CSE 373: Data Structures & Algorithms
Lecture 16: Topological Sort / Graph Traversals

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Spring 2014

Midterm

- This Friday in class
- Closed books, closed notes
- Practice midterms posted online

Graphs

- A graph is a formalism for representing relationships among items
  - Very general definition because very general concept

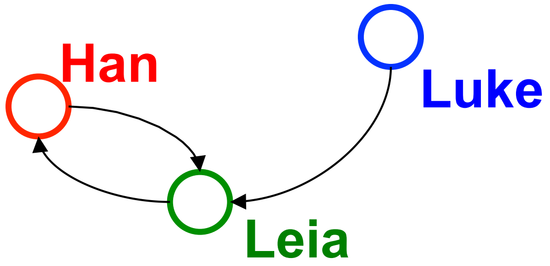
- A graph is a pair

  G = (V,E)
  - A set of vertices, also known as nodes

    $\mathbf{V = \{v_1, v_2, \ldots, v_n\}}$
  - A set of edges

    $\mathbf{E = \{e_1, e_2, \ldots, e_m\}}$
    - Each edge $\mathbf{e_i}$ is a pair of vertices

      $\mathbf{(v_j, v_k)}$
    - An edge "connects" the vertices



V = {Han,Leia,Luke}
E = {(Luke,Leia),
(Han,Leia),
(Leia,Han)}

- Graphs can be directed or undirected

Density / Sparsity

- Recall: In an undirected graph, $0 \leq |E| < |V|^2$
- Recall: In a directed graph: $0 \leq |E| \leq |V|^2$
- So for any graph, $O(|E|+|V|^2)$ is $O(|V|^2)$
- Another fact: If an undirected graph is *connected*, then $|V|-1 \leq |E|$
- Because $|E|$ is often much smaller than its maximum size, we do not always approximate $|E|$ as $O(|V|^2)$
  - This is a correct bound, it just is often not tight
  - If it is tight, i.e., $|E|$ is $\Theta(|V|^2)$ we say the graph is dense
    - More sloppily, dense means "lots of edges"
  - If $|E|$ is $O(|V|)$ we say the graph is sparse
    - More sloppily, sparse means "most possible edges missing"

What is the Data Structure?

- So graphs are really useful for lots of data and questions
  - For example, “what’s the lowest-cost path from x to y”

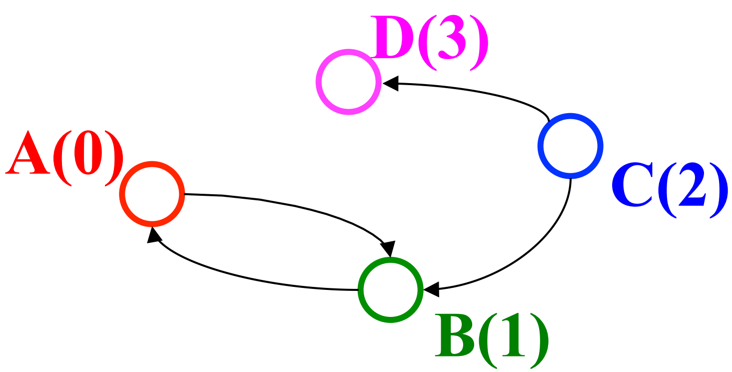
- But we need a data structure that represents graphs

- The “best one” can depend on:
  - Properties of the graph (e.g., dense versus sparse)
  - The common queries (e.g., “is **`(u,v)`** an edge?” versus “what are the neighbors of node **`u`**?”)

- So we’ll discuss the two standard graph representations
  - Adjacency Matrix and Adjacency List
  - Different trade-offs, particularly time versus space

Adjacency Matrix

- Assign each node a number from `0` to `|V|-1`
- A `|V|` x `|V|` matrix (i.e., 2-D array) of Booleans (or 1 vs. 0)
  - If **`M`** is the matrix, then **`M[u][v]`** being **`true`** means there is an edge from **`u`** to **`v`**



	0	1	2	3
0	F	T	F	F
1	T	F	F	F
2	F	T	F	T
3	F	F	F	F

Adjacency Matrix Properties

- Running time to:
  - Get a vertex's out-edges: $O(|V|)$
  - Get a vertex's in-edges: $O(|V|)$
  - Decide if some edge exists: $O(1)$
  - Insert an edge: $O(1)$
  - Delete an edge: $O(1)$

- Space requirements:
  - $|V|^2$ bits

- Best for sparse or dense graphs?
  - Best for dense graphs

	0	1	2	3
0	F	T	F	F
1	T	F	F	F
2	F	T	F	T
3	F	F	F	F

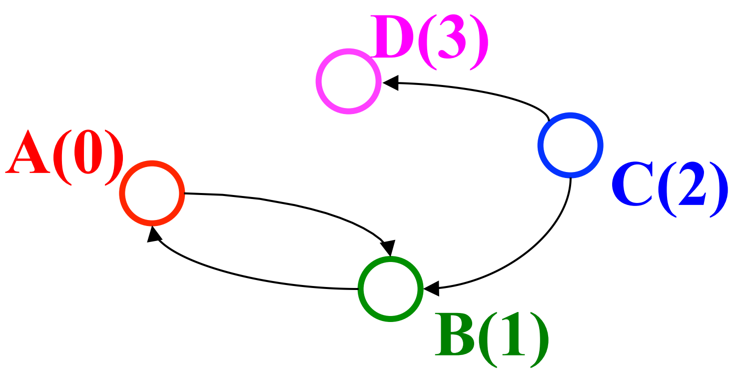
Adjacency Matrix Properties

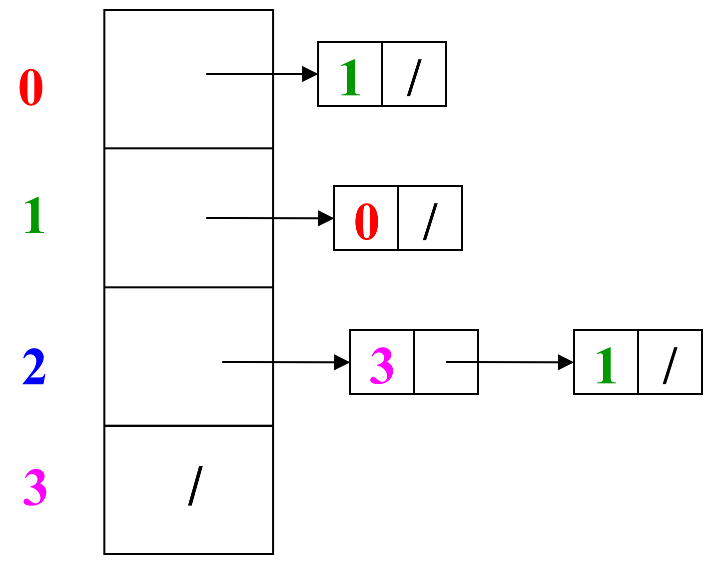
- How will the adjacency matrix vary for an *undirected graph*?
  - Undirected will be symmetric around the diagonal

- How can we adapt the representation for *weighted graphs*?
  - Instead of a Boolean, store a number in each cell
  - Need some value to represent ‘not an edge’
    - In *some* situations, 0 or -1 works

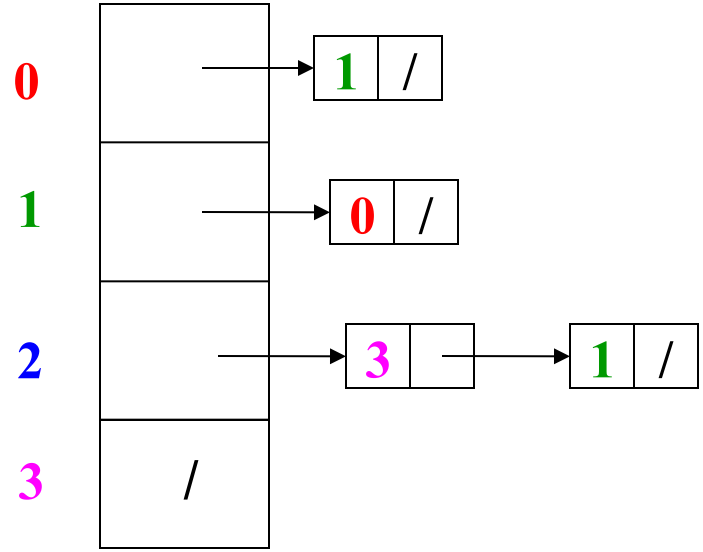
Adjacency List

- Assign each node a number from `0` to `|V|-1`
- An array of length `|V|` in which each entry stores a list of all adjacent vertices (e.g., linked list)





Adjacency List Properties



- Running time to:
  - Get all of a vertex's out-edges:
    O(*d*) where *d* is out-degree of vertex
  - Get all of a vertex's in-edges:
    O(|E|) (but could keep a second adjacency list for this!)
  - Decide if some edge exists:
    O(*d*) where *d* is out-degree of source
  - Insert an edge:
    O(1) (unless you need to check if it's there)
  - Delete an edge:
    O(*d*) where *d* is out-degree of source
- Space requirements:
  - *O*(|V|+|E|)
- Good for sparse graphs

Algorithms

Okay, we can represent graphs

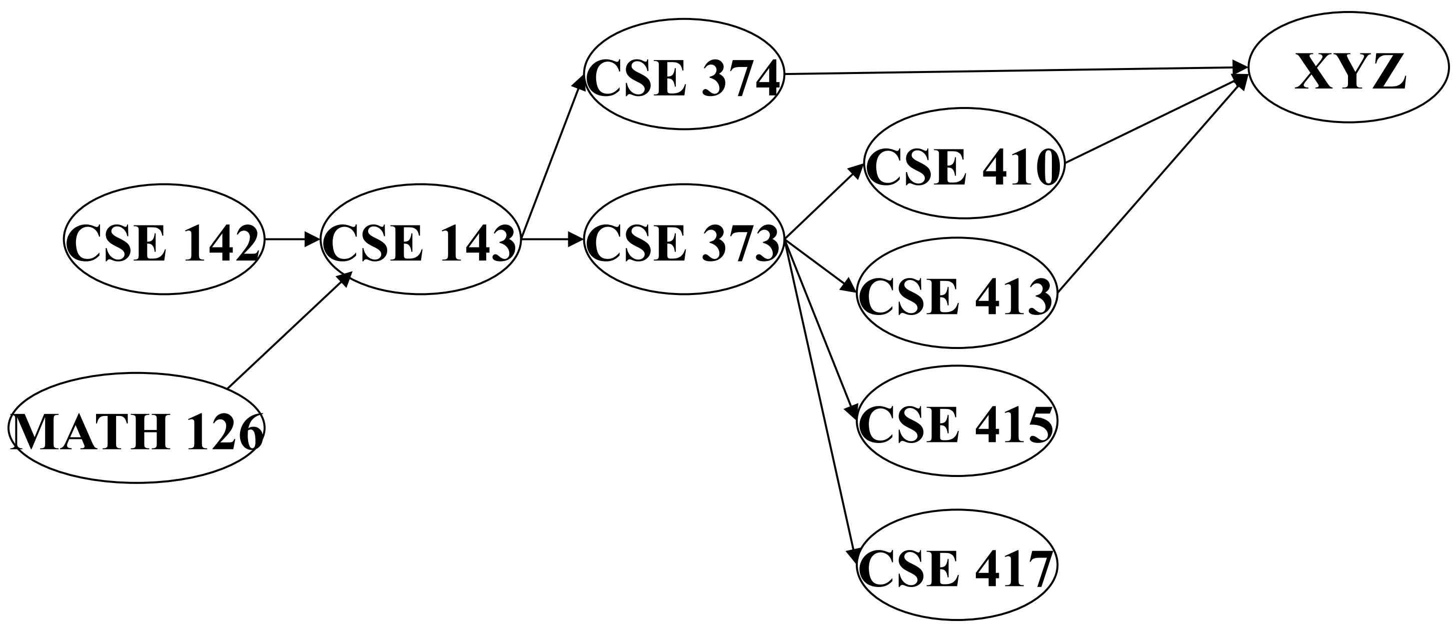
Now we'll implement some useful and non-trivial algorithms

- Topological sort: Given a DAG, order all the vertices so that every vertex comes before all of its neighbors
- Shortest paths: Find the shortest or lowest-cost path from x to y
  - Related: Determine if there even is such a path

Topological Sort

Disclaimer: Do not use for official advising purposes !

Problem: Given a DAG `G=(V,E)`, output all vertices in an order such that no vertex appears before another vertex that has an edge to it

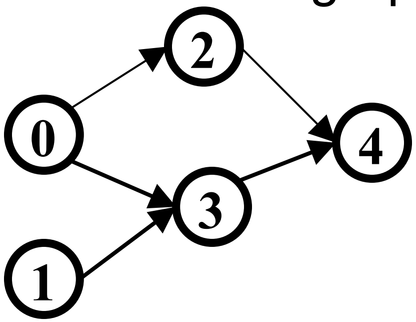


One example output:

126, 142, 143, 374, 373, 417, 410, 413, XYZ, 415

Questions and comments

- Why do we perform topological sorts only on DAGs?
  - Because a cycle means there is no correct answer
- Is there always a unique answer?
  - No, there can be 1 or more answers; depends on the graph
- Do some DAGs have exactly 1 answer?
  - Yes, including all lists



- Terminology: A DAG represents a partial order and a topological sort produces a total order that is consistent with it

Uses

- Figuring out how to graduate
- Computing an order in which to recompute cells in a spreadsheet
- Determining an order to compile files using a Makefile
- In general, taking a dependency graph and finding an order of execution
- …

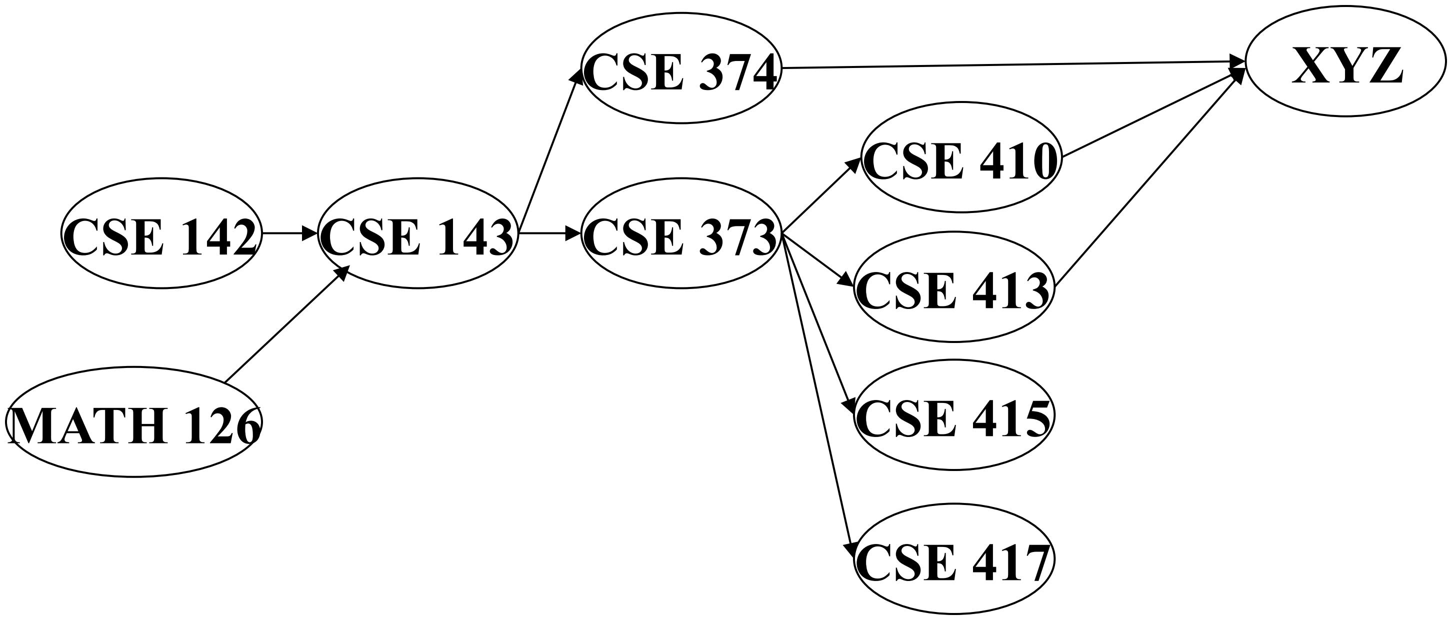
A First Algorithm for Topological Sort

1. Label ("mark") each vertex with its in-degree
   – Think "write in a field in the vertex"
   – Could also do this via a data structure (e.g., array) on the side

2. While there are vertices not yet output:
   a) Choose a vertex **v** with labeled with in-degree of 0
   b) Output **v** and *conceptually* remove it from the graph
   c) For each vertex **u** adjacent to **v** (i.e. **u** such that (**v**,**u**) in **`E`**), decrement the in-degree of **u**

Example

Output:

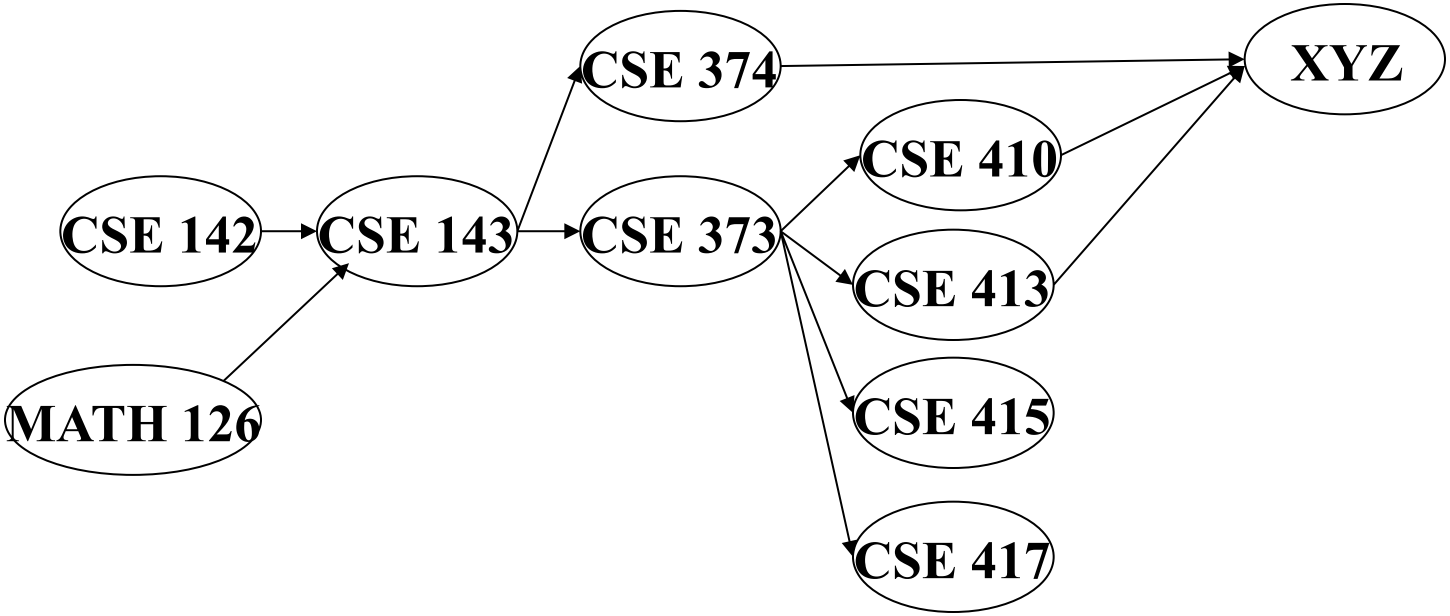


Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?										
In-degree:	0	0	2	1	1	1	1	1	1	3

Example

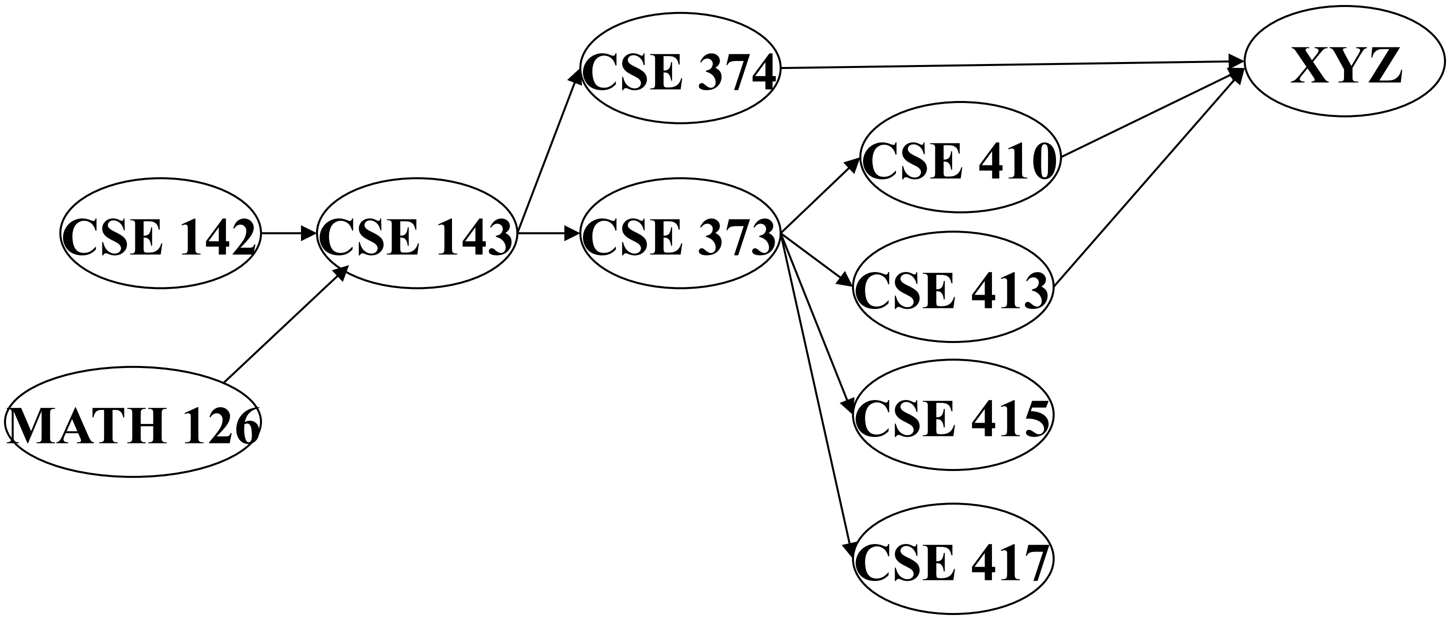
Output:

126



Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x									
In-degree:	0	0	2	1	1	1	1	1	1	3
			1							

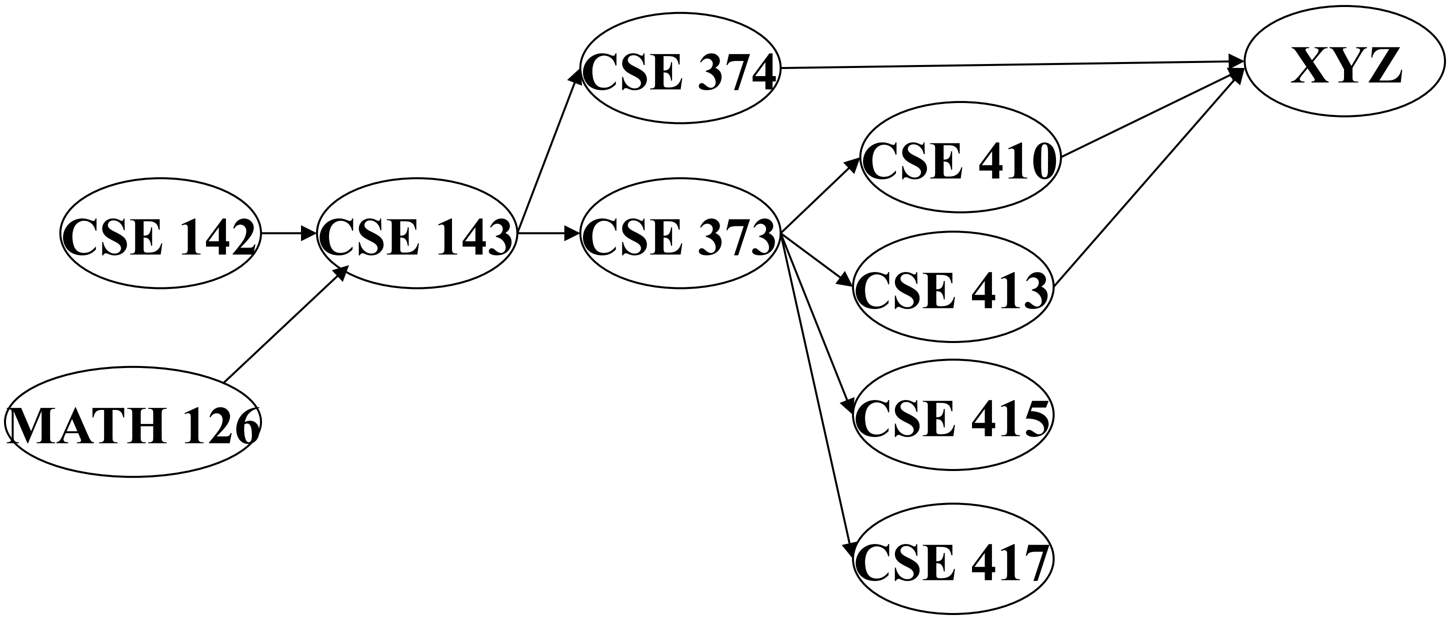
Example



Output:
126
142

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x								
In-degree:	0	0	2	1	1	1	1	1	1	3
			1							
			0							

Example



Output:
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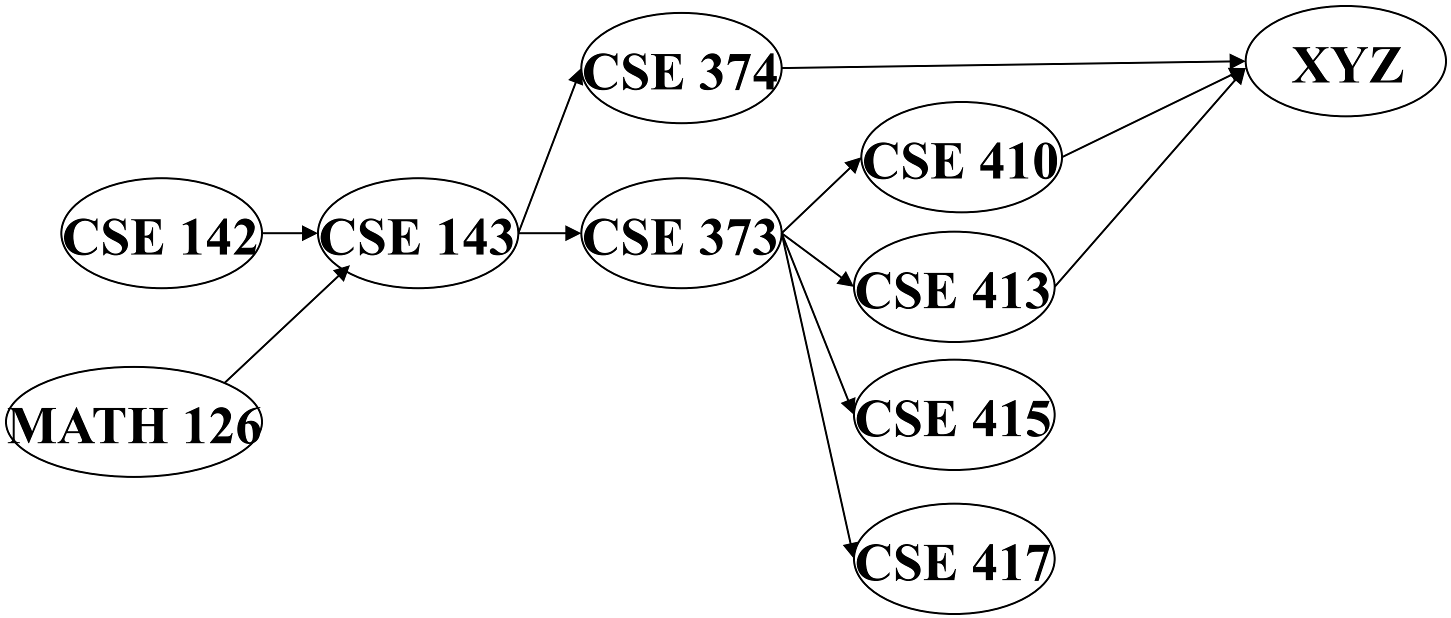
Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x							
In-degree:	0	0	2	1	1	1	1	1	1	3
			1	0	0					
			0							

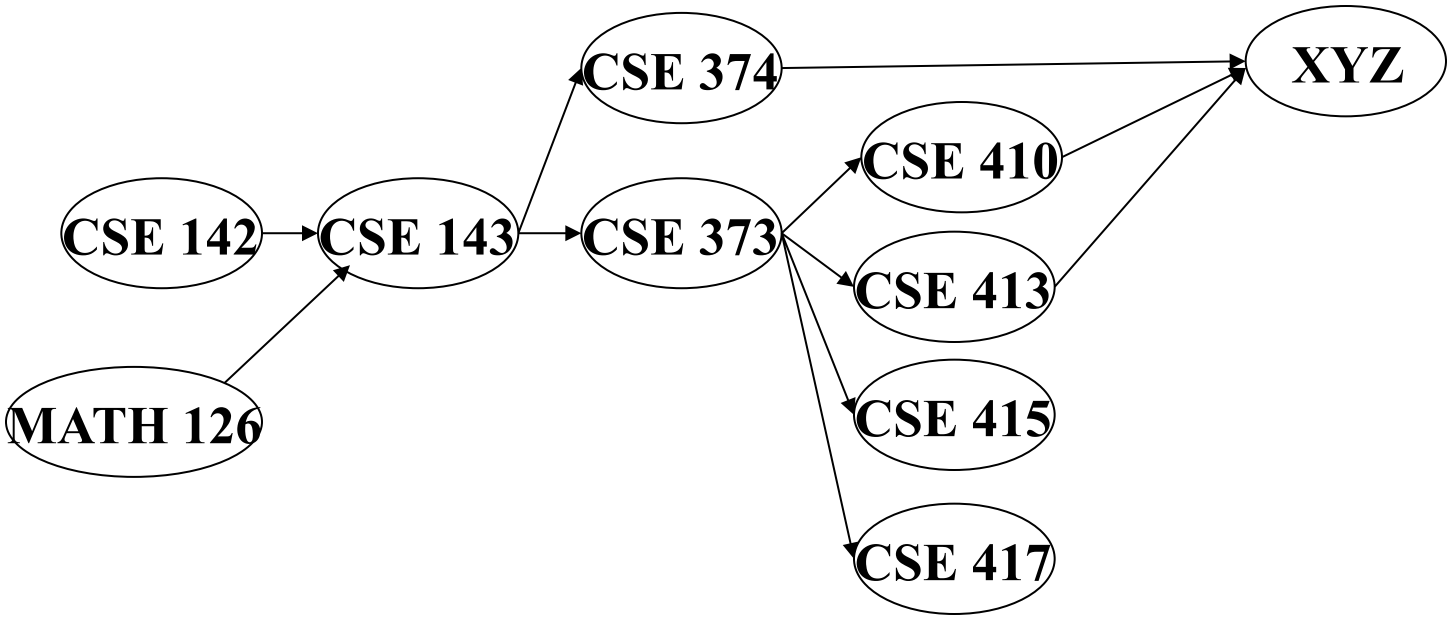
Example



Output:
126
142
143
374

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x						
In-degree:	0	0	2	1	1	1	1	1	1	3
			1	0	0					2
			0							

Example



Output:

126

142

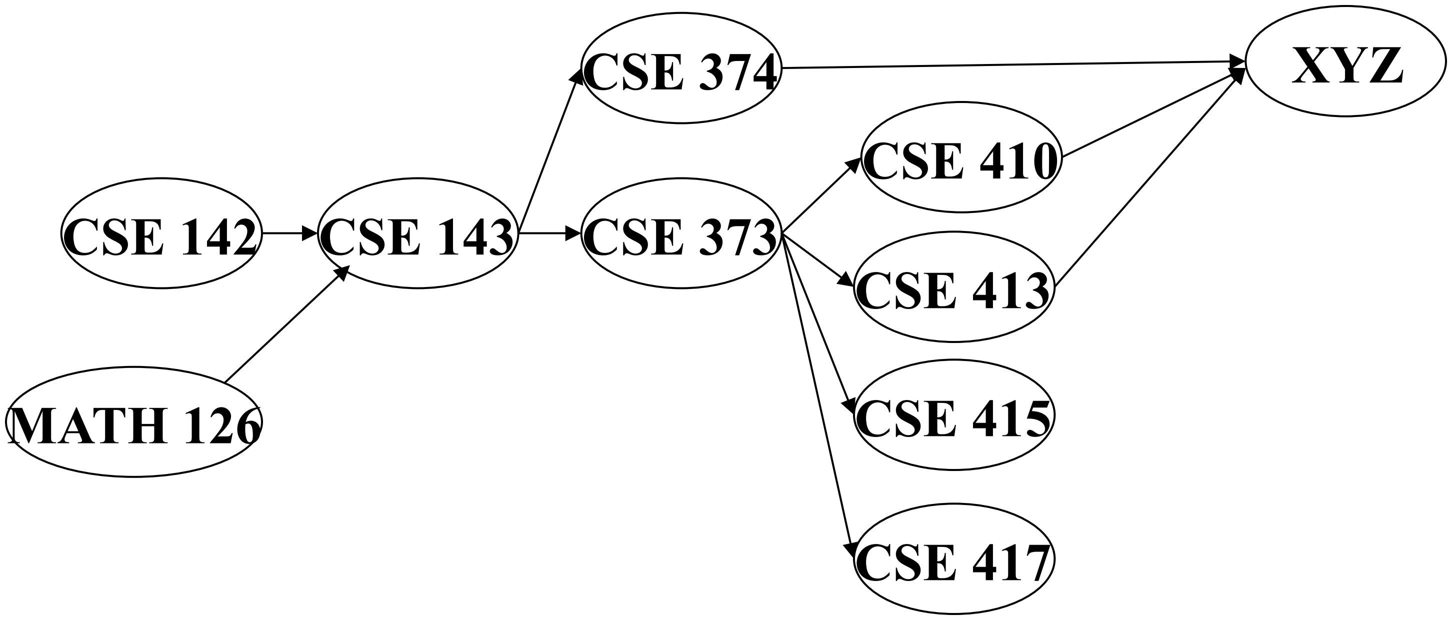
143

374

373

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x					
In-degree:	0	0	2	1	1	1	1	1	1	3
			1	0	0	0	0	0	0	2
			0							

Example



Output:

126

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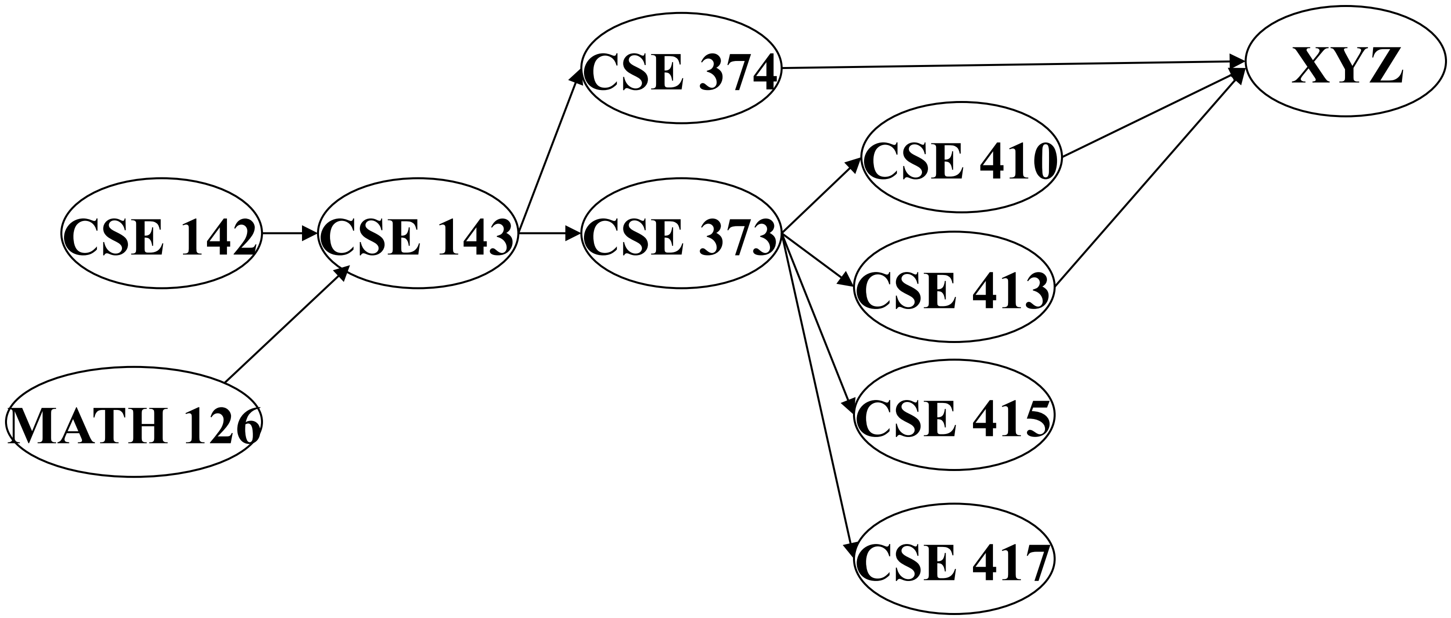
417

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x				x	
In-degree:	0	0	2	1	1	1	1	1	1	3
			1	0	0	0	0	0	0	2
			0							

Example



Output:
126
142
143
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373
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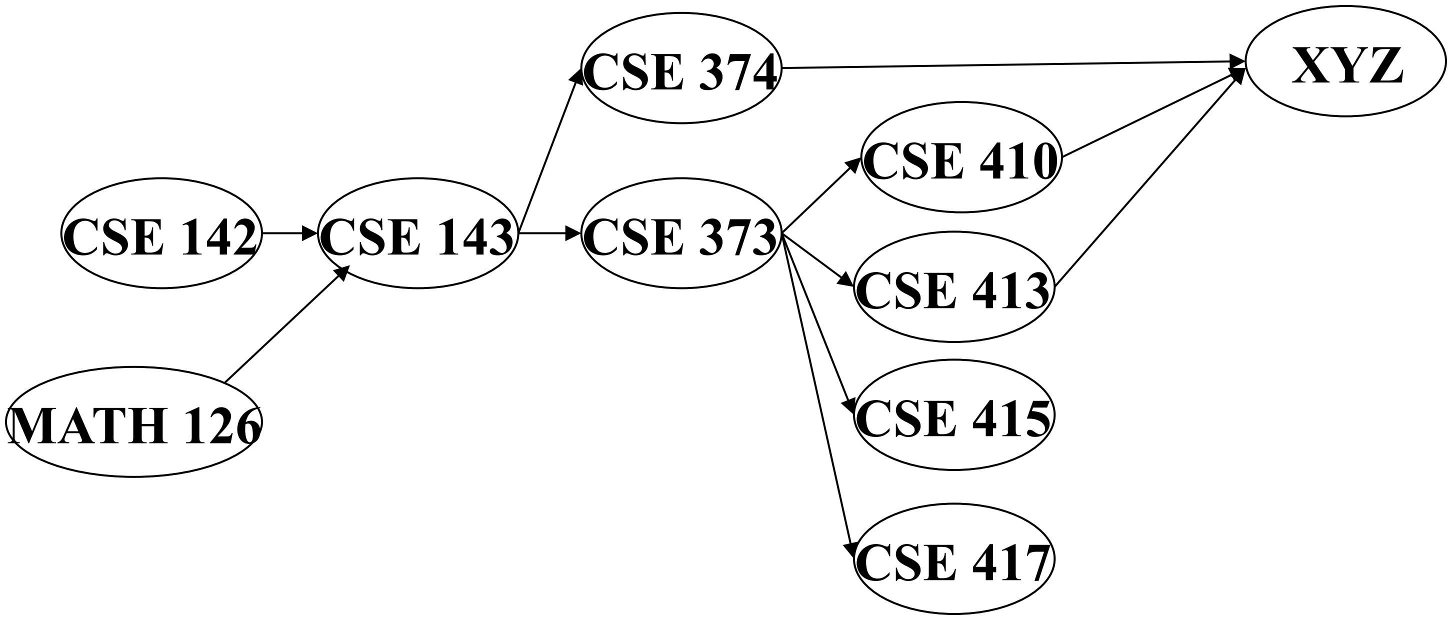
410

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x	x			x	
In-degree:	0	0	2	1	1	1	1	1	1	3
			1	0	0	0	0	0	0	2
			0							1

Example



Output:
126
142
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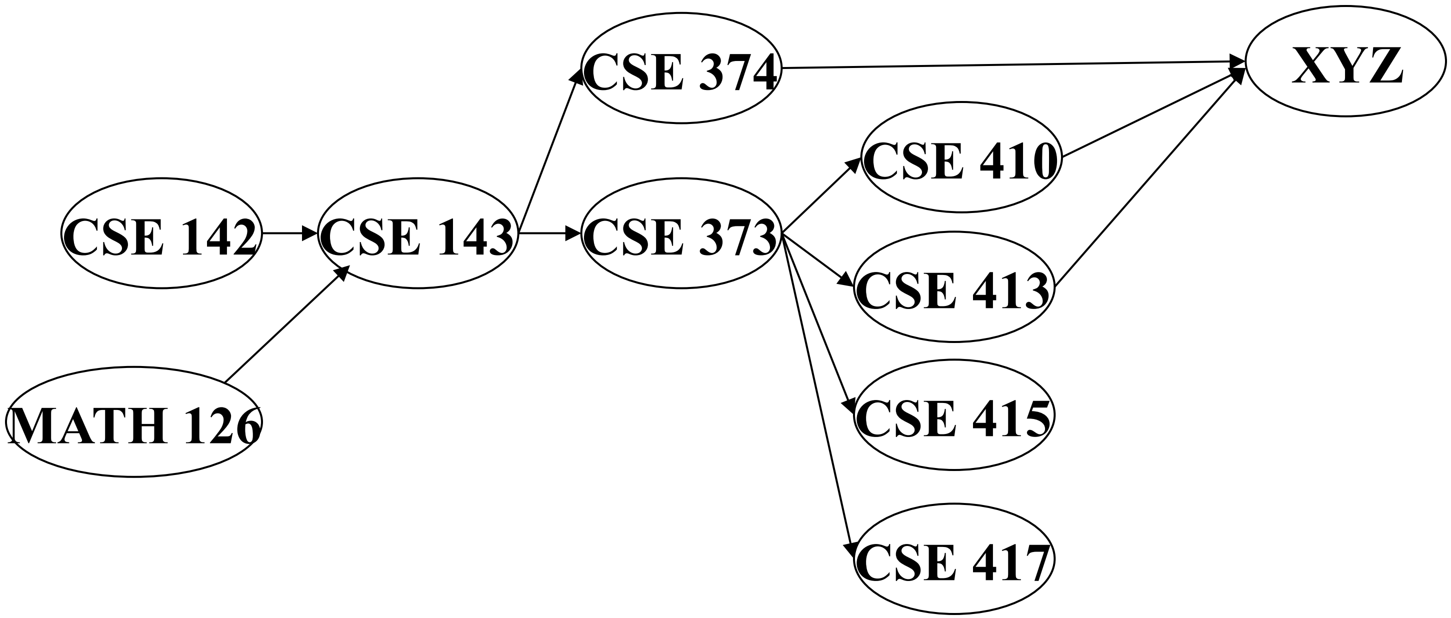
417
410
413

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x	x	x		x	
In-degree:	0	0	2	1	1	1	1	1	1	3
			1	0	0	0	0	0	0	2
			0							1
										0

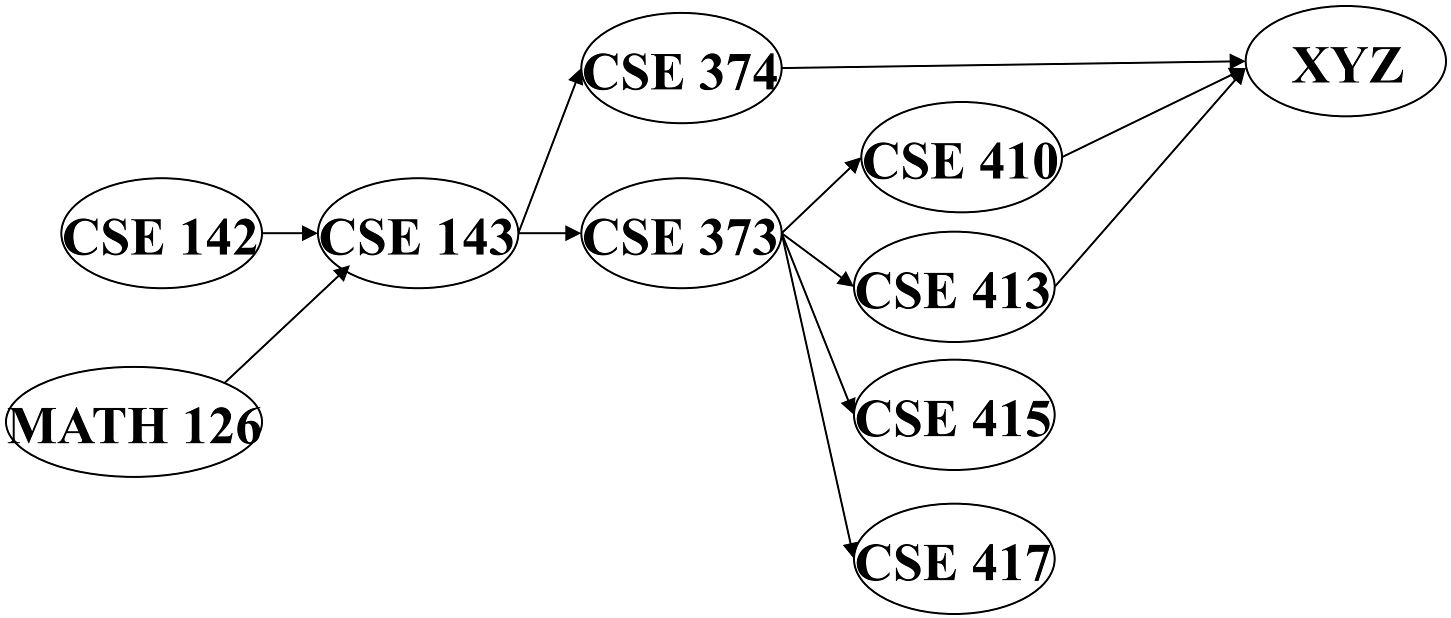
Example



Output:
126
142
143
374
373
417
410
413
XYZ

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x	x	x		x	x
In-degree:	0	0	2	1	1	1	1	1	1	3
			1	0	0	0	0	0	0	2
			0							1
										0

Example



Output:

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XYZ
415

Node:	126	142	143	374	373	410	413	415	417	XYZ
Removed?	x	x	x	x	x	x	x	x	x	x
In-degree:	0	0	2	1	1	1	1	1	1	3
			1	0	0	0	0	0	0	2
			0							1
										0

Notice

- Needed a vertex with in-degree 0 to start
  - Will always have at least 1 because no cycles

- Ties among vertices with in-degrees of 0 can be broken arbitrarily
  - Can be more than one correct answer, by definition, depending on the graph

Running time?

```
labelEachVertexWithItsInDegree();
for(ctr=0; ctr < numVertices; ctr++){
  v = findNewVertexOfDegreeZero();
  put v next in output
  for each w adjacent to v
    w.indegree--;
}
```

- What is the worst-case running time?
  - Initialization $O(|V|+|E|)$ (assuming adjacency list)
  - Sum of all find-new-vertex $O(|V|^2)$ (because each $O(|V|)$)
  - Sum of all decrements $O(|E|)$ (assuming adjacency list)
  - So total is $O(|V|^2)$ – not good for a sparse graph!

Doing better

The trick is to avoid searching for a zero-degree node every time!

- Keep the "pending" zero-degree nodes in a list, stack, queue, bag, table, or something
- Order we process them affects output but not correctness or efficiency provided add/remove are both $O(1)$

Using a queue:

1. Label each vertex with its in-degree, enqueue 0-degree nodes
2. While queue is not empty
   a) **v** = dequeue()
   b) Output **v** and remove it from the graph
   c) For each vertex **u** adjacent to **v** (i.e. **u** such that (**v**,**u**) in **`E`**), decrement the in-degree of **u**, if new degree is 0, enqueue it

Running time?

```
labelAllAndEnqueueZeros();
for(ctr=0; ctr < numVertices; ctr++){
  v = dequeue();
  put v next in output
  for each w adjacent to v {
    w.indegree--;
    if(w.indegree==0)
      enqueue(v);
  }
}
```

- What is the worst-case running time?
  - Initialization: *O*(|V|+|E|) (assuming adjacency list)
  - Sum of all enqueues and dequeues: *O*(|V|)
  - Sum of all decrements: *O*(|E|) (assuming adjacency list)
  - So total is *O*(|E| + |V|) – much better for sparse graph!

Graph Traversals

Next problem: For an arbitrary graph and a starting node **v**, find all nodes *reachable* from **v** (i.e., there exists a path from **v**)

- Possibly "do something" for each node
- Examples: print to output, set a field, etc.

- Subsumed problem: Is an undirected graph connected?
- Related but different problem: Is a directed graph strongly connected?
  - Need cycles back to starting node

Basic idea:

- Keep following nodes
- But "mark" nodes after visiting them, so the traversal terminates and processes each reachable node exactly once

Abstract Idea

```
traverseGraph(Node start) {
   Set pending = emptySet()
   pending.add(start)
   mark start as visited
   while(pending is not empty) {
     next = pending.remove()
     for each node u adjacent to next
        if(u is not marked) {
          mark u
          pending.add(u)
        }
   }
}
```

Running Time and Options

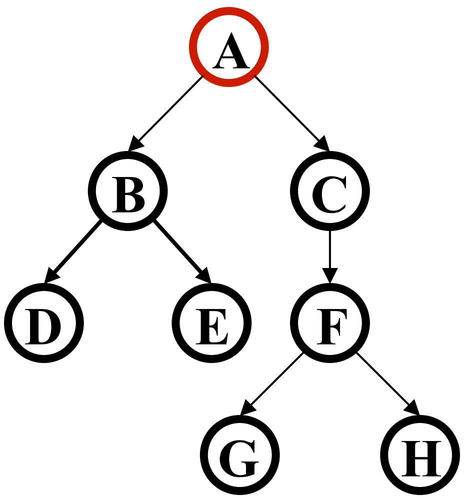
- Assuming **`add`** and **`remove`** are $O(1)$, entire traversal is $O(|E|)$
  - Use an adjacency list representation

- The order we traverse depends entirely on **`add`** and **`remove`**
  - Popular choice: a stack "depth-first graph search" "DFS"
  - Popular choice: a queue "breadth-first graph search" "BFS"

- DFS and BFS are "big ideas" in computer science
  - Depth: recursively explore one part before going back to the other parts not yet explored
  - Breadth: explore areas closer to the start node first

Example: Depth First Search

- A tree is a graph and DFS and BFS are particularly easy to "see"

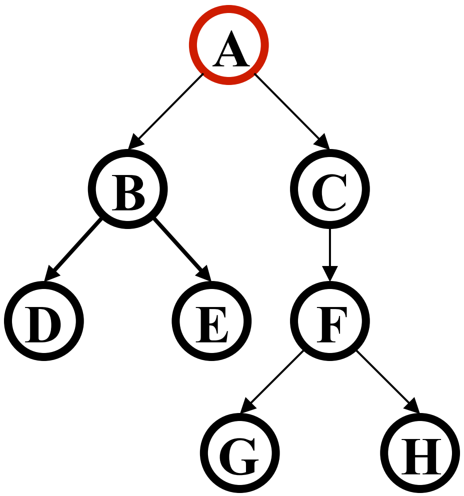



```
DFS(Node start) {
  mark and process start
  for each node u adjacent to start
    if u is not marked
      DFS(u)
}
```

- A B D E C F G H
- Exactly what we called a "pre-order traversal" for trees
  - The marking is because we support arbitrary graphs and we want to process each node exactly once

Example: Another Depth First Search

- A tree is a graph and DFS and BFS are particularly easy to “see”

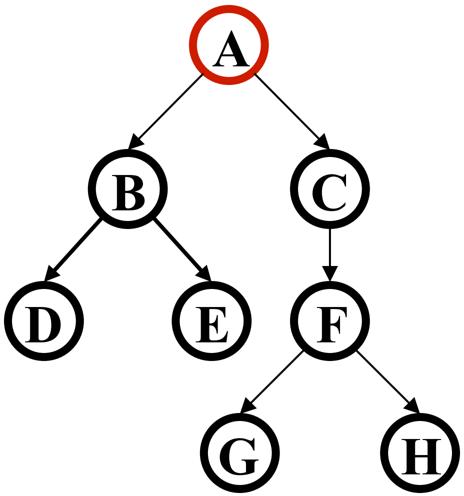



```
DFS2(Node start) {
  initialize stack s and push start
  mark start as visited
  while(s is not empty) {
    next = s.pop() // and “process”
    for each node u adjacent to next
     if(u is not marked)
       mark u and push onto s
  }
}
```

- A C F H G B E D
- A different but perfectly fine traversal

Example: Breadth First Search

- A tree is a graph and DFS and BFS are particularly easy to "see"




```
BFS(Node start) {
  initialize queue q and enqueue start
  mark start as visited
  while(q is not empty) {
    next = q.dequeue() // and "process"
    for each node u adjacent to next
     if(u is not marked)
       mark u and enqueue onto q
  }
}
```

- A B C D E F G H
- A "level-order" traversal

Comparison

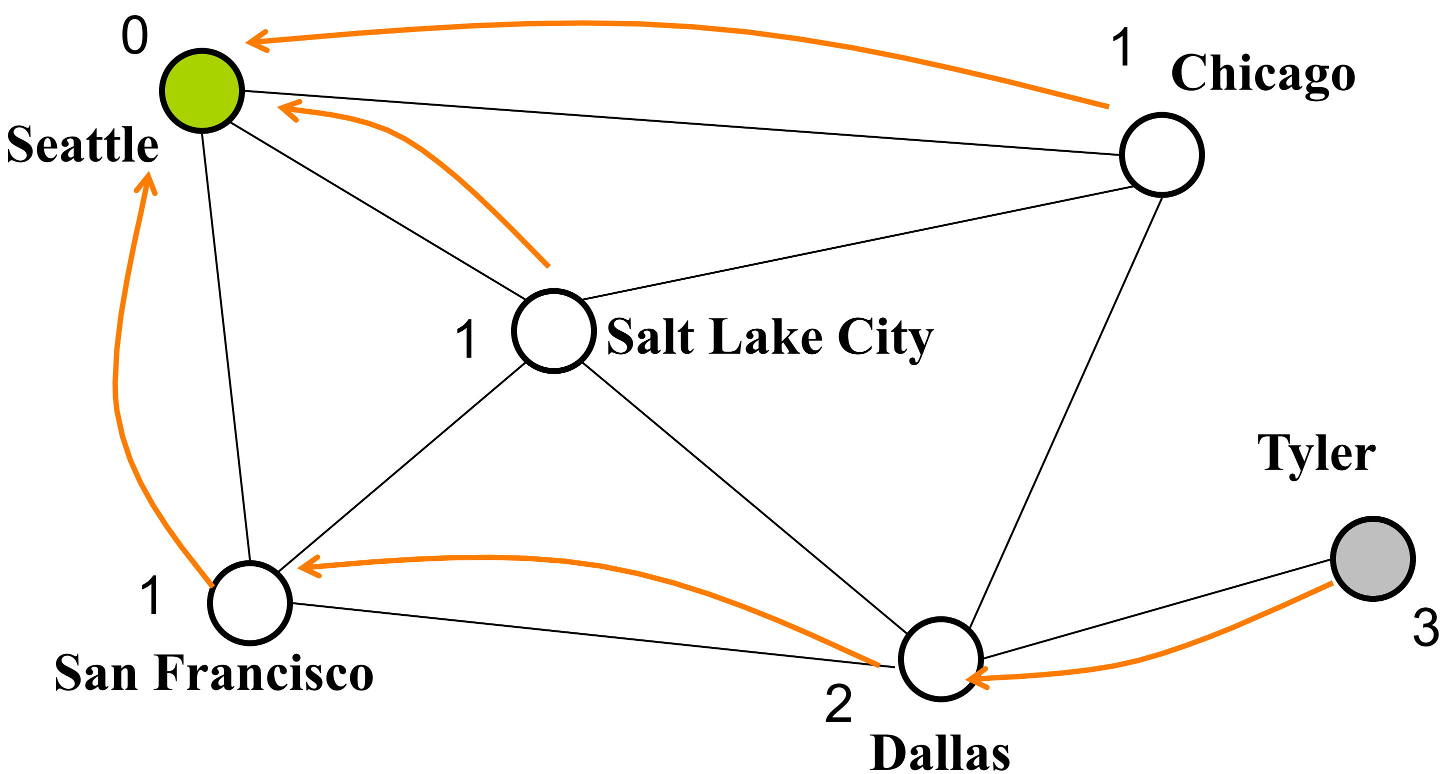
- Breadth-first always finds shortest paths, i.e., "optimal solutions"
  - Better for "what is the shortest path from **`x`** to **`y`**"
- But depth-first can use less space in finding a path
  - If *longest path* in the graph is **`p`** and highest out-degree is **`d`** then DFS stack never has more than **`d*p`** elements
  - But a queue for BFS may hold $O(|V|)$ nodes
- A third approach:
  - *Iterative deepening (IDFS)*:
    - Try DFS but disallow recursion more than **`K`** levels deep
    - If that fails, increment **`K`** and start the entire search over
  - Like BFS, finds shortest paths. Like DFS, less space.

Saving the Path

- Our graph traversals can answer the reachability question:
  - "Is there a path from node x to node y?"
- But what if we want to actually output the path?
  - Like getting driving directions rather than just knowing it's possible to get there!
- How to do it:
  - Instead of just "marking" a node, store the previous node along the path (when processing **`u`** causes us to add **`v`** to the search, set **`v.path`** field to be **`u`**)
  - When you reach the goal, follow **`path`** fields back to where you started (and then reverse the answer)
  - If just wanted path *length*, could put the integer distance at each node instead

Example using BFS

What is a path from Seattle to Tyler
- Remember marked nodes are not re-enqueued
- Note shortest paths may not be unique



Single source shortest paths

- Done: BFS to find the minimum path length from **v** to **u** in $O(|E|+|V|)$

- Actually, can find the minimum path length from **v** to *every node*
  - Still $O(|E|+|V|)$
  - No faster way for a "distinguished" destination in the worst-case

- Now: Weighted graphs

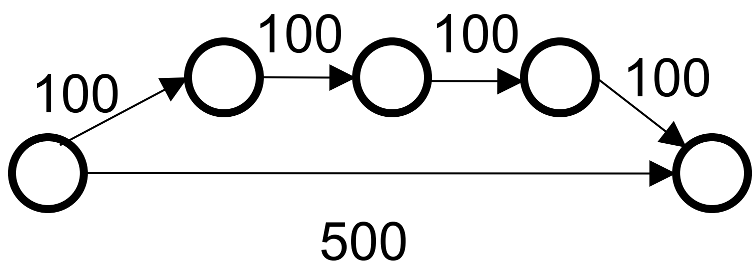
Given a weighted graph and node **v**,
find the minimum-cost path from **v** to every node

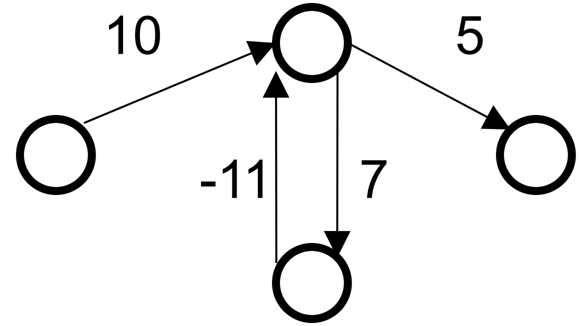
- As before, asymptotically no harder than for one destination

Applications

- Driving directions
- Cheap flight itineraries
- Network routing
- Critical paths in project management

Not as easy as BFS





Why BFS won't work: Shortest path may not have the fewest edges

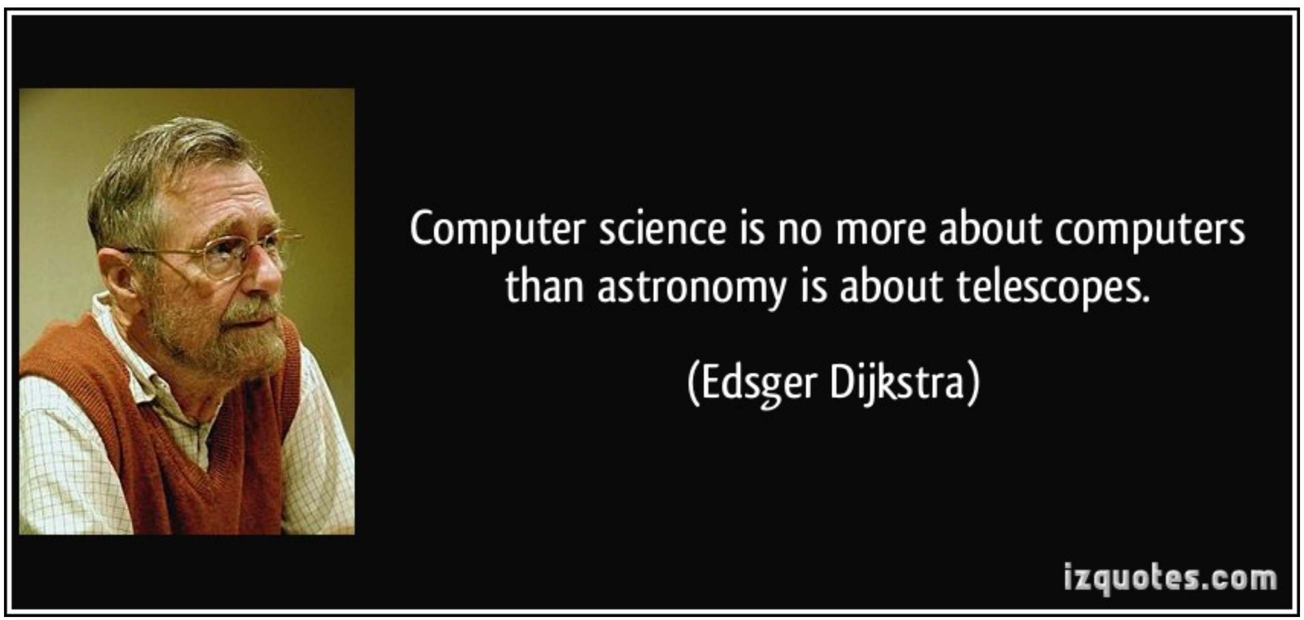
– Annoying when this happens with costs of flights

We will assume there are no negative weights

- *Problem* is *ill-defined* if there are negative-cost *cycles*
- *Today's algorithm* is *wrong* if *edges* can be negative
  – There are other, slower (but not terrible) algorithms

Dijkstra's Algorithm

- Named after its inventor Edsger Dijkstra (1930-2002)
  - Truly one of the "founders" of computer science; this is just one of his many contributions
  - Many people have a favorite Dijkstra story, even if they never met him



Dijkstra's Algorithm

- The idea: reminiscent of BFS, but adapted to handle weights
  - Grow the set of nodes whose shortest distance has been computed
  - Nodes not in the set will have a "best distance so far"
  - A priority queue will turn out to be useful for efficiency
- An example of a greedy algorithm
  - A series of steps
  - At each one the locally optimal choice is made