Minimum Spanning Trees

CSE 373
Data Structures & Algorithms
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Spring 2013

Today's Outline

- **Announcements:**
  - **HW 5 due Friday, May 31.**
- **Today's Topics:**
  - **Weiss 9.5, 9.6**

Recall Spanning Tree

- Given (connected) graph G(V,E), a spanning tree T(V',E'):
  - Spans the graph (V' = V)
  - Forms a tree (no cycle);
  - E' has |V| -1 edges

Minimum Spanning Tree

- Edges are weighted: find minimum cost spanning tree
- Applications
  - Find cheapest way to wire your house
  - Find minimum cost to send a message on the Internet

Strategy for Minimum Spanning Tree

- For any spanning tree T, inserting an edge e_{new} not in T creates a cycle
- But
  - Removing any edge e_{old} from the cycle gives back a spanning tree
  - If e_{new} has a lower cost than e_{old} we have progressed!

Strategy

- Strategy for construction:
  - Add an edge of minimum cost that does not create a cycle (greedy algorithm)
  - Repeat |V| -1 times
  - Correct since if we could replace an edge with one of lower cost, the algorithm would have picked it up

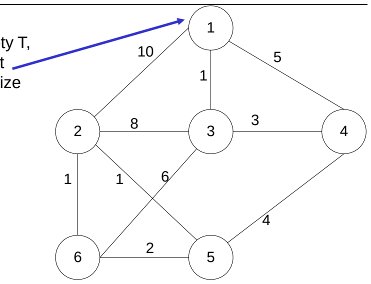
Two Algorithms

- Prim: (build tree incrementally)
  - Pick lower cost edge connected to known (incomplete) spanning tree that does not create a cycle and expand to include it in the tree
- Kruskal: (build forest that will finish as a tree)
  - Pick lower cost edge not yet in a tree that does not create a cycle and expand to include it somewhere in the forest

Prim's algorithm

Starting from empty T, choose a vertex at random and initialize

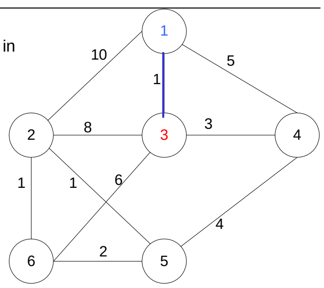
V = {1), E' ={}



Prim's algorithm

Choose the vertex u not in V such that edge weight from u to a vertex in V is minimal (greedy!)

V={1,3} E'= {(1,3) }



Prim's algorithm

Repeat until all vertices have been chosen

Choose the vertex u not in V such that edge weight from v to a vertex in V is minimal (greedy!)

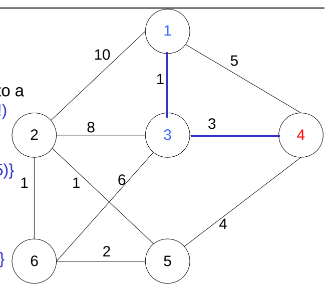
V= {1,3,4} E'= {(1,3),(3,4)}

V={1,3,4,5} E'={(1,3),(3,4),(4,5)}

….

V={1,3,4,5,2,6}

E'={(1,3),(3,4),(4,5),(5,2),(2,6)}



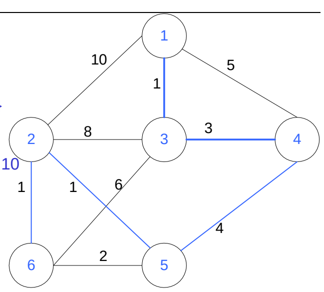
Prim's algorithm

Repeat until all vertices have been chosen

V={1,3,4,5,2,6}

E'={(1,3),(3,4),(4,5),(5,2),(2,6)}

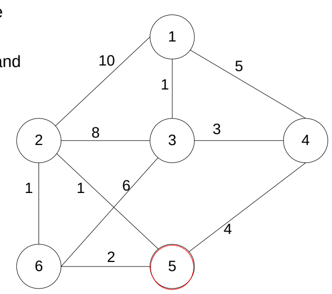
Final Cost: 1 + 3 + 4 + 1 + 1 = 10



Prim's algorithm

Repeat until all vertices have been chosen.

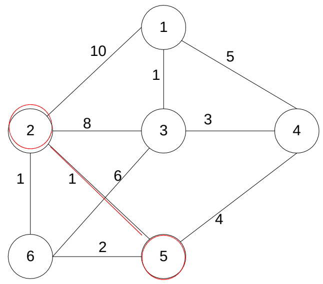
1. Choose an initial vertex and put in tree.



Prim's algorithm

Repeat until all vertices have been chosen.

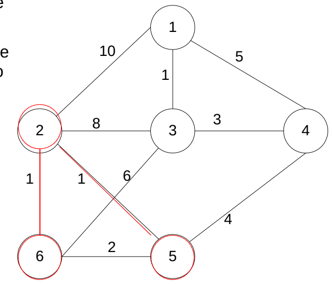
2. Choose a vertex not in tree with minimal cost edge to add.



Prim's algorithm

Repeat until all vertices have been chosen.

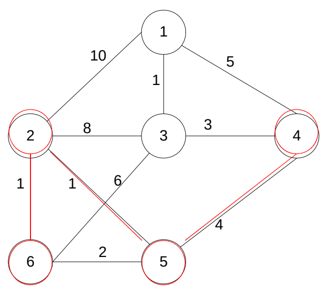
3. Choose a vertex not in tree with minimal cost edge to add.



Prim's algorithm

Repeat until all vertices have been chosen.

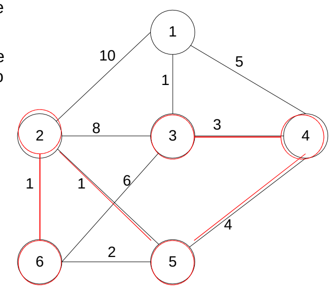
4. Choose a vertex not in tree with minimal cost edge to add.



Prim's algorithm

Repeat until all vertices have been chosen.

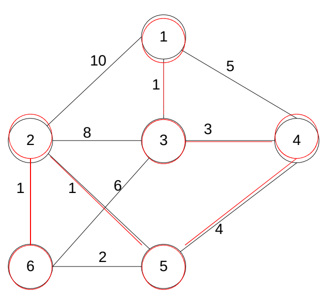
5. Choose a vertex not in tree with minimal cost edge to add.



Prim's algorithm

Repeat until all vertices have been chosen.

6. Choose a vertex not in tree with minimal cost edge to add.



Prim's Algorithm Implementation

- Assume adjacency list representation

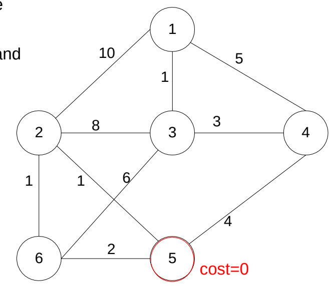
Initialize connection cost of each node to "inf" and "unmark" them
Choose one node, say v and set cost[v] = 0 and prev[v] =0
While there are unmarked nodes
  Select the unmarked node **u** with minimum cost; mark it
  For each unmarked node **w** adjacent to **u**
    if cost(u,w) < cost(w) then {
      cost(w) := cost (u,w)
      prev[w] = u}

- Looks a lot like Dijkstra's algorithm!

Prim’s algorithm

Repeat until all vertices have been chosen.

1. Choose an initial vertex and put in tree.

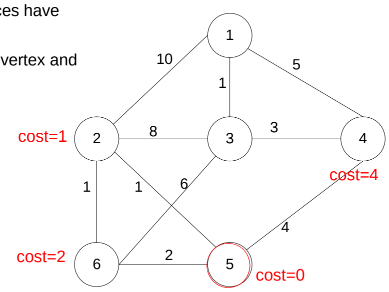


Prim’s algorithm

Repeat until all vertices have been chosen.

1. Choose an initial vertex and put in tree.

Update the costs of its neighbors.

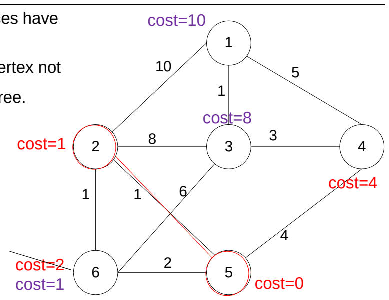


Prim’s algorithm

Repeat until all vertices have been chosen.

2. Choose min cost vertex not in tree and put in tree.

Update the costs of its neighbors.

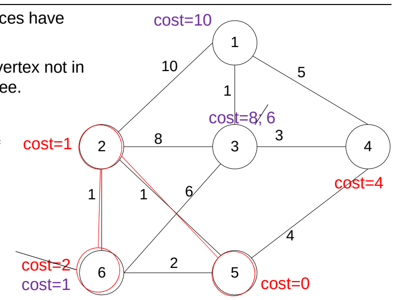


Prim’s algorithm

Repeat until all vertices have been chosen.

3. Choose min cost vertex not in tree and put in tree.

Update the costs of its neighbors.

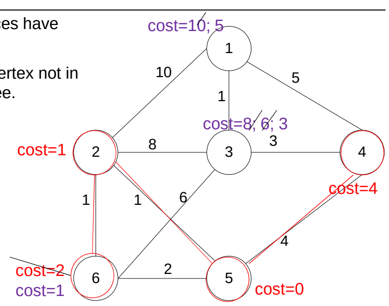


Prim’s algorithm

Repeat until all vertices have been chosen.

4. Choose min cost vertex not in tree and put in tree.

Update the costs of its neighbors.



Prim’s algorithm Analysis

- Like Dijkstra’s algorithm
- If the “Select the unmarked node u with minimum cost” is done with binary heap then O((n+m)logn)

Kruskal's Algorithm

- Select edges in order of increasing cost
- Accept an edge to expand tree or forest only if it does not cause a cycle
- Implementation using adjacency list, priority queues and disjoint sets

Kruskal's Algorithm

Initialize a forest of trees, each tree being a single node
Build a priority queue of edges with priority being lowest cost

Repeat until |V| -1 edges have been accepted {
    Deletemin edge from priority queue
    If it forms a cycle then discard it
    else accept the edge – It will join 2 existing trees yielding a larger tree and reducing the forest by one tree
}

The accepted edges form the minimum spanning tree

Detecting Cycles

- If the edge to be added (u,v) is such that vertices u and v belong to the same tree, then by adding (u,v) you would form a cycle
  - Therefore to check, Find(u) and Find(v). If they are the same discard (u,v)
  - If they are different Union(Find(u),Find(v))

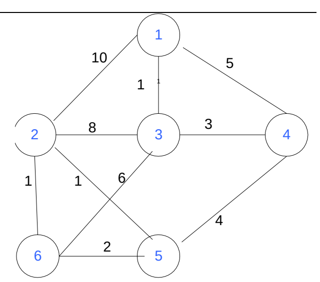
Properties of trees in K's algorithm

- Vertices in different trees are disjoint
  - True at initialization and Union won't modify the fact for remaining trees
- Trees form equivalent classes under the relation "is connected to"
  - u connected to u (reflexivity)
  - u connected to v implies v connected to u (symmetry)
  - u connected to v and v connected to w implies a path from u to w so u connected to w (transitivity)

K's Algorithm Data Structures

- Adjacency list for the graph
  - To perform the initialization of the data structures below
- Disjoint Set ADT's for the trees (recall Up tree implementation of Union-Find)
- Binary heap for edges

Example

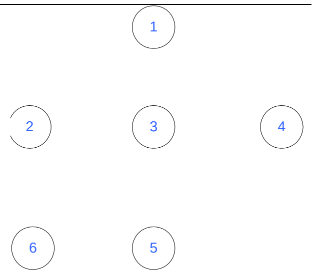


Initialization

Initially, Forest of 6 trees

F= {{1},{2},{3},{4},{5},{6}}

Edges in a heap (not shown)



Step 1

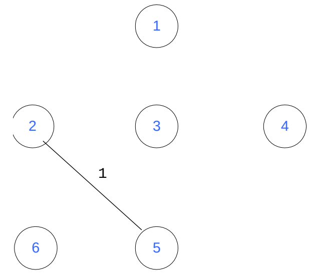
Select edge with lowest cost (2,5)

Find(2) = 2, Find (5) = 5

Union(2,5)

F= {{1},{2,5},{3},{4},{6}}

1 edge accepted



Step 2

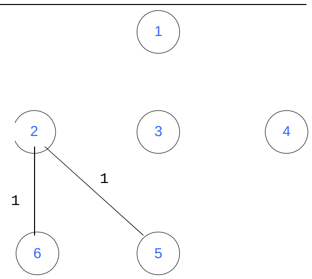
Select edge with lowest cost (2,6)

Find(2) = 2, Find (6) = 6

Union(2,6)

F= {{1},{2,5,6},{3},{4}}

2 edges accepted



Step 3

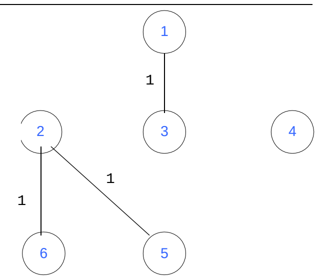
Select edge with lowest cost (1,3)

Find(1) = 1, Find (3) = 3

Union(1,3)

F= {{1,3},{2,5,6},{4}}

3 edges accepted



Step 4

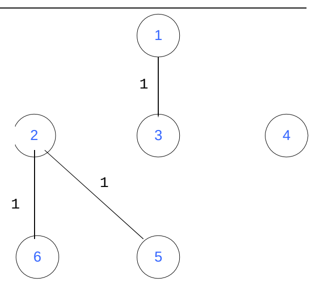
Select edge with lowest cost (5,6)

Find(5) = 2, Find (6) = 2

Do nothing

F= {{1,3},{2,5,6},{4}}

3 edges accepted



Step 5

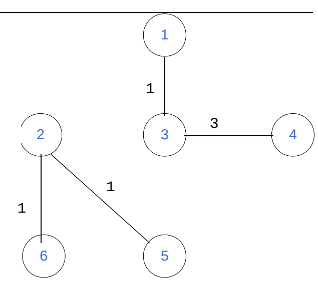
Select edge with lowest cost (3,4)

Find(3) = 1, Find (4) = 4

Union(1,4)

F= {{1,3,4},{2,5,6}}

4 edges accepted



Step 6

Select edge with lowest cost (4,5)

Find(4) = 1, Find (5) = 2

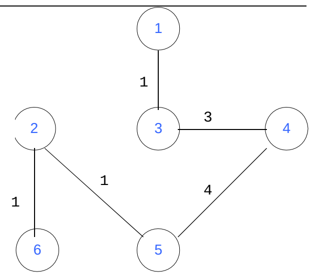
Union(1,2)

F= {{1,3,4,2,5,6}}

5 edges accepted : end

Total cost = 10

Although there is a unique spanning tree in this example, this is not generally the case



Kruskal's Algorithm Analysis

- Initialize forest O(n)
- Initialize heap O(m), m = |E|
- Loop performed m times
  - › In the loop one Deletemin O(logm)
  - › Two Find, each O(logn)
  - › One Union (at most) O(1)
- So worst case O(mlogm) = O(mlogn)

Time Complexity Summary

- Recall that m = |E| = $O(V^2)$ = $O(n^2)$
- Prim's runs in O((n+m) log n)
- Kruskal's runs in O(m log m)
- In practice, Kruskal has a tendency to run faster since graphs might not be dense and not all edges need to be looked at in the Deletemin operations