Math Review

CSE 373
Data Structures & Algorithms
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Spring 2013

Today's Outline

- **Announcements**
  - Assignment #1 due Friday, April 12 at 11pm
- **Math Review**
  - **Proof by Induction**
  - **Powers of 2**
  - **Binary numbers**
  - **Exponents and Logs**
- **Algorithm Analysis**

Mathematical Induction

Suppose we wish to prove that:

For all $n \geq n_0$, some predicate $P(n)$ is true.

We can do this by proving two things:

1. $P(n_0)$ - this is called the "base case" or "basis."
2. If $P(k)$, then $P(k+1)$ - this is called the "induction step" or "inductive case"

Note: We prove 2. by assuming $P(k)$ is true.

Putting these together, we show that $P(n)$ is true.

Example

Prove: for all $n \geq 1$, the sum of first n powers of 2 = $2^n - 1$

$$2^0 + 2^1 + 2^2 + \ldots + 2^{n-1} = 2^n - 1.$$

in other words: $1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1.$

P(n) = " the sum of the first *n* powers of 2 (starting at 2^0) is 2^n-1 "

n=1: $2^0 = 1 = 2^1-1$

n=2: $2^0 + 2^1 = 1 + 2 = 3 = 2^2-1$

P(n) = " the sum of the first *n* powers of 2 (starting at 2^0) is 2^n-1 "

Example Proof by Induction

Theorem: *P(n)* holds for all $n \geq 1$

Proof: By induction on *n*

- Base case, *n*=1: $2^0 = 1 = 2^1 - 1$
- Induction step:
  - Inductive hypothesis: Assume the sum of the first *k* powers of 2 is 2^k-1
  - Given the hypothesis, show that: the sum of the first (*k*+1) powers of 2 is 2^{k+1}-1

From our inductive hypothesis we know:

$$1 + 2 + 4 + \ldots + 2^{k-1} = 2^k - 1$$

Add the next power of 2 to both sides…

$$1 + 2 + 4 + \ldots + 2^{k-1} + 2^k = 2^k - 1 + 2^k$$

We have what we want on the left; massage the right a bit:

$$1 + 2 + 4 + \ldots + 2^{k-1} + 2^k = 2(2^k) - 1 = 2^{k+1} - 1$$

Example: Putting it all together

- *Inductive hypothesis*: (We assumed this was true)
  $1 + 2 + 4 + \ldots + 2^{k-1} = 2^k - 1$
- *Induction step*: (Adding 2^k to both sides)
  $1 + 2 + 4 + \ldots\ 2^{k-1} + \mathbf{2^k} = 2^k - 1 + \mathbf{2^k} = 2(2^k) - 1 = 2^{k+1} - 1$
  Therefore if the equation is valid for n = k, it must also be valid for n = k+1.

Summary: Our theorem is valid for n=1 (base case) and by the induction step it is therefore valid for n=2, n=3, …

Thus, it is valid for all integers greater than or equal to 1.

Powers of 2

- Many of the numbers we use in Computer Science are powers of 2
- Binary numbers (base 2) are easily represented in digital computers
  - each "bit" is a 0 or a 1
  - an n-bit wide field can represent how many different things?

0000000000101011

N bits can represent how many things?

# Bits	Patterns	# of patterns
1		
2		

Unsigned binary numbers

- For **unsigned** numbers in a fixed width field
  - the minimum value is 0
  - the maximum value is 2^n-1, where n is the number of bits in the field
  - The value is $\sum_{i=0}^{i=n-1} a_i 2^i$
- Each bit position represents a power of 2 with $a_i = 0$ or $a_i = 1$

Powers of 2

- A bit is 0 or 1
- A sequence of *n* bits can represent 2^n distinct things
  - For example, the numbers 0 through 2^n-1
- 2^{10} is 1024 ("about a thousand", kilo in CSE speak)
- 2^{20} is "about a million", mega in CSE speak
- 2^{30} is "about a billion", giga in CSE speak

Java:

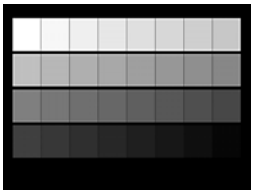
- an `int` is 32 bits and signed, so "max int" is "about 2 billion"
- a `long` is 64 bits and signed, so "max long" is 2^{63}-1

Logarithms and Exponents

- Definition: $\log_2 x = y$ if and only if $x = 2^y$
  $8 = 2^3$, so $\log_2 8 = 3$
  $65536 = 2^{16}$, so $\log_2 65536 = 16$
- Notice that $\log_2 n$ tells you how many bits are needed to distinguish among n different values.
  8 bits can hold any of 256 numbers, for example: 0 to 2^8-1, which is 0 to 255
  $\log_2 256 = 8$

Grayscale Images

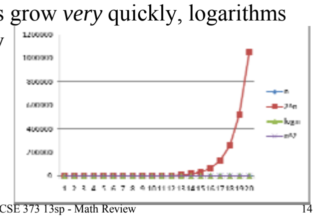
- 8 bits per pixel; 00000000 = black; 11111111 = white



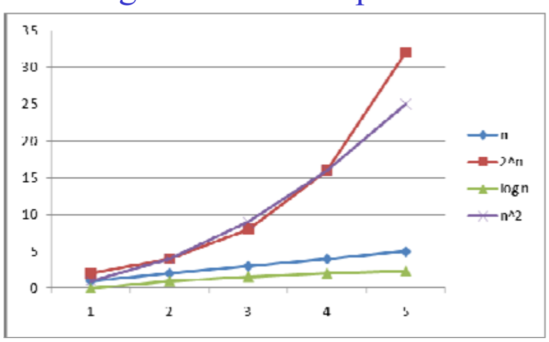
Logarithms and Exponents

- Since so much is binary in CS, `log` almost always means $\log_2$
- Definition: $\log_2 x = y$ if $x = 2^y$
- So, $\log_2$ 1,000,000 = "a little under 20"
- Just as exponents grow *very* quickly, logarithms grow *very* slowly

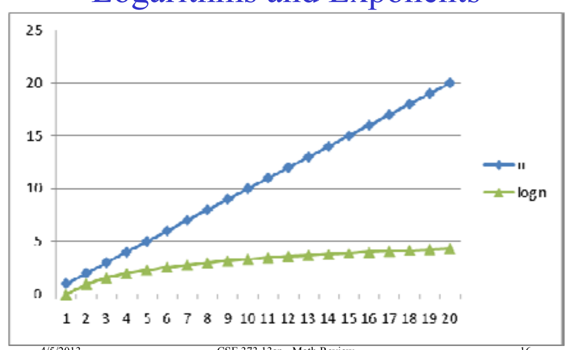
See Excel file for plot data – play with it!



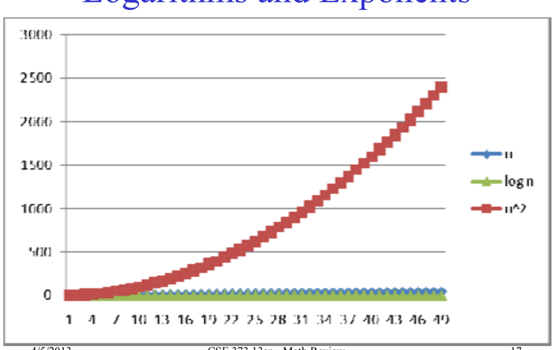
Logarithms and Exponents



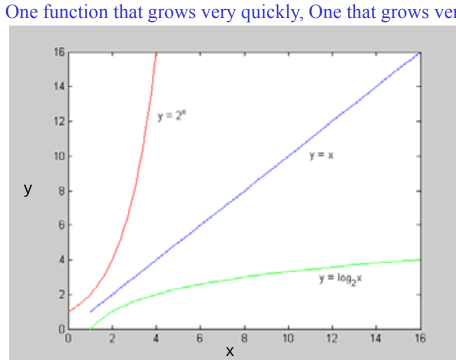
Logarithms and Exponents



Logarithms and Exponents



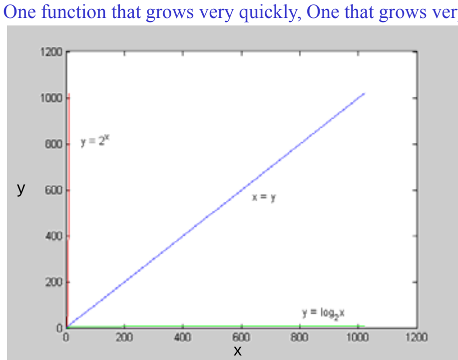
One function that grows very quickly, One that grows very slowly




```
x = 0:.1:4
y = 2.^x
plot(x,y,'r')
hold on
plot(y,x,'g')
plot(y,y,'b')
```

x, 2^x and $\log_2 x$

One function that grows very quickly, One that grows very slowly




```
x = 0:10
y = 2.^x
plot(x,y,'r')
hold on
plot(y,x,'g')
plot(y,y,'b')
```

2^x and $\log_2 x$

Floor and Ceiling

$\lfloor X \rfloor$ Floor function: the largest integer $\leq$ X

$\lfloor 2.7 \rfloor = 2 \qquad \lfloor -2.7 \rfloor = -3 \qquad \lfloor 2 \rfloor = 2$

$\lceil X \rceil$ Ceiling function: the smallest integer $\geq$ X

$\lceil 2.3 \rceil = 3 \qquad \lceil -2.3 \rceil = -2 \qquad \lceil 2 \rceil = 2$

Properties of logs

- We will assume logs to base 2 unless specified otherwise.
- $x = \log_2 2^x$
- $8 = 2^3$, so $\log_2 8 = 3$, so $2^{(\log_2 8)} =$ ________

Show:

$\log (A \bullet B) = \log A + \log B$

$A = 2^{\log_2 A}$ and $B = 2^{\log_2 B}$

$A \bullet B = 2^{\log_2 A} \bullet 2^{\log_2 B} = 2^{\log_2 A + \log_2 B}$

So: $\log_2 AB = \log_2 A + \log_2 B$

- Note: $\log AB \neq \log A \bullet \log B$!!

Also, it follows that `log(N^k)= k log N`

Other log properties

- $\log A/B = \log A - \log B$
- $\log (A^B) = B \log A$
- $\log \log X < \log X < X$ for all $X > 0$
  - $\log \log X = Y$ means: $2^{2^Y} = X$
  - Ex. $\log_2 \log_2 4billion \sim \log_2 \log_2 2^{32} = \log_2 32 = 5$
- log X grows more slowly than X
  - called a “sub-linear” function
- $(\log x)(\log x)$ is written $\log^2 x$ (aka “log-squared”)
  - It is greater than log x for all $x > 2$
- Note: $\log \log X = \log (\log X) \neq \log^2 X$

A log is a log is a log

- “Any base B log is equivalent to base 2 log within a constant factor.”

$B = 2^{\log_2 B}$

$x = 2^{\log_2 x}$

$\log_B X = \log_B X$

substitution $B^{\log_B X} = X$ $B^{\log_B X} = X$ by def. of logs

$(2^{\log_2 B})^{\log_B X} = 2^{\log_2 X}$

$2^{\log_2 B \log_B X} = 2^{\log_2 X}$

$\log_2 B \log_B X = \log_2 X$

$\boxed{\log_B X = \dfrac{\log_2 X}{\log_2 B}}$ **useful**

Log base doesn’t matter (much)

“Any base *B* log is equivalent to base 2 log within a constant factor”

- And we are about to stop worrying about constant factors!
- In particular, $\log_2 x = 3.22 \log_{10} x$
- In general, we can convert log bases via a constant multiplier
- To convert from base B to base A:
  $\log_B x = (\log_A x) / (\log_A B)$

Arithmetic Sequences

N = {0, 1, 2, ... } = natural numbers

[0, 1, 2, ...] is an infinite arithmetic sequence

[a, a+d, a+2d, a+3d, ...] is a general infinite arith. sequence.

There is a *constant difference* between terms.

$$1+2+3+\ldots+N=\sum_{i=1}^{N} i=\frac{N(N+1)}{2}$$ **useful**

Algorithm Analysis Examples

- Consider the following program segment:

```
x:= 0;
for i = 1 to N do
  for j = 1 to i do
    x := x + 1;
```

- What is the value of x at the end?

Analyzing the Loop

- Total number of times x is incremented is executed =

$$1+2+3+\ldots+N=\sum_{i=1}^{N} i=\frac{N(N+1)}{2}$$

- Congratulations - You've just analyzed your first program!
  - Running time of the program is proportional to N(N+1)/2 for all N
  - Big-O ??