Disjoint Sets
Union-Find Algorithm

CSE 373
Data Structures & Algorithms
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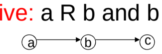
Today's Outline

- **Announcements**
- **Today's Topics:**
  - **Disjoint Sets (Weiss Chapter 8, except Section 6)**

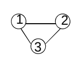
Equivalence Relations

- A relation R is defined on set S if for every pair of elements a, b ∈ S, a R b is either true or false.
- An equivalence relation is a relation R that satisfies the 3 properties:
  - Reflexive: a R a for all a ∈ S
  - Symmetric: a R b iff b R a; a, b ∈ S
  - Transitive: a R b and b R c implies a R c



Equivalence Classes

- Given an equivalence relation R, decide whether a pair of elements a, b ∈ S is such that a R b.
- The equivalence class of an element a is the subset of S of all elements related to a.
- Equivalence classes are disjoint sets



Dynamic Equivalence Problem

- Starting with each element in a singleton set, and an equivalence relation, build the equivalence classes
- Requires two operations:
  - Find the equivalence class (set) of a given element
  - Union of two sets
- It is a dynamic (on-line) problem because the sets change during the operations and Find must be able to cope!

Disjoint Union - Find

- Maintain a set of pairwise disjoint sets.
  - {3,5,7} , {4,2,8}, {9}, {1,6}
- Each set has a unique name, one of its members
  - {3,<u>5</u>,7} , {4,2,<u>8</u>}, {<u>9</u>}, {<u>1</u>,6}

Union

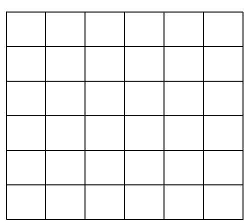
- Union(x,y) – take the union of two sets named x and y
  - {3,5,7} , {4,2,8}, {9}, {1,6}
  - Union(5,1)
    {3,5,7,1,6}, {4,2,8}, {9},

Find

- Find(x) – return the name of the set containing x.
  - {3,5,7,1,6}, {4,2,8}, {9},
  - Find(1) = 5
  - Find(4) = 8
  - Find(9) = ?

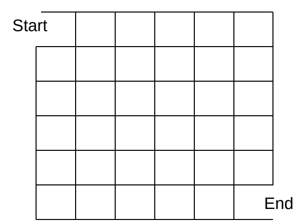
An Application

- Build a random maze by erasing edges.



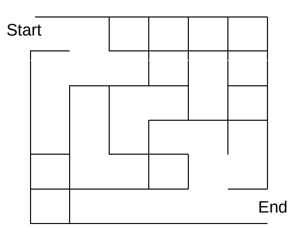
An Application (ct'd)

- Pick Start and End



An Application (ct'd)

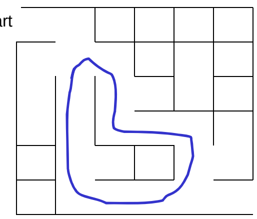
- Repeatedly pick random edges to delete.



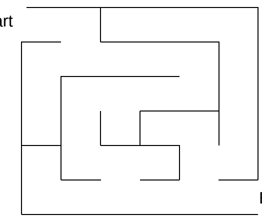
Desired Properties

- None of the boundary is deleted
- Every cell is reachable from every other cell.
- There are no cycles – no cell can reach itself by a path unless it retraces some part of the path.

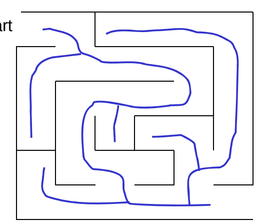
A Cycle (we don't want that)



A Good Solution



Good Solution : A Hidden Tree



Number the Cells

We have disjoint sets S ={ {1}, {2}, {3}, {4},… {36} } each cell is unto itself.
We have all possible edges E ={ (1,2), (1,7), (2,8), (2,3), … } 60 edges total.

Start

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

End

Basic Algorithm

- S = set of sets of connected cells
- E = set of edges
- Maze = set of maze edges initially empty

```
While there is more than one set in S
  pick a random edge (x,y) and remove from E
  u := Find(x);  v := Find(y);
  if u ≠ v then
    Union(u,v)  //knock down the wall between the cells (cells in
                //the same set are connected)
  else
    add (x,y) to Maze //don't remove because there is already
                      // a path between x and y
All remaining members of E together with Maze form the maze
```


Example Step

Pick (8,14)

Start

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

End

S
{1,2,7,8,9,13,19}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{14,20,26,27}
{15,16,21}
.
.
{22,23,24,29,30,32
33,34,35,36}

Example

S
{1,2,7,8,9,13,19}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{14,20,26,27}
{15,16,21}
.
.
{22,23,24,29,39,32
33,34,35,36}

Find(8) = 7
Find(14) = 20

Union(7,20)

S
{1,2,7,8,9,13,19,14,20 26,27}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{15,16,21}
.
.
{22,23,24,29,39,32
33,34,35,36}

Example

Pick (19,20)

Start							
	1	2	3	4	5	6	
	7	8	9	10	11	12	
	13	14	15	16	17	18	
	19	20	21	22	23	24	
	25	26	27	28	29	30	
	31	32	33	34	35	36	End

S
{1,2,7,8,9,13,19
14,20,26,27}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{15,16,21}
.
.
{22,23,24,29,39,32
33,34,35,36}

Example at the End

S
{1,2,3,4,5,6,7,… 36}

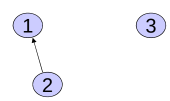
Start							
	1	2	3	4	5	6	
	7	8	9	10	11	12	
	13	14	15	16	17	18	
	19	20	21	22	23	24	
	25	26	27	28	29	30	
	31	32	33	34	35	36	End

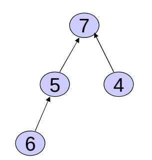
E
Maze

Up-Tree for DU/F

Initial state

Intermediate state

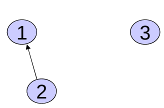


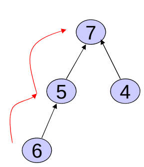


Roots are the names of each set.

Find Operation

- Find(x) follow x to the root and return the root (which is the name of the class).



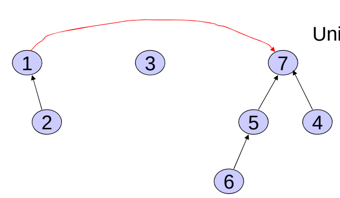


Find(6) = 7

Union Operation

- Union(i,j) - assuming i and j roots, point i to j.

Union(1,7)

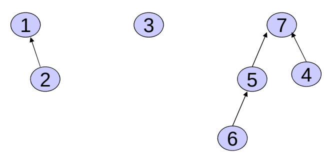


Simple Implementation

- Array of indices (Up[i] is parent of i)

	1	2	3	4	5	6	7
up	0	1	0	7	7	5	0

Up [x] = 0 means x is a root.



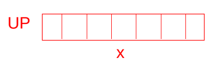
Union

```
Union(up[] : integer array, x,y : integer) : {
//precondition: x and y are roots//
Up[x] := y
}
```

Constant Time!

Find

- Design Find operator
  - › Recursive version
  - › Iterative version




```
Find(up[] : integer array, x : integer) : integer {
//precondition: x is in the range 1 to size//
???
}
```

if up[x] = 0 then return x
else

Find Solutions

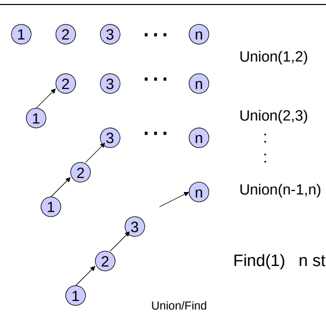
Recursive

```
Find(up[] : integer array, x : integer) : integer {
//precondition: x is in the range 1 to size//
if up[x] = 0 then return x
else return Find(up,up[x]);
}
```

Iterative

```
Find(up[] : integer array, x : integer) : integer {
//precondition: x is in the range 1 to size//
while up[x] ≠ 0 do
  x := up[x];
return x;
}
```

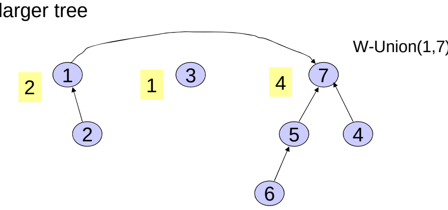

A Bad Case



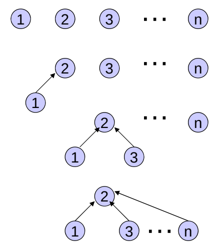
Find(1) n steps!!

Weighted Union

- Weighted Union (weight = number of nodes)
  - › Always point the smaller tree to the root of the larger tree



Example Again



Union(1,2)

Union(2,3)

:
:

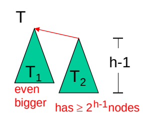
Union(n-1,n)

Find(1) constant time

Analysis of Weighted Union

- With weighted union an up-tree of height h has weight at least 2^h.
- Proof by induction
  - Basis: h = 0. The up-tree has one node, $2^0 = 1$
  - Inductive step: Assume true for all h' < h.

Minimum weight up-tree of height h formed by weighted unions



$W(T_1) \geq W(T_2) \geq 2^{h-1}$

Weighted union

Induction hypothesis

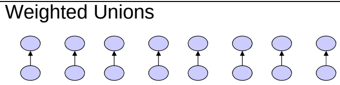
$W(T) \geq 2^{h-1} + 2^{h-1} = 2^h$

Analysis of Weighted Union

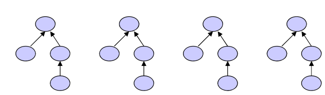
- Let T be an up-tree of weight n formed by weighted union. Let h be its height.
- $n \geq 2^h$
- $\log_2 n \geq h$
- Find(x) in tree T takes O(log n) time.
- Can we do better?

Worst Case for Weighted Union

n/2 Weighted Unions

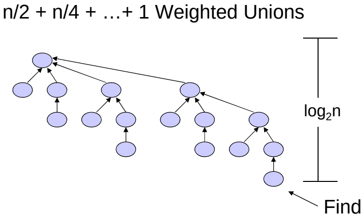


n/4 Weighted Unions



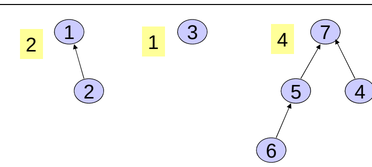
Example of Worst Cast (cont')

After n -1 = n/2 + n/4 + …+ 1 Weighted Unions



If there are $n = 2^k$ nodes then the longest path from leaf to root has length k.

Elegant Array Implementation



	1	2	3	4	5	6	7
up	0	1	0	7	7	5	0
weight	2		1				4
up2							

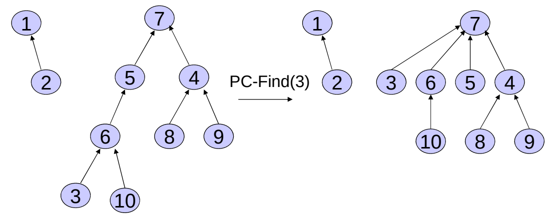
Can save the extra space by storing the complement of weight in the space reserved for the root

Weighted Union

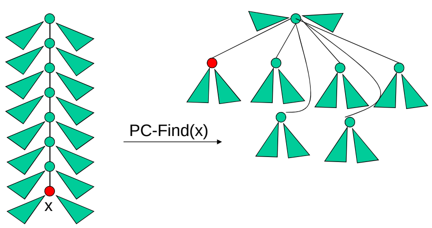
```
W-Union(i,j : index){
//i and j are roots//
  wi := weight[i];
  wj := weight[j];
  if wi < wj then
    up[i] := j;
    weight[j] := wi + wj;
  else
    up[j] :=i;
    weight[i] := wi +wj;
}
```


Path Compression

- On a Find operation point all the nodes on the search path directly to the root.



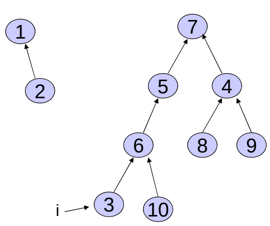
Self-Adjustment Works



Path Compression Find

```
PC-Find(i : index) {
  r := i;
  while up[r] ≠ 0 do //find root//
    r := up[r];
  if i ≠ r then  //compress path//
    k := up[i];
    while k ≠ r do
      up[i] := r;
      i := k;
      k := up[k]
  return(r)
}
```


Example



Disjoint Union / Find with Weighted Union and PC

- Worst case time complexity for a W-Union is O(1) and for a PC-Find is O(log n).
- Time complexity for m ≥ n operations on n elements is O(m log* n) where log* n is a very slow growing function.
  - log * n < 7 for all reasonable n. Essentially constant time per operation!

Amortized Complexity

- For disjoint union / find with weighted union and path compression.
  - average time per operation is essentially a constant.
  - worst case time for a PC-Find is O(log n).
- An individual operation can be costly, but over time the average cost per operation is not.