Binary Heaps

CSE 373
Data Structures & Algorithms
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Spring 2013

Today's Outline

- **Announcements**
  - Assignment #3 is due May 1 at 11:00pm.
- **Today's Topics:**
  - **Binary Heaps (Weiss Ch. 6: 6.1-6.3)**

Revisiting FindMin

- Application: Find the smallest (or highest priority) item quickly
  - Operating system needs to schedule jobs according to priority instead of FIFO
  - Event simulation (bank customers arriving and departing, ordered according to when the event happened)
  - Find student with highest grade, employee with highest salary etc.

Priority Queue ADT

- Priority Queue can efficiently do:
  - FindMin (and DeleteMin)
  - Insert
- What if we use…
  - Lists: If sorted, what is the run time for Insert and FindMin? Unsorted?
  - Binary Search Trees: What is the run time for Insert and FindMin?
  - Hash Tables: What is the run time for Insert and FindMin?

Less flexibility → More speed

- Lists
  - If sorted: FindMin is O(1) but Insert is O(N)
  - If not sorted: Insert is O(1) but FindMin is O(N)
- Balanced Binary Search Trees (BSTs)
  - Insert is O(log N) and FindMin is O(log N)
- Hash Tables
  - Insert O(1) but no hope for FindMin
- BSTs look good but…
  - BSTs are efficient for all Finds, not just FindMin
  - We only need FindMin

Better than a speeding BST

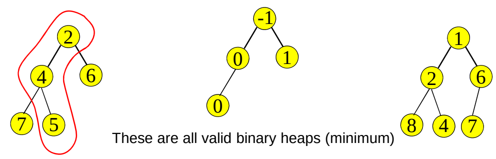
- We can do better than Balanced Binary Search Trees?
  - Very limited requirements: Insert, FindMin, DeleteMin. The goals are:
  - FindMin is O(1)
  - Insert is O(log N)
  - DeleteMin is O(log N)

Binary Heaps

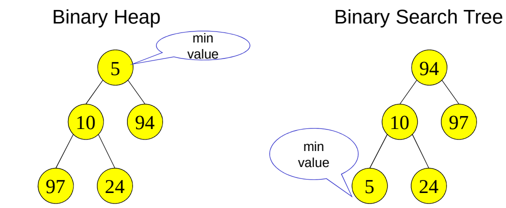
- A binary heap is a binary tree (NOT a BST) that is:
  - › Complete: the tree is completely filled except possibly the bottom level, which is filled from left to right
  - › Satisfies the heap order property
    - every node is less than or equal to its children
    - or every node is greater than or equal to its children
- The root node is always the smallest node
  - › or the largest, depending on the heap order

Heap order property

- A heap provides limited ordering information
- Each *path* is sorted, but the subtrees are not sorted relative to each other
  - › A binary heap is NOT a binary search tree



Binary Heap vs Binary Search Tree

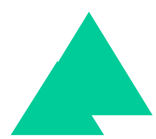


Parent is less than both left and right children

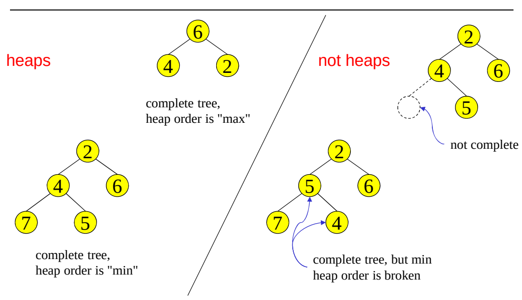
Parent is greater than left child, less than right child

Structure property

- A binary heap is a complete tree
  - › All nodes are in use except for possibly the right end of the bottom row



Examples



How to Implement

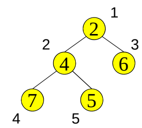
- Heaps are not linked structures.
- They are stored in arrays.
- They are *extremely* efficient, both in time and in space.

Array Implementation of Heaps (Implicit Pointers)

- Root node = A[1]
- Children of A[i] = A[2i], A[2i + 1]
- Parent of A[j] = A[j/2]
- Keep track of current size N (number of nodes)

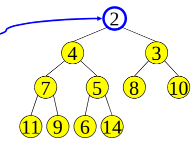
value	-	2	4	6	7	5		
index	0	1	2	3	4	5	6	7

N = 5



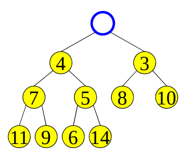
FindMin and DeleteMin

- FindMin: Easy!
  - Return root value A[1]
  - Run time = ?
- DeleteMin:
  - Delete (and return) value at root node



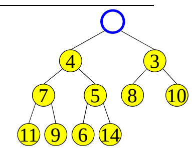
DeleteMin

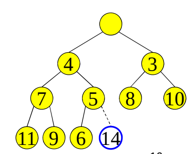
- Delete (and return) value at root node



Maintain the Structure Property

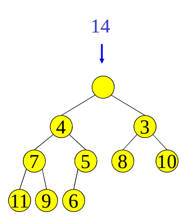
- We now have a “Hole” at the root
  - Need to fill the hole with another value
- When we get done, the tree will have one less node and must still be complete



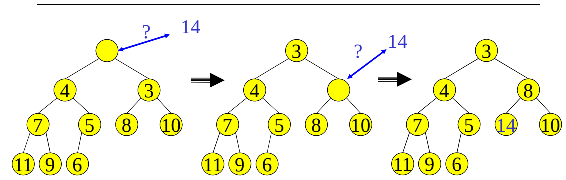


Maintain the Heap Property

- The last value has lost its node
  - we need to find a new place for it
- We can do a simple insertion sort operation to find the correct place for it in the tree



DeleteMin: Percolate Down



- Keep comparing with children A[2i] and A[2i + 1]
- Copy smaller child up and go down one level
- Done if both children are ≥ item or reached a leaf node
- What is the run time?

1 2 3 4 5 6

~~6~~ | 10 | 8 | 13 | 14 | 25

Percolate Down

```
     PercDown(i:integer, x: integer): {
     // N is the number elements, i is the hole,
        x is the value to insert
     Case{
no children   2i > N : A[i] := x; //at bottom//
one child     2i = N : if A[2i] < x then
at the end                  A[i] := A[2i]; A[2i] := x;
                         else A[i] := x;
2 children    2i < N : if A[2i] < A[2i+1] then j := 2i;
                         else j := 2i+1;
                         if A[j] < x then
                            A[i] := A[j]; PercDown(j,x);
                         else A[i] := x;
     }}
```


Bigger Example Shown Pictorially in Text

1	2	3	4	5	6	7	8	9	10	11
13	14	16	19	21	19	68	65	26	32	31

deleteMin deletes the root (13), so the 31 is the input To Percolate Down, and N becomes 10 instead of 11.1,

PercDown(1,31)

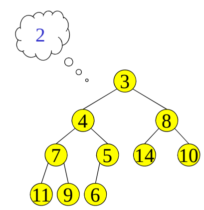
Student Exercise

DeleteMin: Run Time Analysis

- Run time is O(depth of heap)
- A heap is a complete binary tree
- Depth of a complete binary tree of N nodes?
  - depth = $\lfloor \log_2(N) \rfloor$
- Run time of DeleteMin is O(log N)

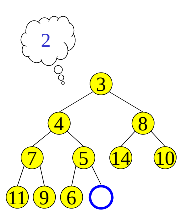
Insert

- Add a value to the tree
- Structure and heap order properties must still be correct when we are done



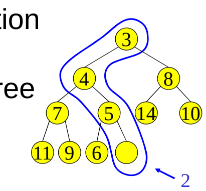
Maintain the Structure Property

- The only valid place for a new node in a complete tree is at the end of the array
- We need to decide on the correct value for the new node, and adjust the heap accordingly

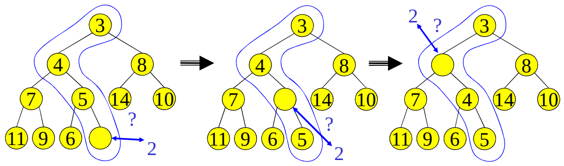


Maintain the Heap Property

- The new value goes where?
- We can do a simple insertion sort operation to find the correct place for it in the tree

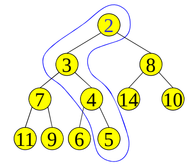


Insert: Percolate Up



- Start at last node and keep comparing with parent A[i/2]
- If parent larger, copy parent down and go up one level
- Done if parent ≤ item or reached top node A[1]

Insert: Done



- Run time?

PercUp

- Define PercUp which percolates new entry to correct spot.
- Note: the parent of i is i/2

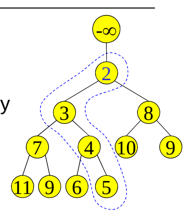
```
PercUp(i : integer, x : integer): {
????
}
```


PercUp Solution

```
PercUp(i : integer, x : integer): {
if i = 1 then A[1] := x
else if A[i/2] < x then
       A[i] := x;
     else
       A[i] := A[i/2];
       Percup(i/2,x);
}
```


Sentinel Values

- Every iteration of Insert needs to test:
  - if it has reached the top node A[1]
  - if parent ≤ item
- Can avoid first test if A[0] contains a very large negative value
  - sentinel -∞ < item, for all items
- Second test alone always stops at top



value	-∞	2	3	8	7	4	10	9	11	9	6	5		
index	0	1	2	3	4	5	6	7	8	9	10	11	12	13

Binary Heap Analysis

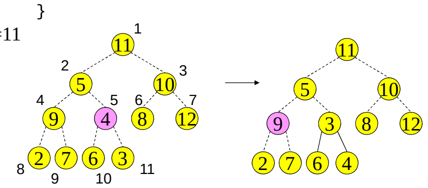
- Space needed for heap of N nodes: O(MaxN)
  - An array of size MaxN, plus a variable to store the size N, plus an array slot to hold the sentinel
- Time
  - FindMin: O(1)
  - DeleteMin and Insert: O(log N)
  - BuildHeap from N inputs : O(N)

How is this possible?

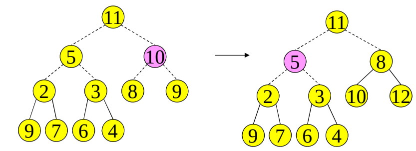
Build Heap from any unsorted array

```
BuildHeap {
for i = N/2 to 1 by –1 PercDown(i, A[i])
}
```

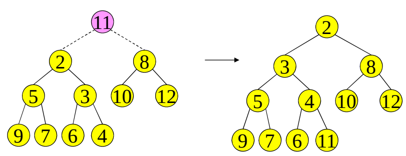
N=11



Build Heap



Build Heap



Analysis of Build Heap

- Assume $N = 2^K - 1$
  - Level 1: k -1 steps for 1 item
  - Level 2: k - 2 steps for 2 items
  - Level 3: k - 3 steps for 4 items
  - Level i : k - i steps for 2^{i-1} items

$$\text{Total Steps} = \sum_{i=1}^{k-1}(k-i)2^{i-1} = 2^k - k - 1$$
$$= O(N)$$

Other Heap Operations

- Find(X, H): Find the element X in heap H of N elements
  - What is the running time? O(N)
- FindMax(H): Find the maximum element in H
- Where FindMin is O(1)
  - What is the running time? O(N)
- We sacrificed performance of these operations in order to get O(1) performance for FindMin

Other Heap Operations

- DecreaseKey(P,Δ,H): Decrease the key value of node at position P by a positive amount Δ, e.g., to increase priority
  - First, subtract Δ from current value at P
  - Heap order property may be violated
  - so percolate up to fix
  - Running Time: O(log N)

Other Heap Operations

- IncreaseKey(P,Δ,H): Increase the key value of node at position P by a positive amount Δ, e.g., to decrease priority
  - First, add Δ to current value at P
  - Heap order property may be violated
  - so percolate down to fix
  - Running Time: O(log N)

Other Heap Operations

- Delete(P,H): E.g. Delete a job waiting in queue that has been preemptively terminated by user
  - Use DecreaseKey(P,∞,H) followed by DeleteMin
  - Running Time: O(log N)

Other Heap Operations

- Merge(H1,H2): Merge two heaps H1 and H2 of size O(N). H1 and H2 are stored in two arrays.
  - Can do O(N) Insert operations: O(N log N) time
  - Better: Copy H2 at the end of H1 and use BuildHeap. Running Time: O(N)

Priority Queues in AI

- The A* algorithm is the classic heuristic search algorithm in Artificial Intelligence.
- Use of a priority queue is key to this algorithm so that the state with the minimum distance to a goal state can be quickly found and removed from the queue.
- The heap is a good data structure for this task.