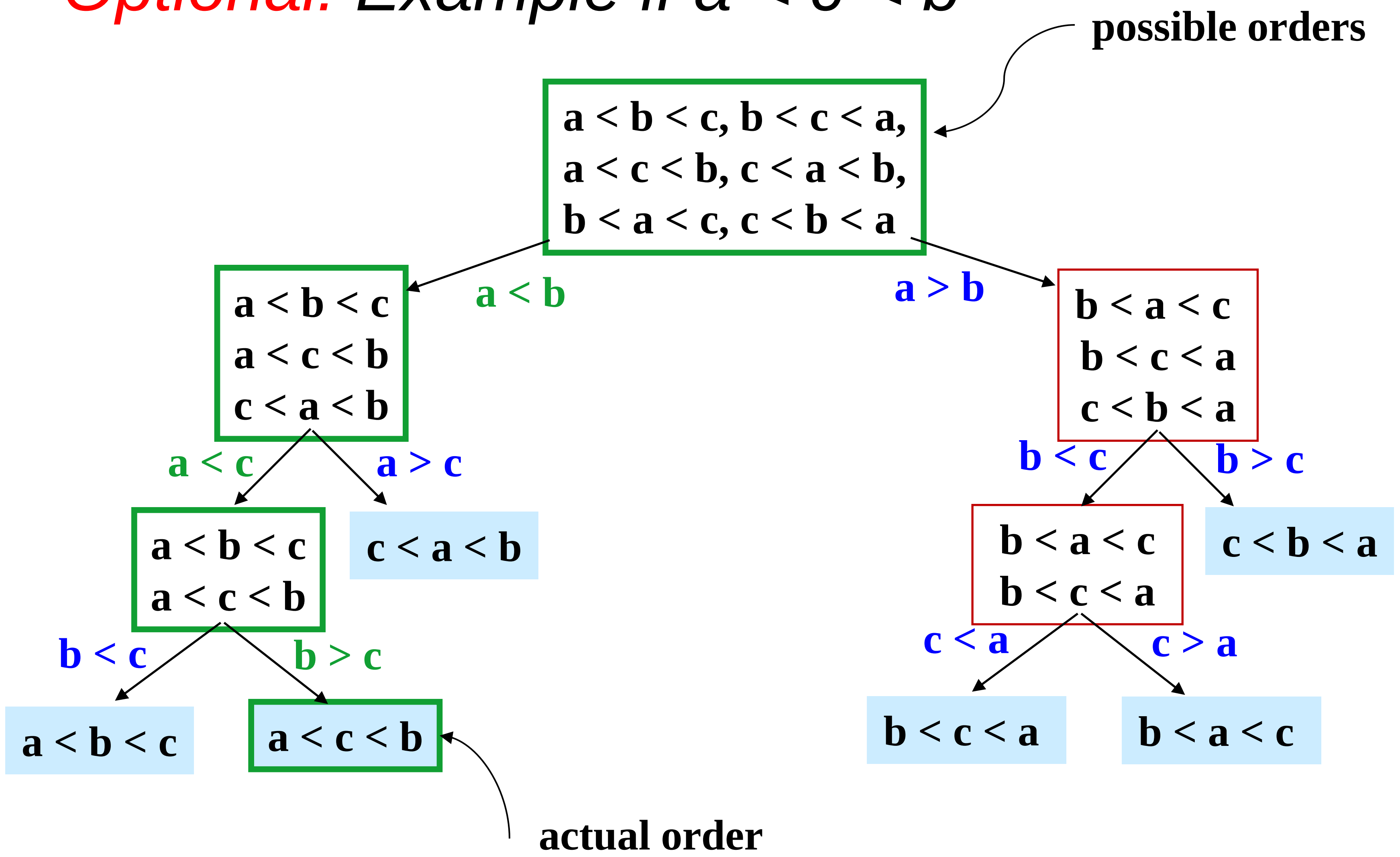
- Can represent this process as a *decision tree*
  - Nodes contain "set of remaining possibilities"
    - Root: None of the n! options yet eliminated
  - Edges are "answers from a comparison"
  - The algorithm does not actually build the tree; it's what our *proof* uses to represent "the most the algorithm could know so far" as the algorithm progresses

Optional: *One Decision Tree for n=3*



- **The leaves contain all the possible orderings of a, b, c**
- **A different algorithm would lead to a different tree**

Optional: *Example if a < c < b*



Optional: What the Decision Tree Tells Us

- A binary tree because each comparison has 2 outcomes
  - (We assume no duplicate elements)
  - (Would have 1 outcome if algorithm asks redundant questions)

- Because any data is possible, any algorithm needs to ask enough questions to produce all n! answers
  - Each answer is a different leaf
  - So the tree must be big enough to have n! leaves
  - Running *any* algorithm on *any* input will *at best* correspond to a root-to-leaf path in *some* decision tree with n! leaves
  - So no algorithm can have worst-case running time better than the height of a tree with n! leaves
    - Worst-case number-of-comparisons for an algorithm is an input leading to a longest path in algorithm's decision tree

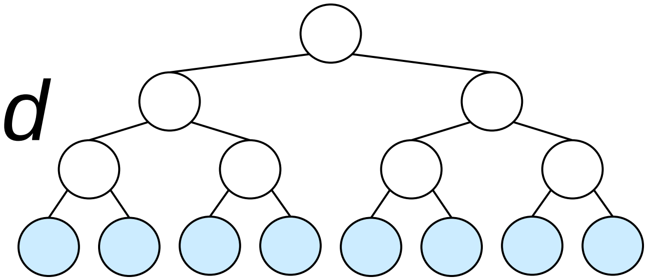
Optional: Where are we

- Proven: No comparison sort can have worst-case running time better than the height of a binary tree with n! leaves
  - A comparison sort could be worse than this height, but it cannot be better

- Now: a binary tree with n! leaves has height $\Omega(n \log n)$
  - Height could be more, but cannot be less
  - Factorial function grows very quickly

- Conclusion: Comparison sorting is $\Omega(n \log n)$
  - An amazing computer-science result: proves all the clever programming in the world cannot comparison-sort in linear time

Optional: *Height lower bound*



- The height of a binary tree with L leaves is at least $\log_2 L$
- So the height of our decision tree, h:

$h \geq \log_2 (n!)$	property of binary trees
$= \log_2 (n*(n-1)*(n-2)\ldots(2)(1))$	definition of factorial
$= \log_2 n + \log_2 (n-1) + \ldots + \log_2 1$	property of logarithms
$\geq \log_2 n + \log_2 (n-1) + \ldots + \log_2 (n/2)$	drop smaller terms (≥ 0)
$\geq \log_2 (n/2) + \log_2 (n/2) + \ldots + \log_2 (n/2)$	shrink terms to $\log_2 (n/2)$
$= (n/2)\log_2 (n/2)$	arithmetic
$= (n/2)(\log_2 n - \log_2 2)$	property of logarithms
$= (1/2)n\log_2 n - (1/2)n$	arithmetic
"=" $\Omega (n \log n)$	

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Simple algorithms: $O(n^2)$	Fancier algorithms: $O(n \log n)$	Comparison lower bound: $\Omega(n \log n)$	Specialized algorithms: $O(n)$	Handling huge data sets
Insertion sort Selection sort Shell sort …	Heap sort Merge sort Quick sort (avg) …		Bucket sort Radix sort	External sorting

How???

- Change the model – assume more than "compare(a,b)"

BucketSort (a.k.a. BinSort)

- If all values to be sorted are known to be integers between 1 and *K* (or any small range):
  - Create an array of size *K*
  - Put each element in its proper bucket (a.k.a. bin)
  - *If* data is only integers, no need to store more than a *count* of how times that bucket has been used
- Output result via linear pass through array of buckets

`count` array	
1	3
2	1
3	2
4	2
5	3

- Example:

  K=5

  input (5,1,3,4,3,2,1,1,5,4,5)

  output: 1,1,1,2,3,3,4,4,5,5,5

Analyzing Bucket Sort

- Overall: $O(n+K)$
  - Linear in n, but also linear in K
  - $\Omega(n \log n)$ lower bound does not apply because this is not a comparison sort

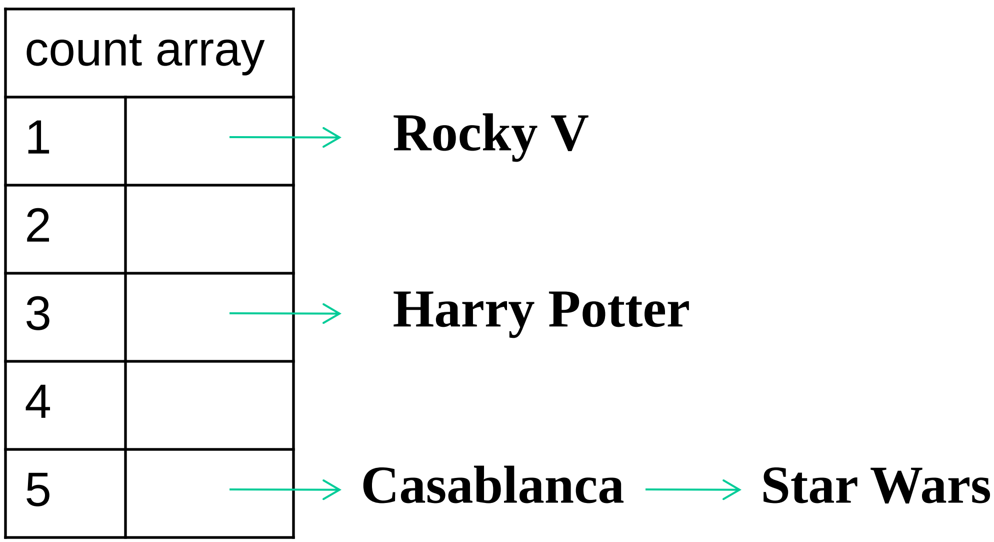
- Good when K is smaller (or not much larger) than n
  - We don't spend time doing comparisons of duplicates

- Bad when K is much larger than n
  - Wasted space; wasted time during linear $O(K)$ pass

- For data in addition to integer keys, use list at each bucket

Bucket Sort with Data

- Most real lists aren't just keys; we have data
- Each bucket is a list (say, linked list)
- To add to a bucket, insert in *O*(1) (at beginning, or keep pointer to last element)



- Example: Movie ratings; scale 1-5;1=bad, 5=excellent

  Input=

  5: Casablanca

  3: Harry Potter movies

  5: Star Wars Original Trilogy

  1: Rocky V

- Result: 1: Rocky V, 3: Harry Potter, 5: Casablanca, 5: Star Wars
- Easy to keep 'stable'; Casablanca still before Star Wars

Radix sort

- Radix = "the base of a number system"
  - Examples will use 10 because we are used to that
  - In implementations use larger numbers
    - For example, for ASCII strings, might use 128

- Idea:
  - Bucket sort on one digit at a time
    - Number of buckets = radix
    - Starting with *least* significant digit
    - Keeping sort *stable*
  - Do one pass per digit
  - Invariant: After *k* passes (digits), the last *k* digits are sorted

- Aside: Origins go back to the 1890 U.S. census

Example

Radix = 10

0	1	2	3	4	5	6	7	8	9
	721		3 143				537 67	478 38	9

Input:
478
537
9
721
3
38
143
67

First pass:

bucket sort by ones digit

Order now:
721
3
143
537
67
478
38
9

Example

Radix = 10

0	1	2	3	4	5	6	7	8	9
	721		3 143				537 67	478 38	9

0	1	2	3	4	5	6	7	8	9
3 9		721	537 38	143		67	478		

Order was: 721
3
143
537
67
478
38
9

Second pass:

stable bucket sort by tens digit

Order now: 3
9
721
537
38
143
67
478

Example

0	1	2	3	4	5	6	7	8	9
3 9		721	537 38	143		67	478		

Radix = 10

0	1	2	3	4	5	6	7	8	9
3 9 38 67	143			478	537		721		

Order was:

3
9
721
537
38
143
67
478

Third pass:

stable bucket sort by 100s digit

Order now:

3
9
38
67
143
478
537
721

Analysis

Input size: n

Number of buckets = Radix: B

Number of passes = "Digits": P

Work per pass is 1 bucket sort: $O(B+n)$

Total work is $O(P(B+n))$

Compared to comparison sorts, sometimes a win, but often not

- Example: Strings of English letters up to length 15
  - Run-time proportional to: 15*(52 + n)
  - This is less than n log n only if n > 33,000
  - Of course, cross-over point depends on constant factors of the implementations
    - And radix sort can have poor locality properties

Sorting massive data

- Need sorting algorithms that minimize disk/tape access time:
  - Quicksort and Heapsort both jump all over the array, leading to expensive random disk accesses
  - Mergesort scans linearly through arrays, leading to (relatively) efficient sequential disk access

- Mergesort is the basis of massive sorting

- Mergesort can leverage multiple disks

Last Slide on Sorting

- Simple $O(n^2)$ sorts can be fastest for small n
  - Selection sort, Insertion sort (latter linear for mostly-sorted)
  - Good for "below a cut-off" to help divide-and-conquer sorts
- $O(n \texttt{ log } n)$ sorts
  - Heap sort, in-place but not stable nor parallelizable
  - Merge sort, not in place but stable and works as external sort
  - Quick sort, in place but not stable and $O(n^2)$ in worst-case
    - Often fastest, but depends on costs of comparisons/copies
- $\Omega(n \texttt{ log } n)$ is worst-case and average lower-bound for sorting by comparisons
- Non-comparison sorts
  - Bucket sort good for small number of possible key values
  - Radix sort uses fewer buckets and more phases
- Best way to sort? It depends!