CSE373: Data Structures & Algorithms

Lecture 8: AVL Trees and Priority Queues

Nicki Dell

Spring 2014

Announcements

- Homework 1 feedback out soon (by Friday)
- Homework 2 due NOW (a few minutes ago!!!)
- Homework 3 out today (due April 30th) ☺

- TA Sessions
  - Tomorrow: BST and AVL Trees
  - Tuesday: Priority Queues and Binary Heaps

- Today
  - Finish AVL Trees
  - Start Priority Queues

The AVL Tree Data Structure

An AVL tree is a self-balancing binary search tree.

Structural properties

1. Binary tree property (same as BST)
2. Order property (same as for BST)
3. Balance property:
   balance of every node is between -1 and 1

Need to keep track of height of every node and maintain balance as we perform operations.

AVL Trees: Insert

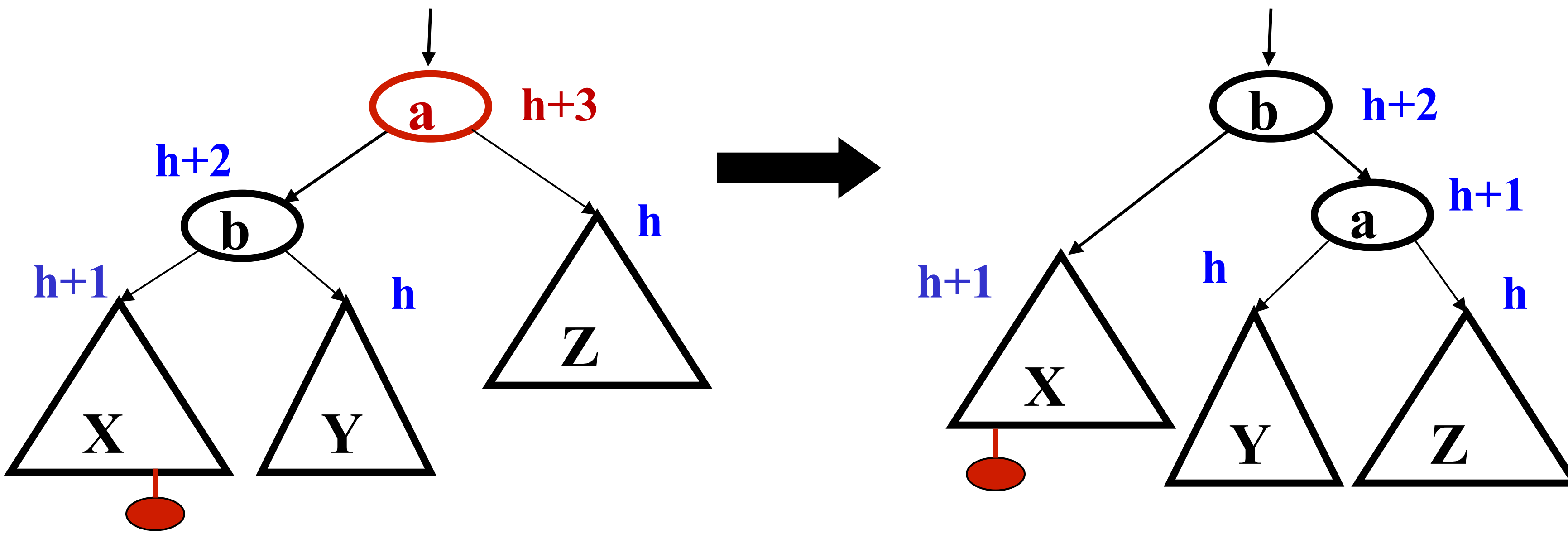
- Insert as in a BST (add a leaf in appropriate position)
- Check back up path for imbalance, which will be 1 of 4 cases:
  - Unbalanced node's left-left grandchild is too tall
  - Unbalanced node's left-right grandchild is too tall
  - Unbalanced node's right-left grandchild is too tall
  - Unbalanced node's right-right grandchild is too tall
- Only one case occurs because tree was balanced before insert
- After the appropriate single or double rotation, the smallest-unbalanced subtree has the same height as before the insertion
  - So all ancestors are now balanced

AVL Trees: Single rotation

- *Single rotation:*
  - The basic operation we'll use to rebalance an AVL Tree
  - Move child of unbalanced node into parent position
  - Parent becomes the "other" child (always okay in a BST!)
  - Other sub-trees move in only way BST allows

The general left-left case

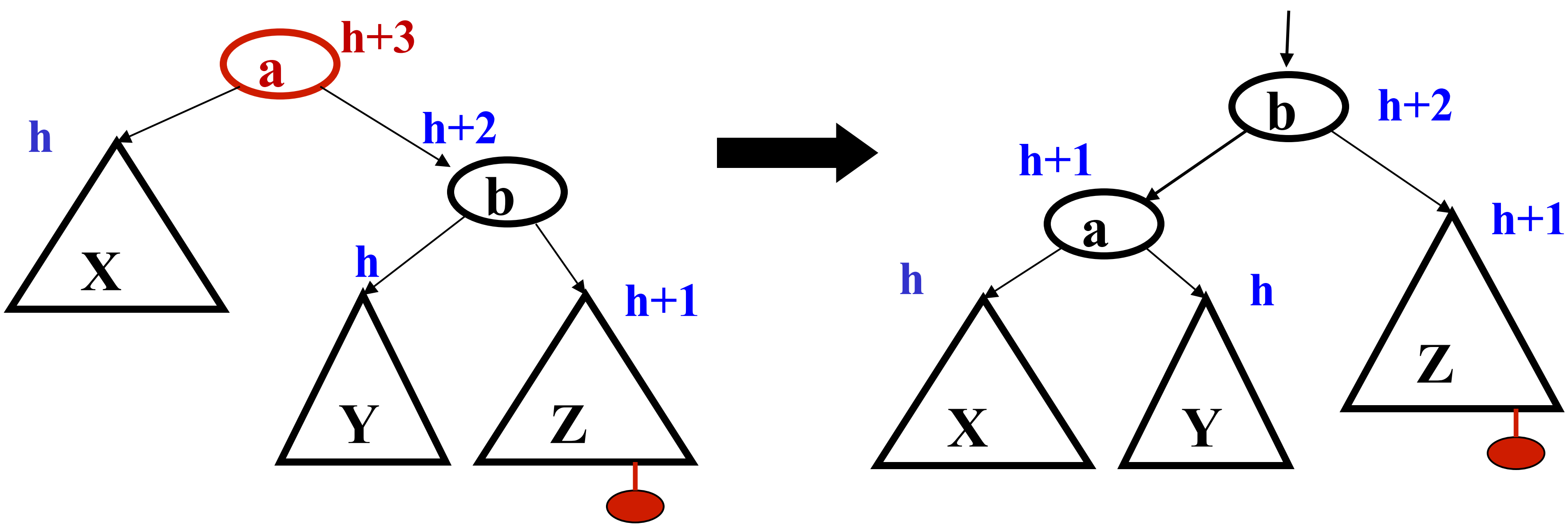
- Insertion into left-left grandchild causes an imbalance at node **a**
  - Move child of unbalanced node into parent position
  - Parent becomes the "other" child
  - Other sub-trees move in the only way BST allows:
    - using BST facts: X < b < Y < a < Z



- A single rotation restores balance at the node
  - To *same height* as before insertion, so ancestors now balanced

The general right-right case

- Mirror image to left-left case, so you rotate the other way
  - Exact same concept, but need different code

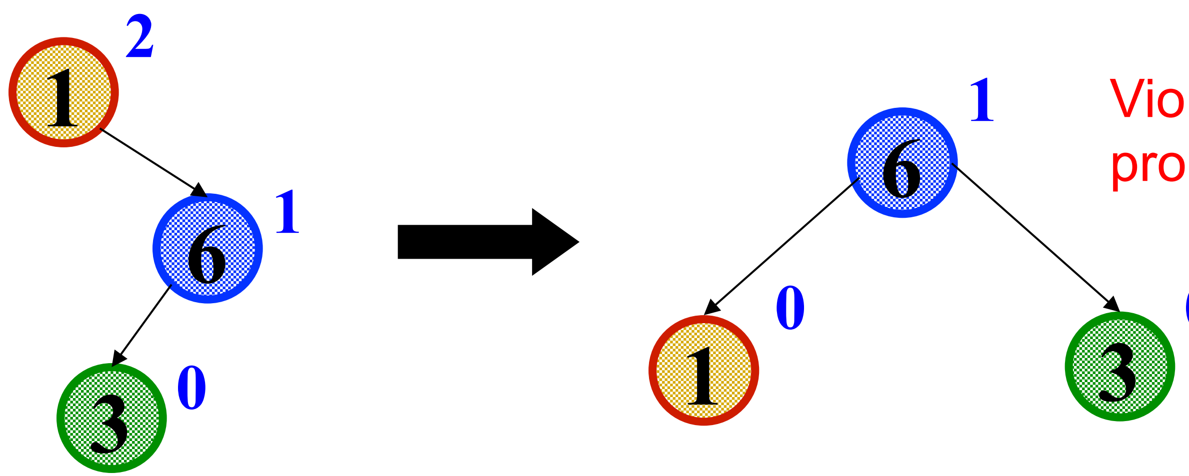


Two cases to go

Unfortunately, single rotations are not enough for insertions in the left-right subtree or the right-left subtree

Simple example: **`insert`**(1), **`insert`**(6), **`insert`**(3)

- First wrong idea: single rotation like we did for left-left



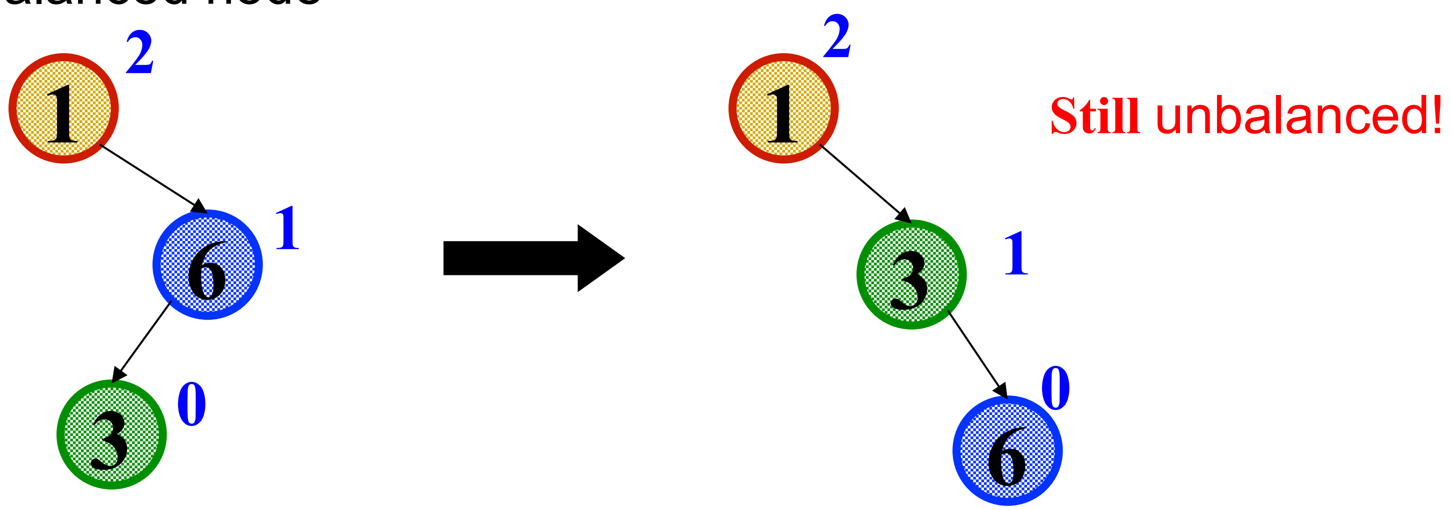
Violates order property!

Two cases to go

Unfortunately, single rotations are not enough for insertions in the left-right subtree or the right-left subtree

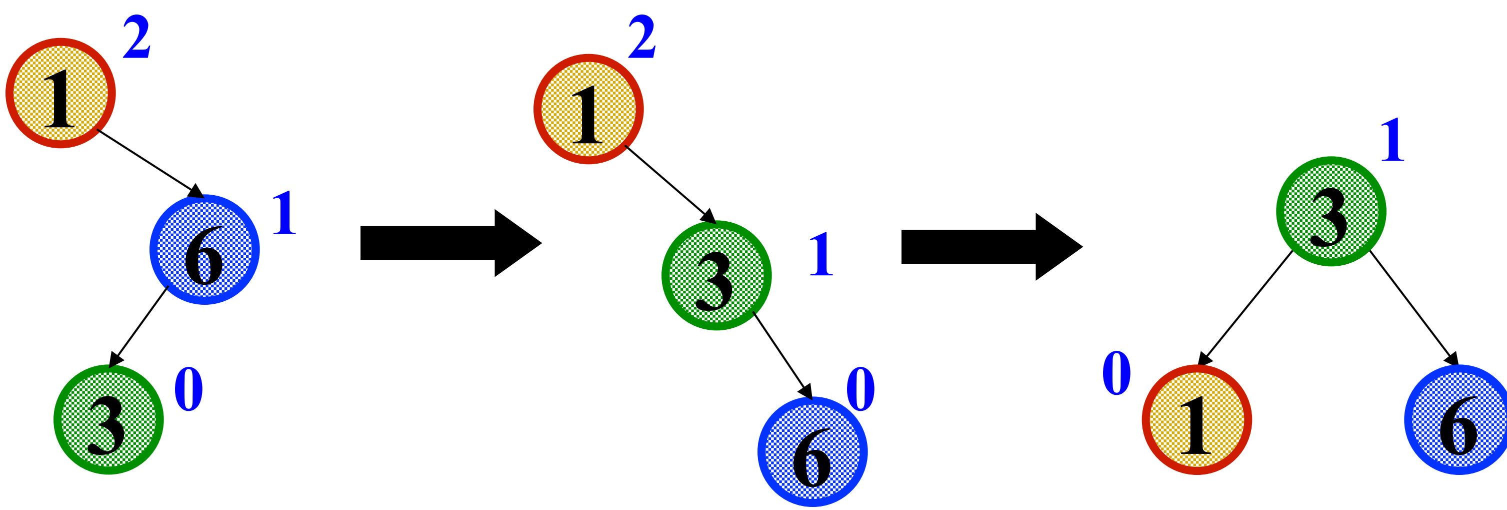
Simple example: **`insert`**(1), **`insert`**(6), **`insert`**(3)

- Second wrong idea: single rotation on the child of the unbalanced node

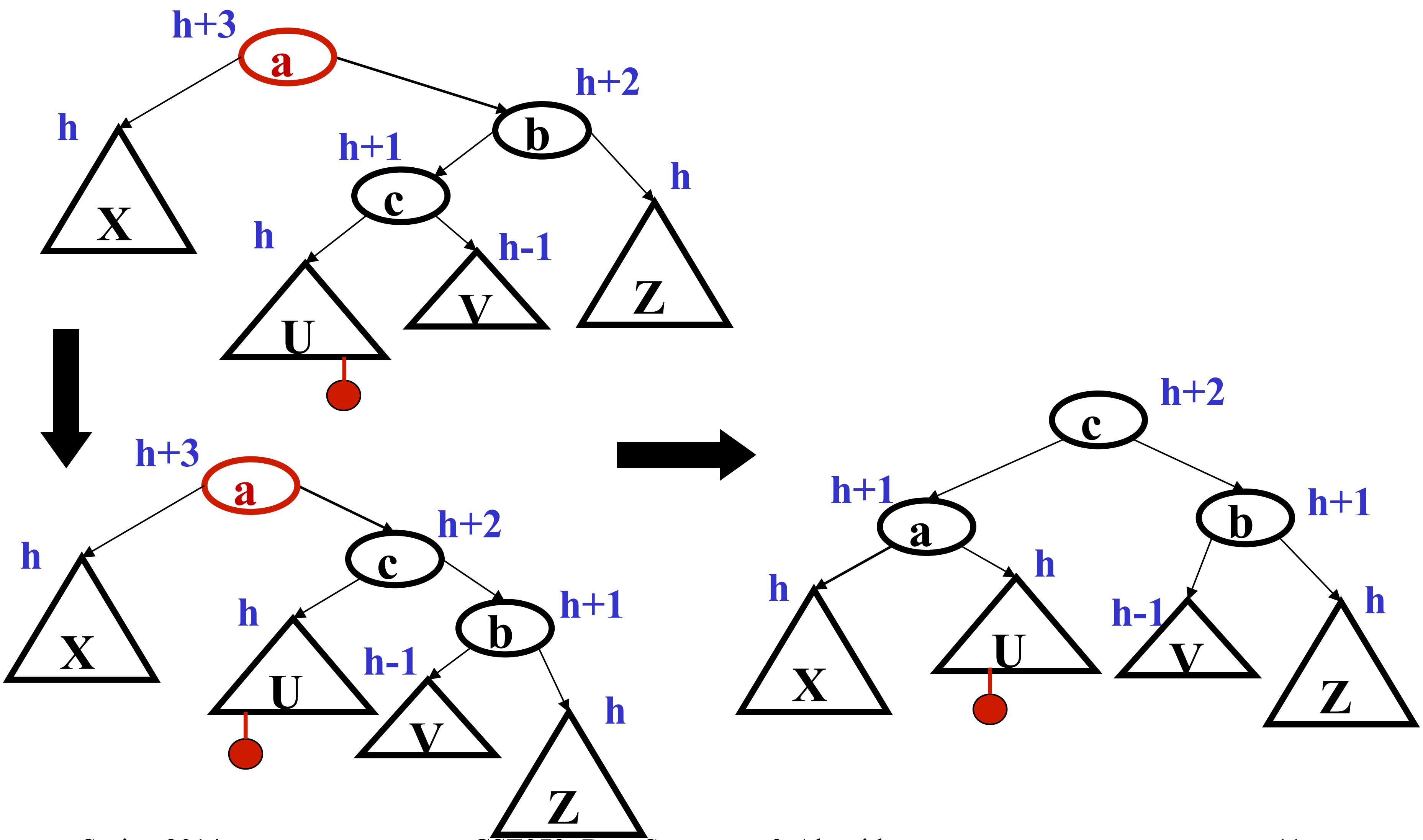


Sometimes two wrongs make a right ☺

- First idea violated the order property
- Second idea didn't fix balance
- But if we do both single rotations, starting with the second, it works! (And not just for this example.)
- Double rotation:
  1. Rotate problematic child and grandchild
  2. Then rotate between self and new child

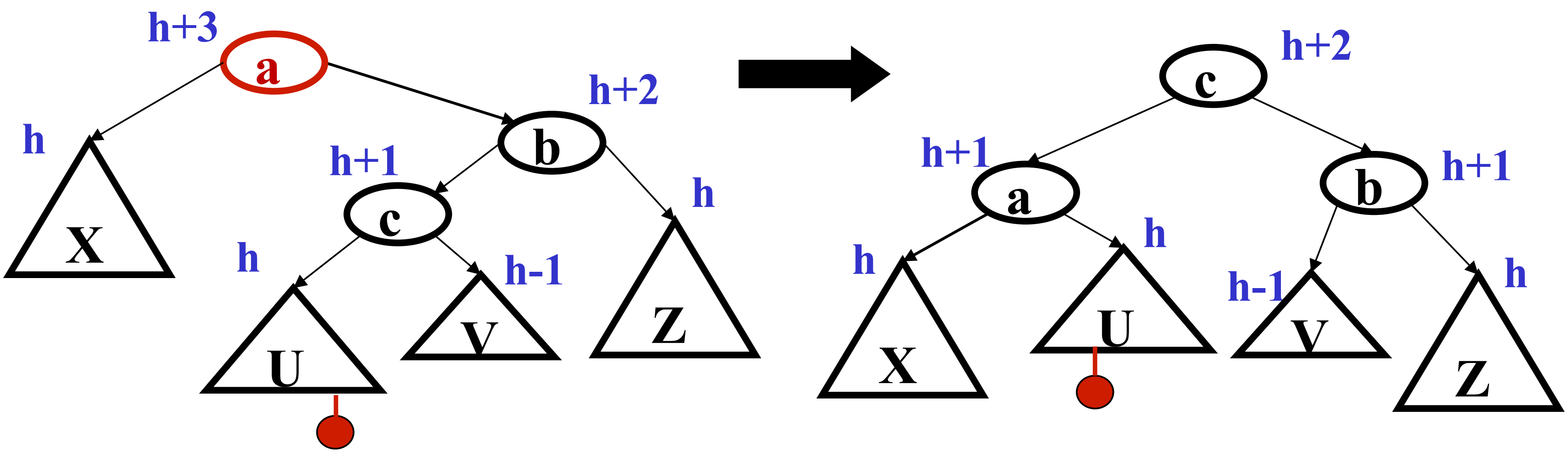


The general right-left case



Comments

- Like in the left-left and right-right cases, the height of the subtree after rebalancing is the same as before the insert
  - So no ancestor in the tree will need rebalancing
- Does not have to be implemented as two rotations; can just do:



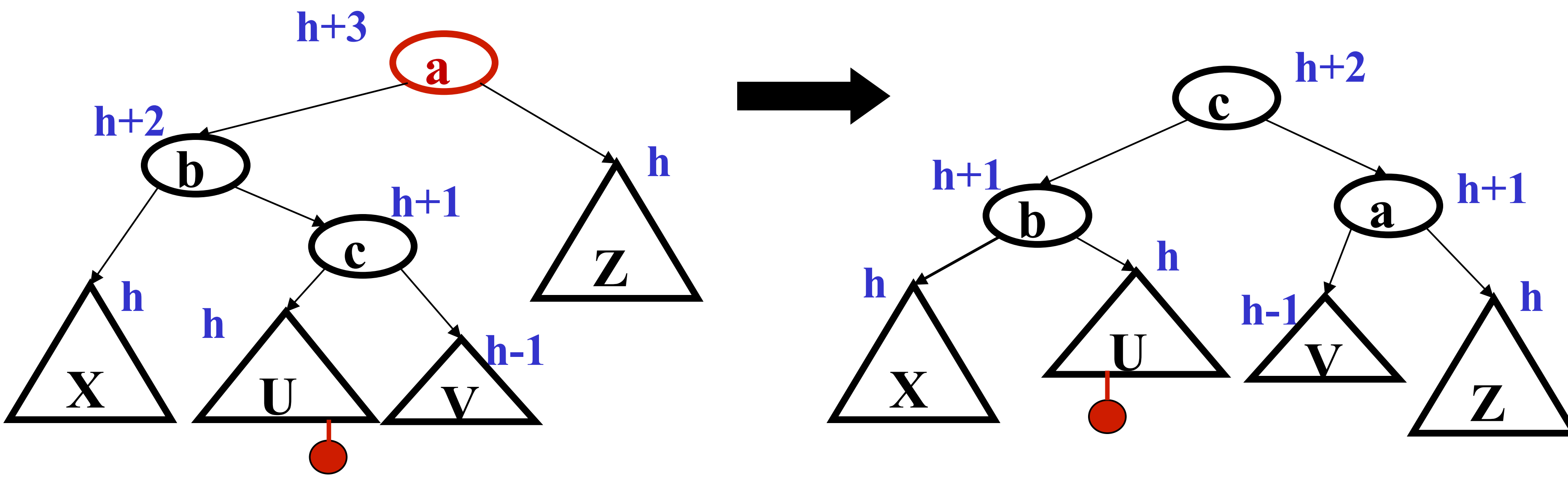
- Easier to remember than you may think:

  Move c to grandparent's position

  Put a, b, X, U, V, and Z in the only legal positions for a BST

The last case: left-right

- Mirror image of right-left
  - Again, no new concepts, only new code to write



AVL Trees: efficiency

- Worst-case complexity of **`find`**: $O(\log n)$
  - Tree is balanced

- Worst-case complexity of **`insert`**: $O(\log n)$
  - Tree starts balanced
  - A rotation is $O(1)$ and there's an $O(\log n)$ path to root
  - Tree ends balanced

- Worst-case complexity of **`buildTree`**: $O(n \log n)$

Takes some more rotation action to handle **`delete`**…

Pros and Cons of AVL Trees

Arguments for AVL trees:

1. All operations logarithmic worst-case because trees are *always* balanced
2. Height balancing adds no more than a constant factor to the speed of **`insert`** and **`delete`**

Arguments against AVL trees:

1. Difficult to program & debug [but done once in a library!]
2. More space for height field
3. Asymptotically faster but rebalancing takes a little time
4. If *amortized* (later, I promise) logarithmic time is enough, use splay trees (in the text)

Done with AVL Trees (….phew!)

next up…

Priority Queues ADT
(Homework 3 ☺)

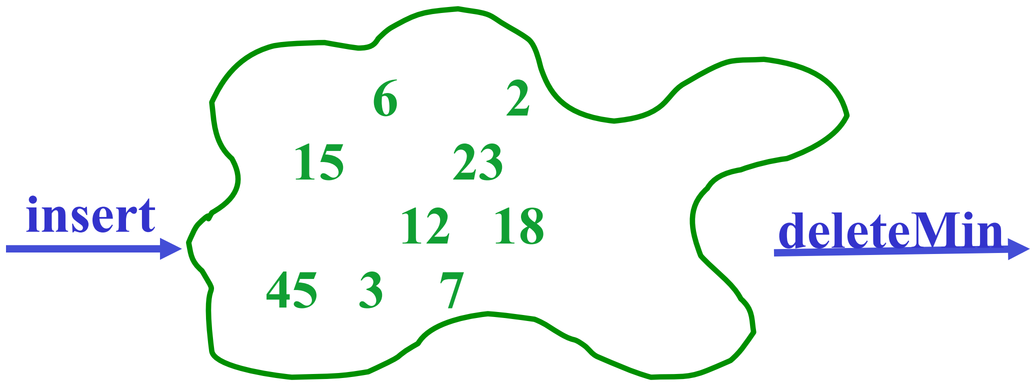
A new ADT: Priority Queue

- A **priority queue** holds *compare-able data*
  - Like dictionaries, we need to *compare items*
    - Given *x* and *y*, is *x* less than, equal to, or greater than *y*
    - Meaning of the ordering can depend on your data
  - Integers are comparable, so will use them in examples
    - But the priority queue ADT is much more general
    - Typically two fields, the *priority* and the *data*

Priorities

- Each item has a "priority"
  - In our examples, the *lesser* item is the one with the *greater* priority
  - So "priority 1" is more important than "priority 4"
  - (Just a convention, think "first is best")

- Operations:
  - **`insert`**
  - **`deleteMin`**
  - **`is_empty`**



- Key property: **`deleteMin`** *returns* and *deletes* the item with greatest priority (lowest priority value)
  - Can resolve ties arbitrarily

Example

`insert` *x1* with priority *5*

`insert` *x2* with priority *3*

`insert` *x3* with priority *4*

a = **`deleteMin // x2`**

b = **`deleteMin // x3`**

`insert` *x4* with priority *2*

`insert` *x5* with priority *6*

c = **`deleteMin // x4`**

d = **`deleteMin // x1`**

- Analogy: **`insert`** is like **`enqueue`**, **`deleteMin`** is like **`dequeue`**
  - But the whole point is to use priorities instead of FIFO

Applications

Like all good ADTs, the priority queue arises often

- Sometimes blatant, sometimes less obvious

- Run multiple programs in the operating system
  - "critical" before "interactive" before "compute-intensive"
  - Maybe let users set priority level
- Treat hospital patients in order of severity (or triage)
- Select print jobs in order of decreasing length?
- Forward network packets in order of urgency
- Select most frequent symbols for data compression
- Sort (first **`insert`** all, then repeatedly **`deleteMin`**)
  - Much like Homework 1 uses a stack to implement reverse

Finding a good data structure

- Will show an efficient, non-obvious data structure for this ADT
  - But first let's analyze some "obvious" ideas for *n* data items
  - All times worst-case; assume arrays "have room"

data	*insert algorithm / time*		*deleteMin algorithm / time*	
unsorted array	add at end	$O(1)$	search	$O(n)$
unsorted linked list	add at front	$O(1)$	search	$O(n)$
sorted circular array	search / shift	$O(n)$	move front	$O(1)$
sorted linked list	put in right place	$O(n)$	remove at front	$O(1)$
binary search tree	put in right place	$O(n)$	leftmost	$O(n)$
AVL tree	put in right place	$O(\texttt{log}\ n)$	leftmost	$O(\texttt{log}\ n)$

More on possibilities

- One more idea: if priorities are 0, 1, …, *k* can use an array of k lists
  - **`insert`**: add to front of list at **`arr[priority]`**, *O*(1)
  - **`deleteMin`**: remove from lowest non-empty list *O(k)*

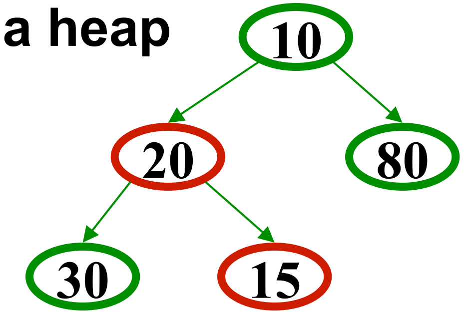
- We are about to see a data structure called a "binary heap"
  - Another binary tree structure with specific properties
  - *O*(**`log`** *n*) **`insert`** and *O*(**`log`** *n*) **`deleteMin`** *worst-case*
    - Possible because we don't support unneeded operations; no need to maintain a full sort
  - *Very* good constant factors
  - *If* items arrive in random order, then **`insert`** is *O*(1) on *average*
    - Because 75% of nodes in bottom two rows

Our data structure

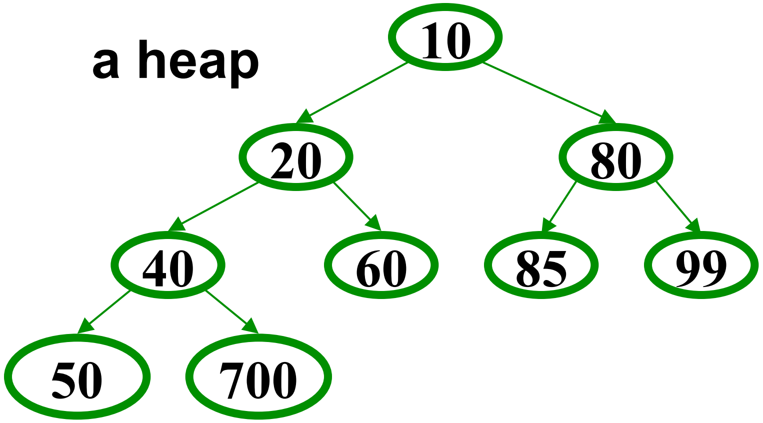
A *binary min-heap* (or just *binary heap* or just *heap*) has:

- Structure property: A *complete* binary tree
- Heap property: The priority of every (non-root) node is less important than the priority of its parent
  - ***Not** a binary search tree*

not a heap



a heap

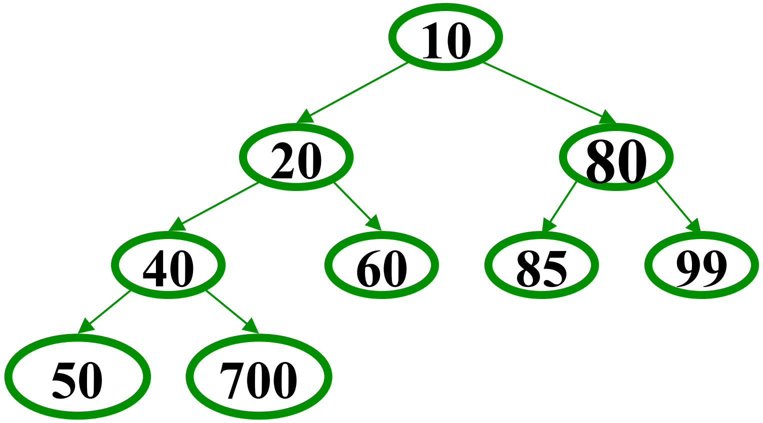


So:

- Where is the highest-priority item?
- What is the height of a heap with *n* items?

Operations: basic idea

- **`findMin`**: return **`root.data`**
- **`deleteMin`**:
  1. **`answer = root.data`**
  2. Move right-most node in last row to root to restore structure property
  3. "Percolate down" to restore heap property
- **`insert:`**
  1. Put new node in next position on bottom row to restore structure property
  2. "Percolate up" to restore heap property

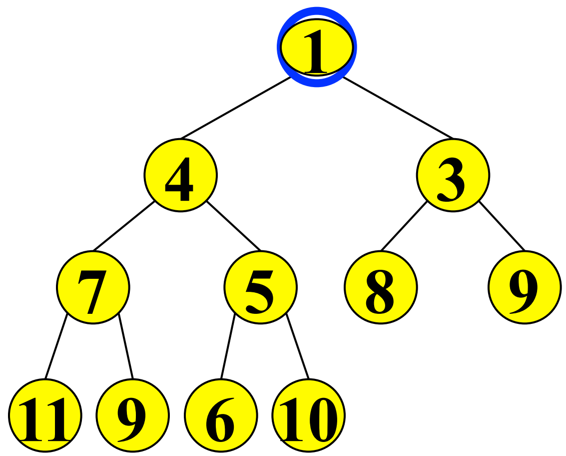


Overall strategy:

- *Preserve structure property*
- *Break and restore heap property*

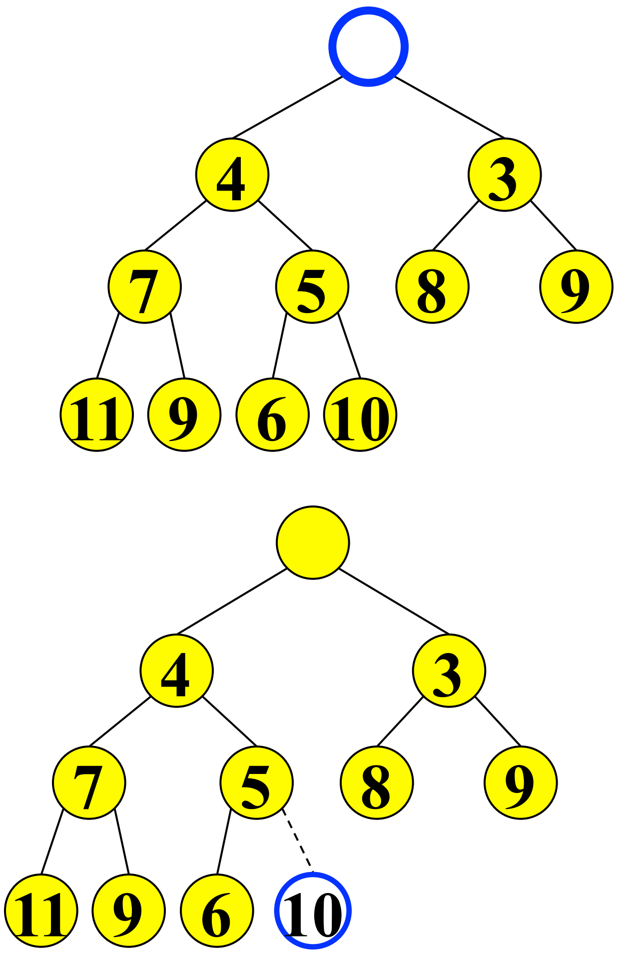
DeleteMin

Delete (and later return) value at root node



DeleteMin: Keep the Structure Property

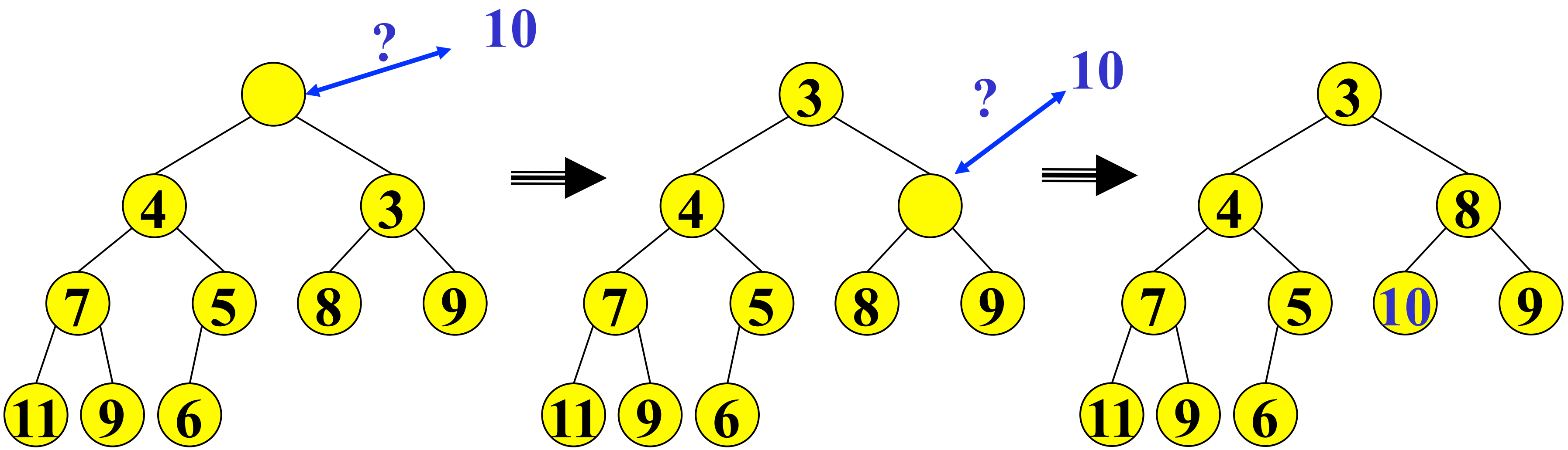
- We now have a "hole" at the root
  - Need to fill the hole with another value
- Keep structure property: When we are done, the tree will have one less node and must still be complete
- Pick the last node on the bottom row of the tree and move it to the "hole"



DeleteMin: Restore the Heap Property

Percolate down:

- Keep comparing priority of item with both children
- If priority is less important, swap with the most important child and go down one level
- Done if both children are less important than the item or we've reached a leaf node



Why is this correct?
What is the run time?

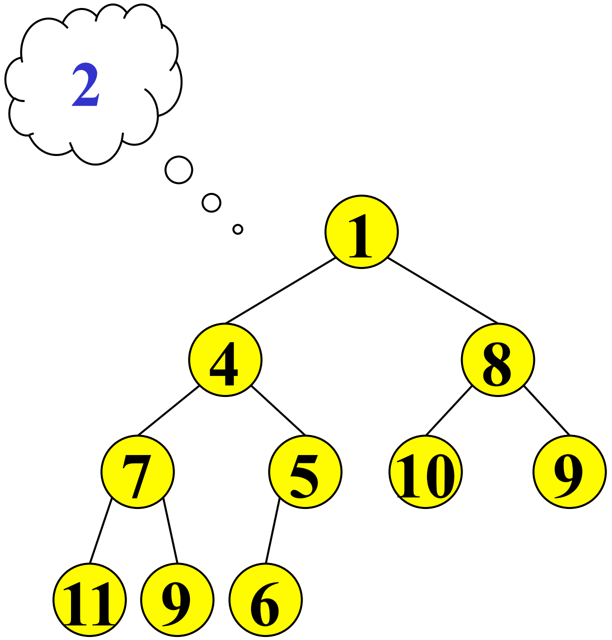
DeleteMin: Run Time Analysis

- Run time is O(height of heap)
- A heap is a complete binary tree
- Height of a complete binary tree of n nodes?
  - height = $\lfloor \texttt{log}_2(n) \rfloor$
- Run time of **`deleteMin`** is $O(\texttt{log}\ n)$

Insert

- Add a value to the tree

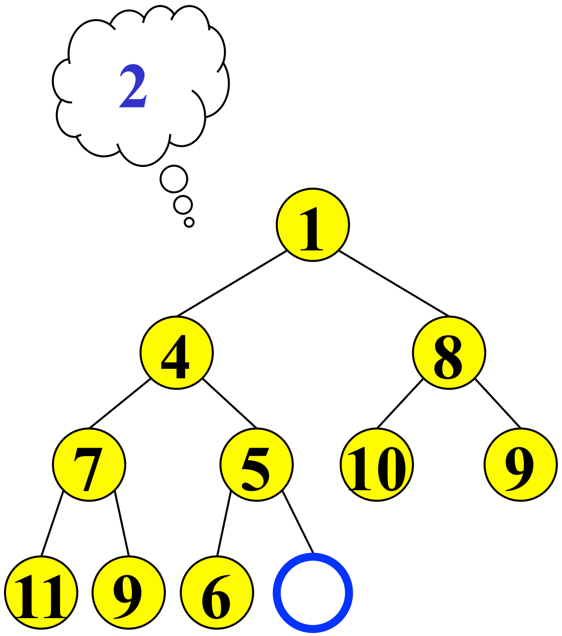
- Afterwards, structure and heap properties must still be correct



Insert: Maintain the Structure Property

- There is only one valid tree shape after we add one more node

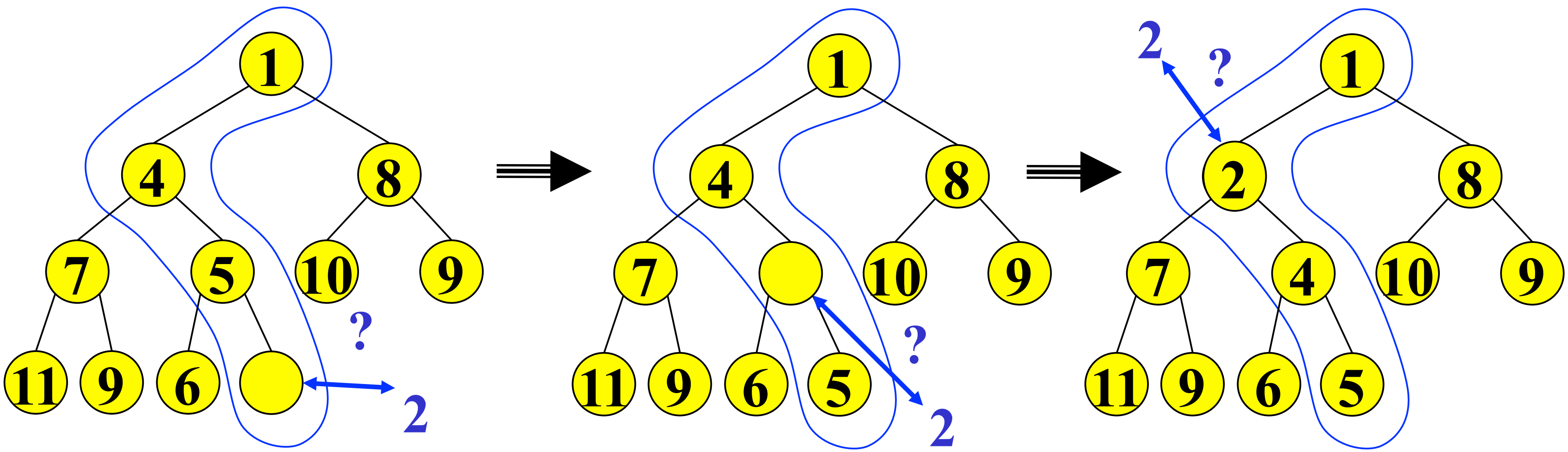
- So put our new data there and then focus on restoring the heap property



Insert: Restore the heap property

Percolate up:

- Put new data in new location
- If parent is less important, swap with parent, and continue
- Done if parent is more important than item or reached root

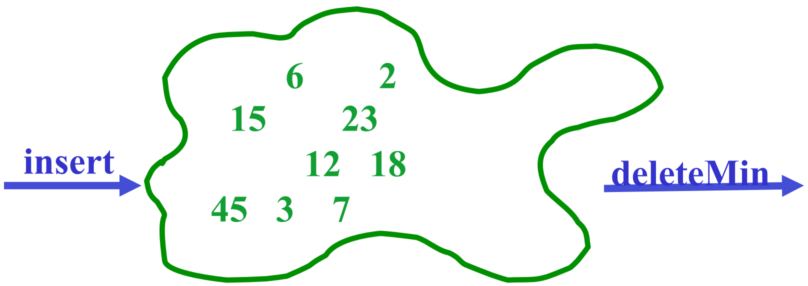


What is the running time?

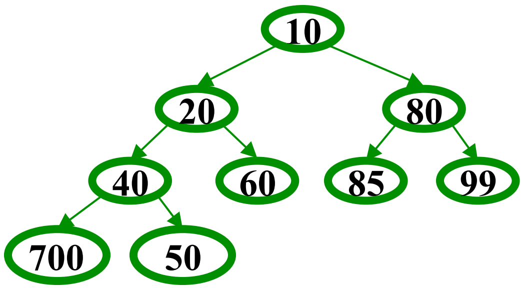
Like `deleteMin`, worst-case time proportional to tree height: $O(\log n)$

Summary

- Priority Queue ADT:
  - **`insert`** comparable object,
  - **`deleteMin`**



- Binary heap data structure:
  - Complete binary tree
  - Each node has less important priority value than its parent



- **`insert`** and **`deleteMin`** operations = *O*(height-of-tree)=*O*(**`log`** *n*)
  - **`insert`**: put at new last position in tree and percolate-up
  - **`deleteMin`**: remove root, put last element at root and percolate-down