CSE373: Data Structures & Algorithms

Lecture 16: Shortest Paths

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Winter 2014

Announcements

- No class on Monday (Presidents' Day)
- HW3 feedback before next lecture
- Midterm 2 will cover material up through next Wednesday
- Midterm info slightly out of date on web page, up-to-date soon

Single source shortest paths

- Done: BFS to find the minimum path length from **v** to **u** in $O(|E|+|V|)$
- Actually, can find the minimum path length from **v** to *every node*
  - Still $O(|E|+|V|)$
  - No faster way for a "distinguished" destination in the worst-case
- Now: Weighted graphs

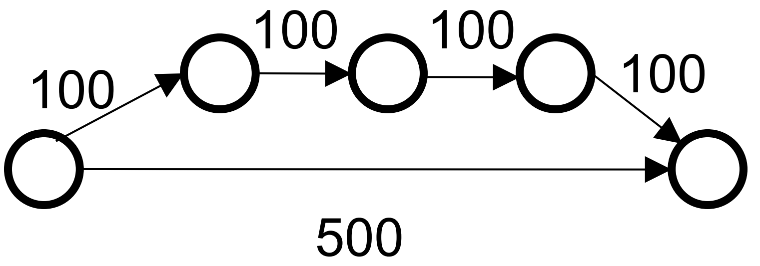
Given a weighted graph and node **v**,
find the minimum-cost path from **v** to every node

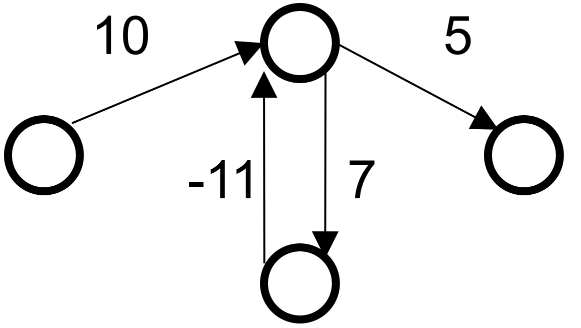
- As before, asymptotically no harder than for one destination
- Unlike before, BFS will not work

Applications

- Driving directions
- Cheap flight itineraries
- Network routing
- Critical paths in project management

Not as easy





Why BFS won't work: Shortest path may not have the fewest edges
- Annoying when this happens with costs of flights

We will assume there are no negative weights
- *Problem* is *ill-defined* if there are negative-cost *cycles*
- *Today's algorithm* is *wrong* if *edges* can be negative
  - There are other, slower (but not terrible) algorithms

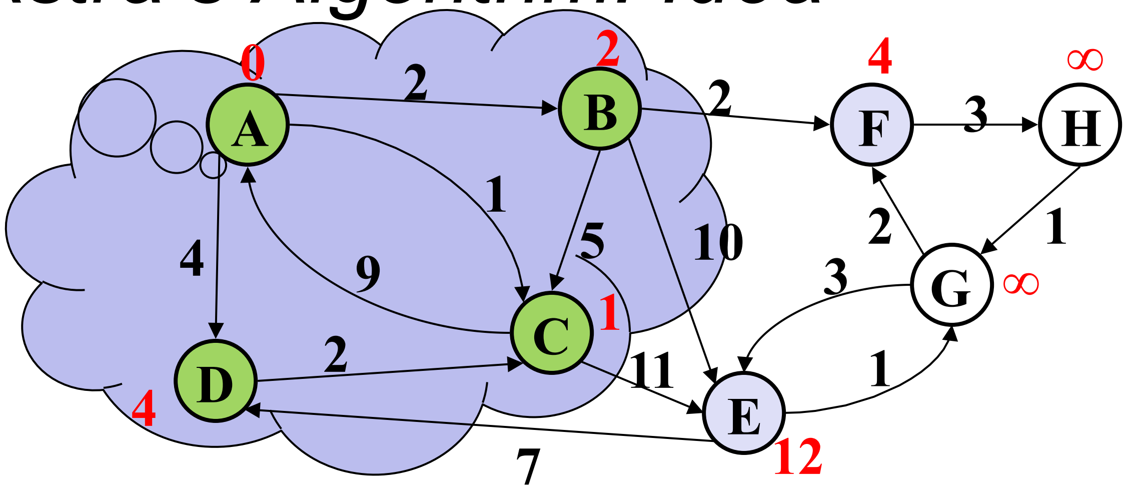
Dijkstra

- Algorithm named after its inventor Edsger Dijkstra (1930-2002)
  - Truly one of the "founders" of computer science; this is just one of his many contributions
  - Many people have a favorite Dijkstra story, even if they never met him
  - My favorite quotation: "computer science is no more about computers than astronomy is about telescopes"

Dijkstra's algorithm

- The idea: reminiscent of BFS, but adapted to handle weights
  - Grow the set of nodes whose shortest distance has been computed
  - Nodes not in the set will have a "best distance so far"
  - A priority queue will turn out to be useful for efficiency

Dijkstra's Algorithm: Idea



- Initially, start node has cost 0 and all other nodes have cost ∞
- At each step:
  - Pick closest unknown vertex **v**
  - Add it to the "cloud" of known vertices
  - Update distances for nodes with edges from **v**
- That's it! (But we need to prove it produces correct answers)

The Algorithm

1. For each node **`v`**, set **`v.cost =`** ∞ and **`v.known = false`**
2. Set **`source.cost = 0`**
3. While there are unknown nodes in the graph
   a) Select the unknown node **`v`** with lowest cost
   b) Mark **`v`** as known
   c) For each edge **`(v,u)`** with weight **`w`**,

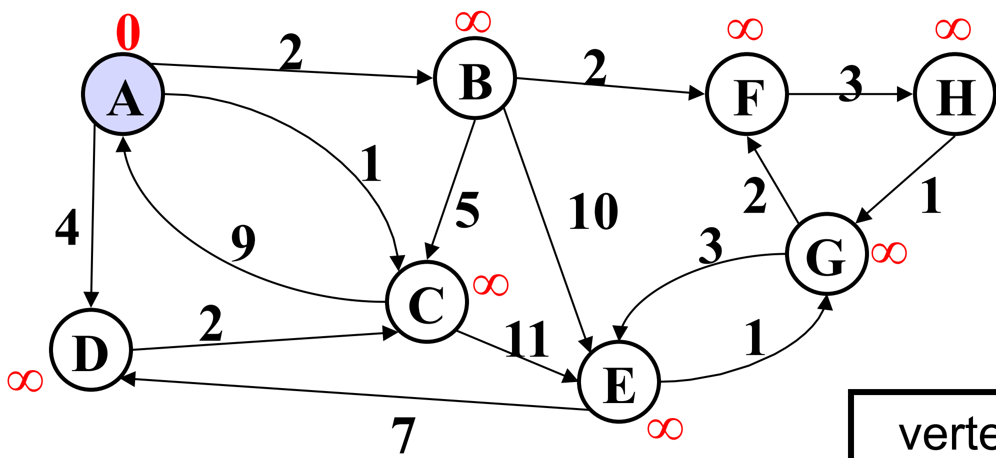
```
c1 = v.cost + w // cost of best path through v to u
c2 = u.cost   // cost of best path to u previously known
if(c1 < c2){ // if the path through v is better
  u.cost = c1
  u.path = v // for computing actual paths
}
```

Important features

- When a vertex is marked known, the cost of the shortest path to that node is known
  - The path is also known by following back-pointers

- While a vertex is still not known, another shorter path to it *might* still be found

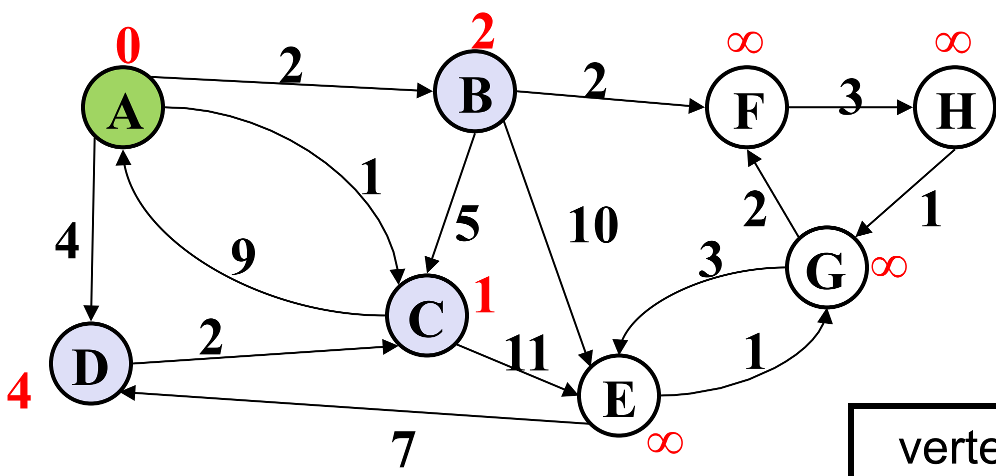
Example #1



vertex	known?	cost	path
A		0	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	
H		??	

<u>Order Added to Known Set:</u>

Example #1

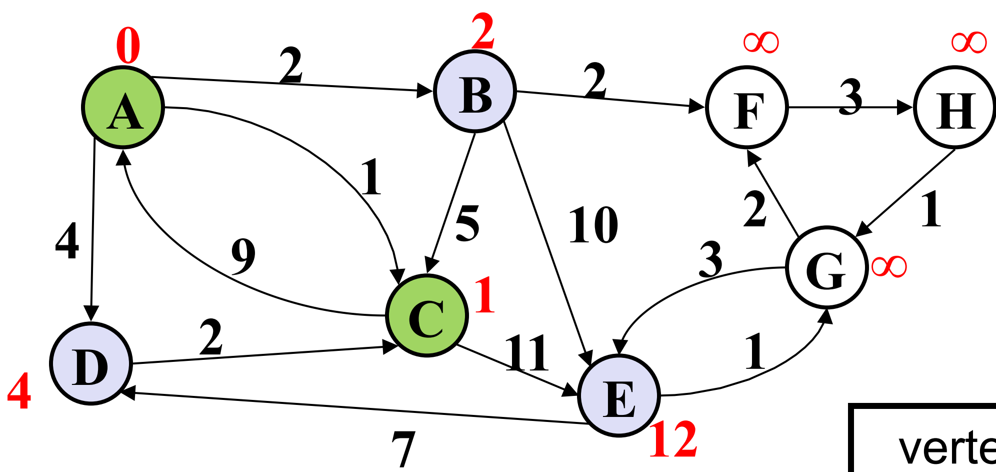


vertex	known?	cost	path
A	Y	0	
B		≤ 2	A
C		≤ 1	A
D		≤ 4	A
E		??	
F		??	
G		??	
H		??	

<u>Order Added to Known Set:</u>

A

Example #1

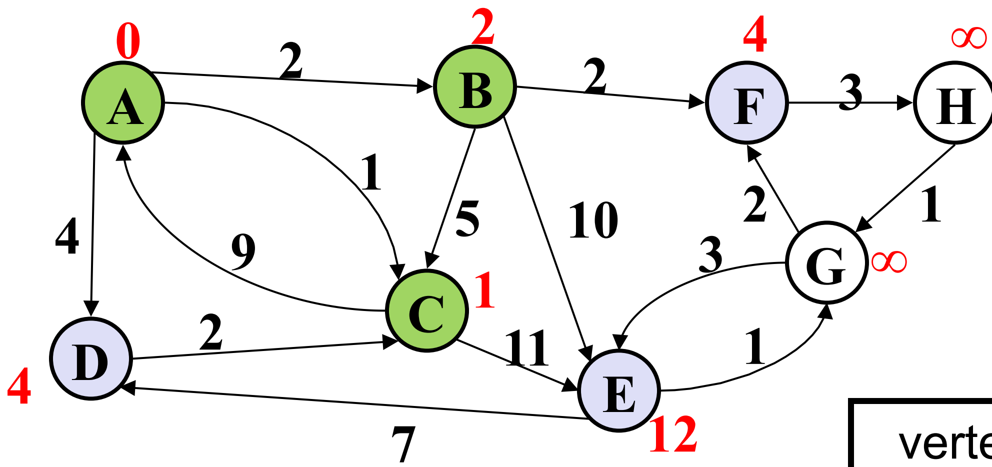


vertex	known?	cost	path
A	Y	0	
B		≤ 2	A
C	Y	1	A
D		≤ 4	A
E		≤ 12	C
F		??	
G		??	
H		??	

<u>Order Added to Known Set:</u>

A, C

Example #1

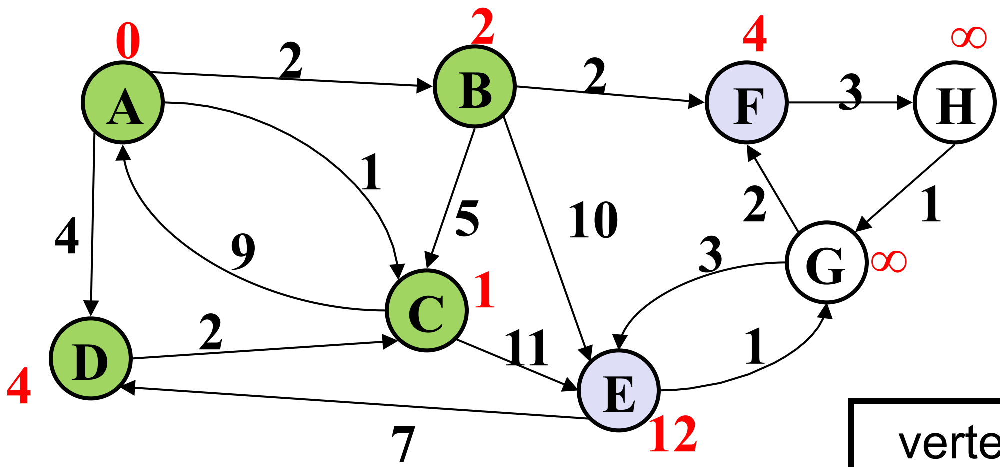


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D		≤ 4	A
E		≤ 12	C
F		≤ 4	B
G		??	
H		??	

<u>Order Added to Known Set:</u>

A, C, B

Example #1

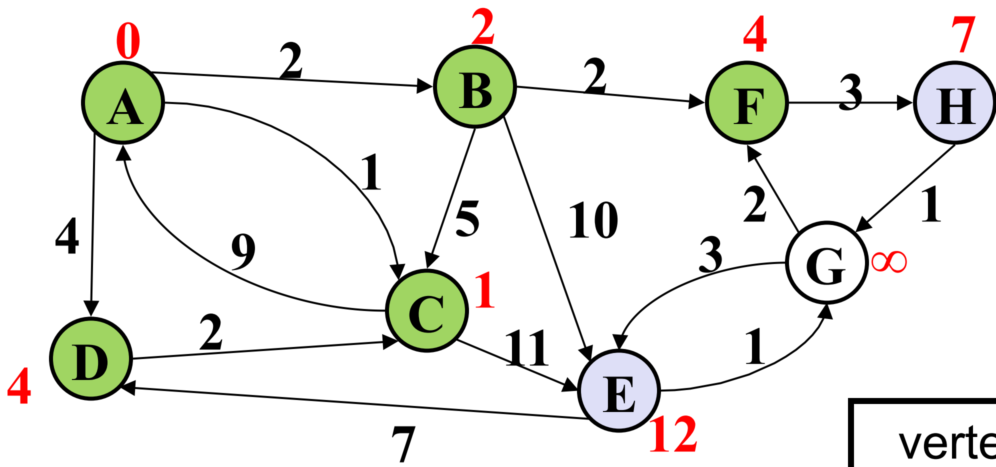


Order Added to Known Set:

A, C, B, D

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F		≤ 4	B
G		??	
H		??	

Example #1

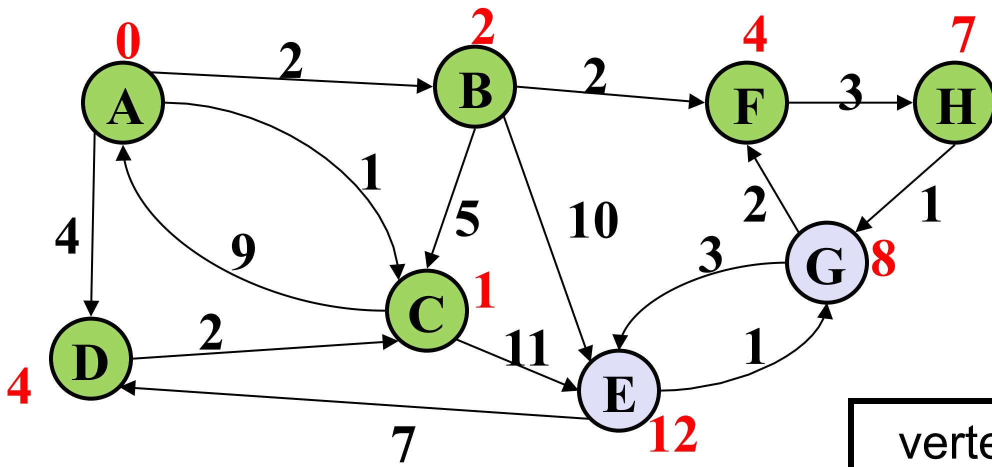


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F	Y	4	B
G		??	
H		≤ 7	F

<u>Order Added to Known Set:</u>

A, C, B, D, F

Example #1

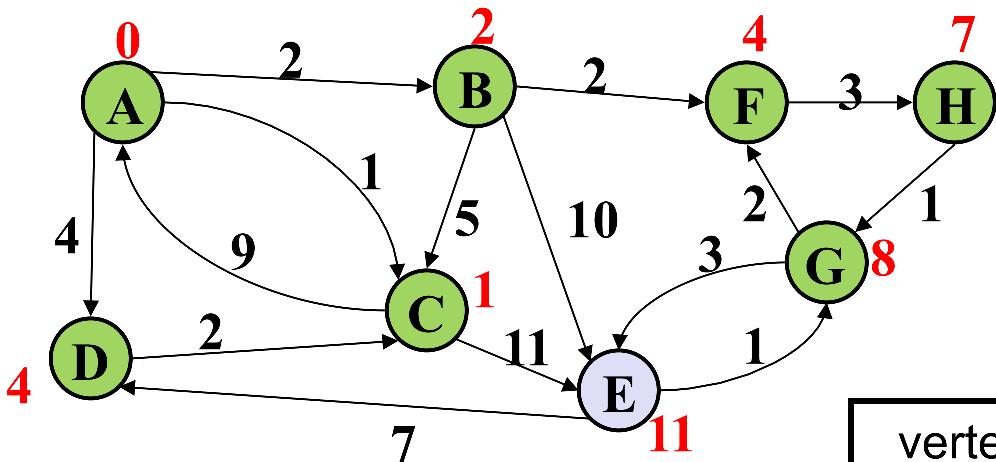


Order Added to Known Set:

A, C, B, D, F, H

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F	Y	4	B
G		≤ 8	H
H	Y	7	F

Example #1

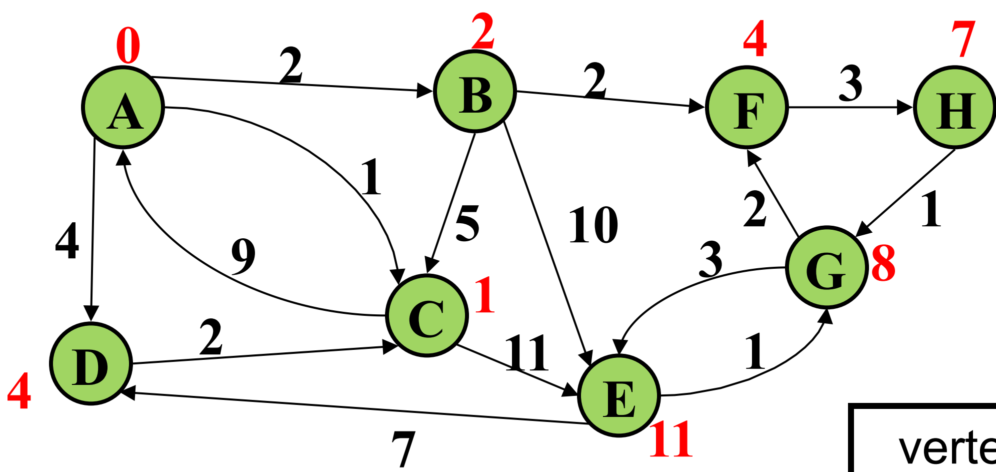


vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

<u>Order Added to Known Set:</u>

A, C, B, D, F, H, G

Example #1



vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

<u>Order Added to Known Set:</u>

A, C, B, D, F, H, G, E

Features

- When a vertex is marked known,
  the cost of the shortest path to that node is known
  - The path is also known by following back-pointers

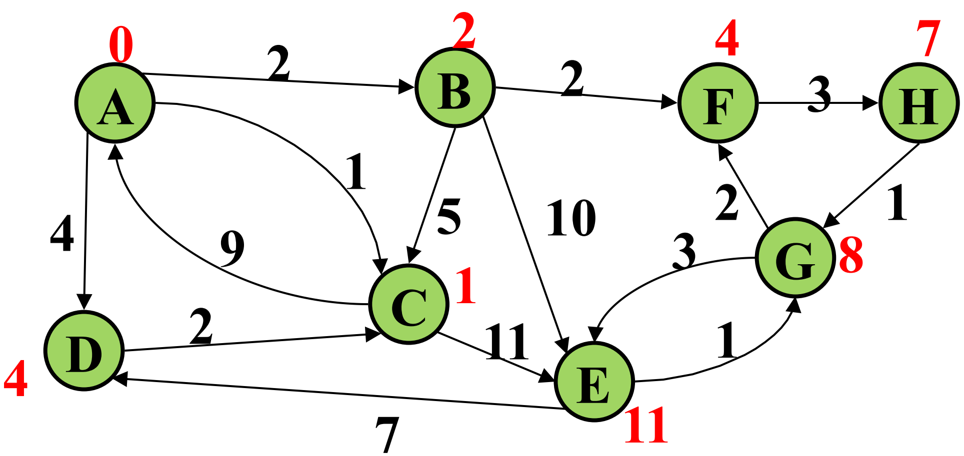
- While a vertex is still not known,
  another shorter path to it might still be found

Note: The “Order Added to Known Set” is not important

- A detail about how the algorithm works (client doesn’t care)
- Not used by the algorithm (implementation doesn’t care)
- It is sorted by path-cost, resolving ties in some way
  - Helps give intuition of why the algorithm works

Interpreting the Results

- Now that we're done, how do we get the path from, say, A to E?



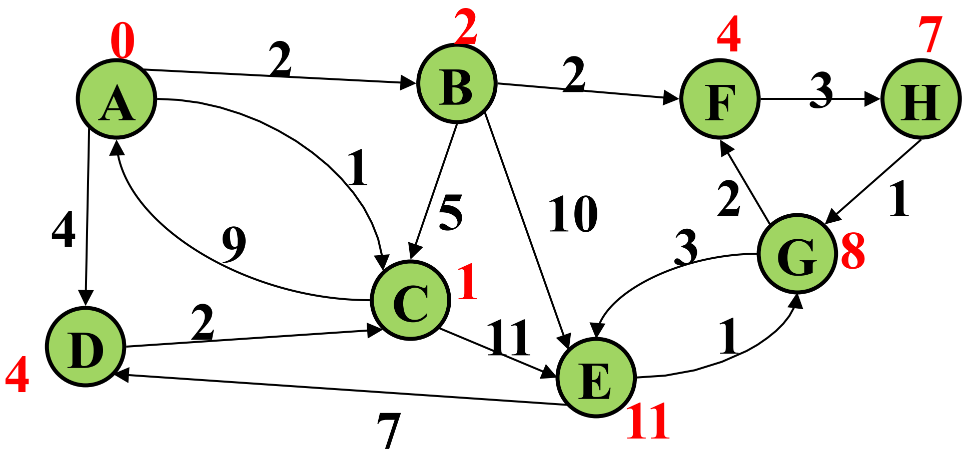
<u>Order Added to Known Set:</u>

A, C, B, D, F, H, G, E

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Stopping Short

- How would this have worked differently if we were only interested in:
  - The path from A to G?
  - The path from A to E?

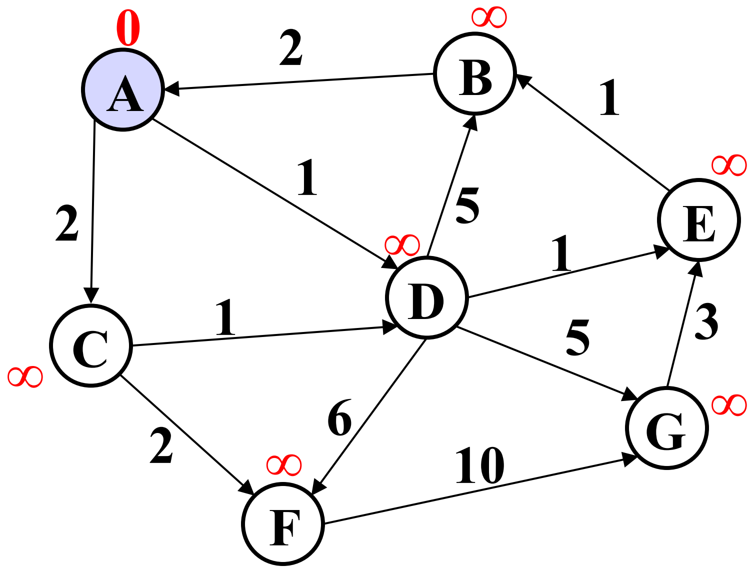


<u>Order Added to Known Set:</u>

A, C, B, D, F, H, G, E

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

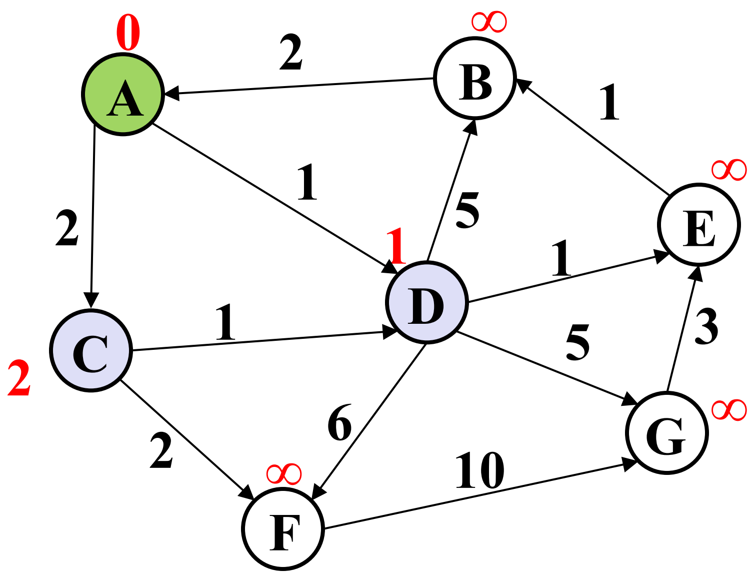
Example #2



vertex	known?	cost	path
A		0	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	

<u>Order Added to Known Set:</u>

Example #2

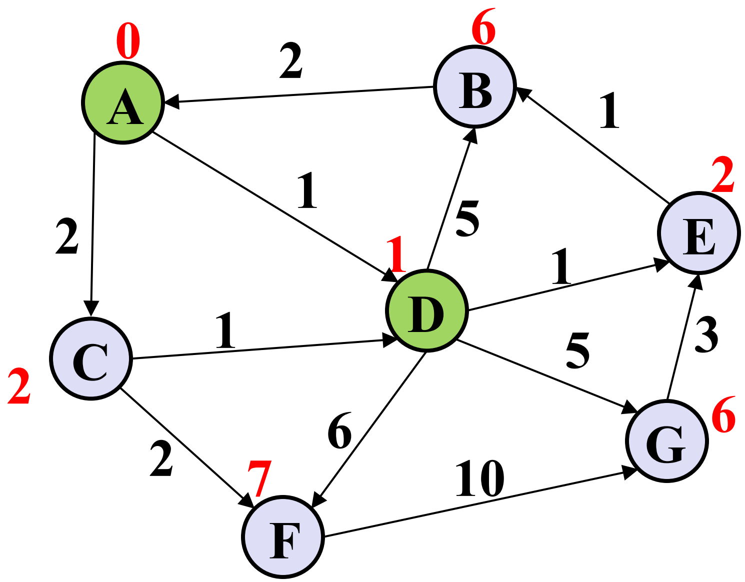


vertex	known?	cost	path
A	Y	0	
B		??	
C		≤ 2	A
D		≤ 1	A
E		??	
F		??	
G		??	

<u>Order Added to Known Set:</u>

A

Example #2

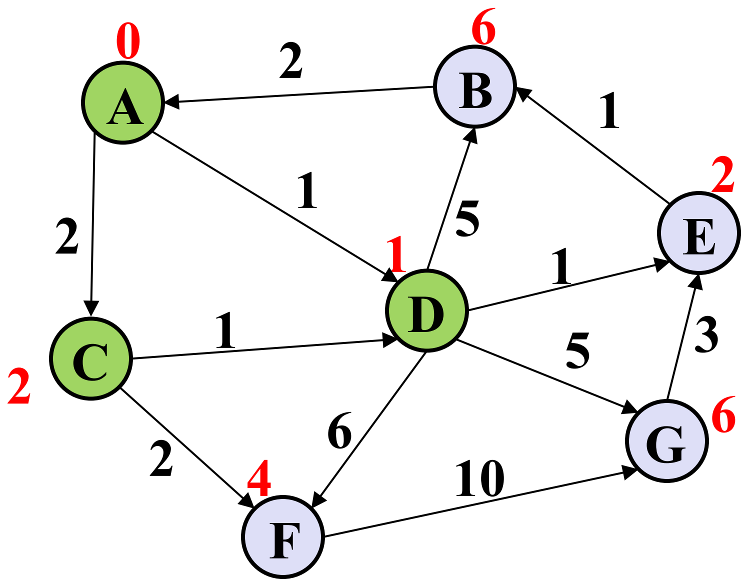


Order Added to Known Set:

A, D

vertex	known?	cost	path
A	Y	0	
B		≤ 6	D
C		≤ 2	A
D	Y	1	A
E		≤ 2	D
F		≤ 7	D
G		≤ 6	D

Example #2

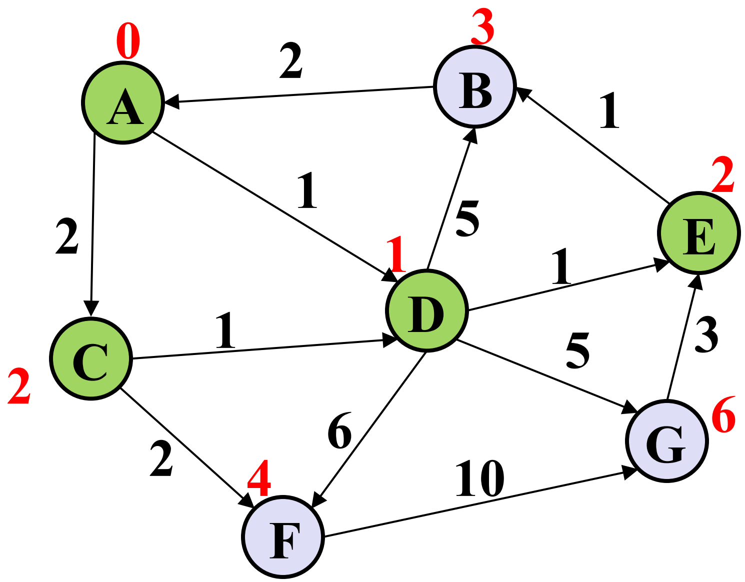


vertex	known?	cost	path
A	Y	0	
B		≤ 6	D
C	Y	2	A
D	Y	1	A
E		≤ 2	D
F		≤ 4	C
G		≤ 6	D

<u>Order Added to Known Set:</u>

A, D, C

Example #2

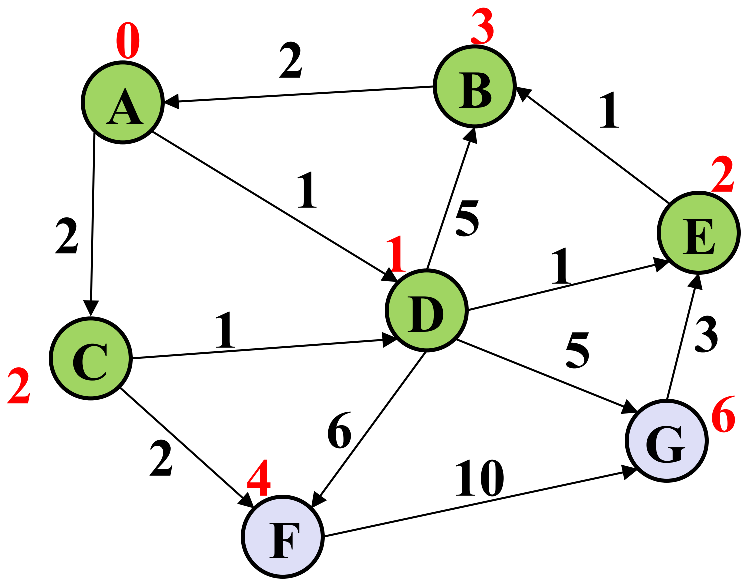


vertex	known?	cost	path
A	Y	0	
B		≤ 3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F		≤ 4	C
G		≤ 6	D

<u>Order Added to Known Set:</u>

A, D, C, E

Example #2

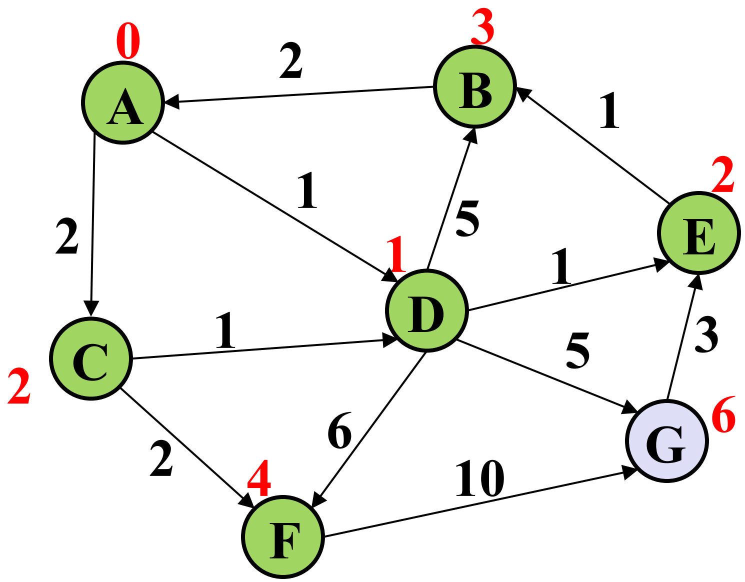


vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F		≤ 4	C
G		≤ 6	D

<u>Order Added to Known Set:</u>

A, D, C, E, B

Example #2

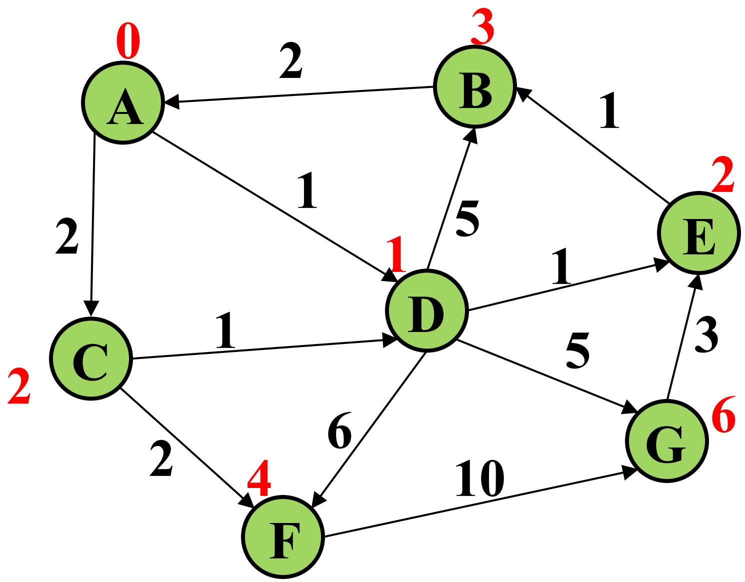


Order Added to Known Set:

A, D, C, E, B, F

vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F	Y	4	C
G		≤ 6	D

Example #2

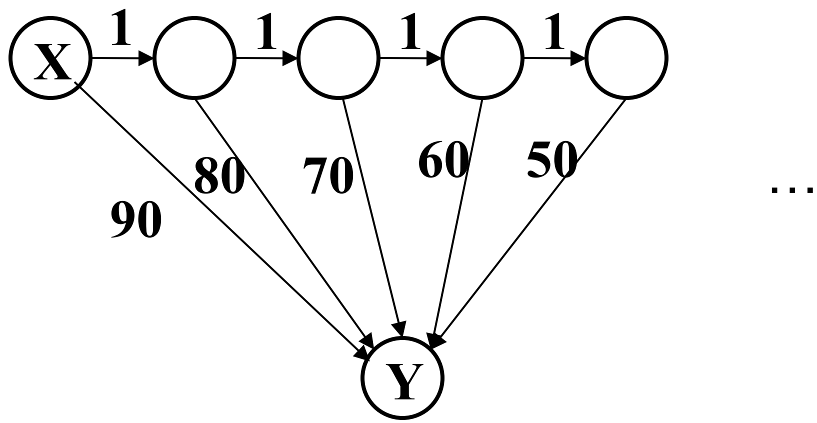


vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F	Y	4	C
G	Y	6	D

<u>Order Added to Known Set:</u>

A, D, C, E, B, F, G

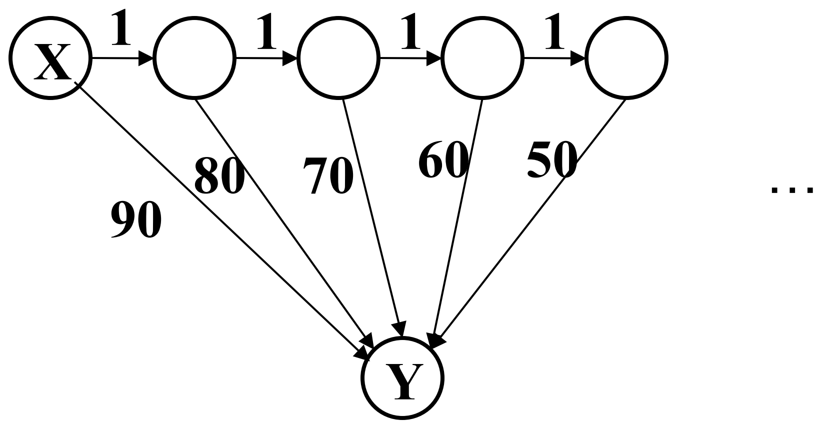
Example #3



How will the best-cost-so-far for Y proceed?

Is this expensive?

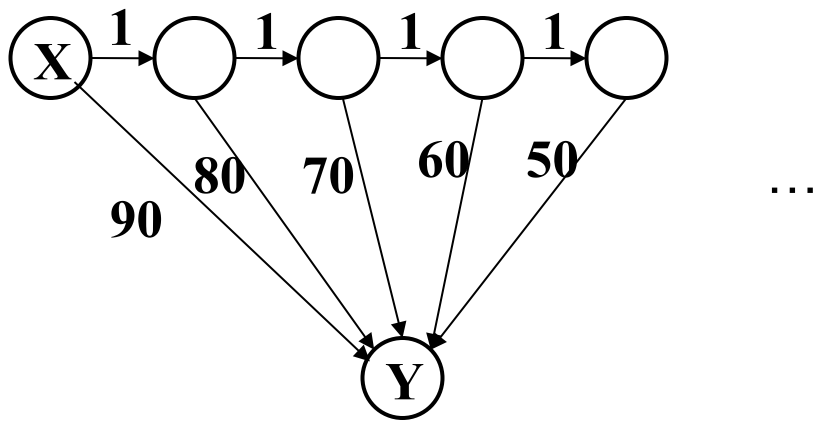
Example #3



How will the best-cost-so-far for Y proceed? 90, 81, 72, 63, 54, …

Is this expensive?

Example #3



How will the best-cost-so-far for Y proceed? 90, 81, 72, 63, 54, …

Is this expensive? No, each *edge* is processed only once

A Greedy Algorithm

- Dijkstra's algorithm
  - For single-source shortest paths in a weighted graph (directed or undirected) with no negative-weight edges
- An example of a *greedy algorithm*:
  - At each step, always does what seems best at that step
    - A locally optimal step, not necessarily globally optimal
  - Once a vertex is known, it is not revisited
    - Turns out to be globally optimal

Where are We?

- Had a problem: Compute shortest paths in a weighted graph with no negative weights
- Learned an algorithm: Dijkstra's algorithm
- What should we do after learning an algorithm?
  - Prove it is correct
    - Not obvious!
    - We will sketch the key ideas
  - Analyze its efficiency
    - Will do better by using a data structure we learned earlier!

Correctness: Intuition

Rough intuition:

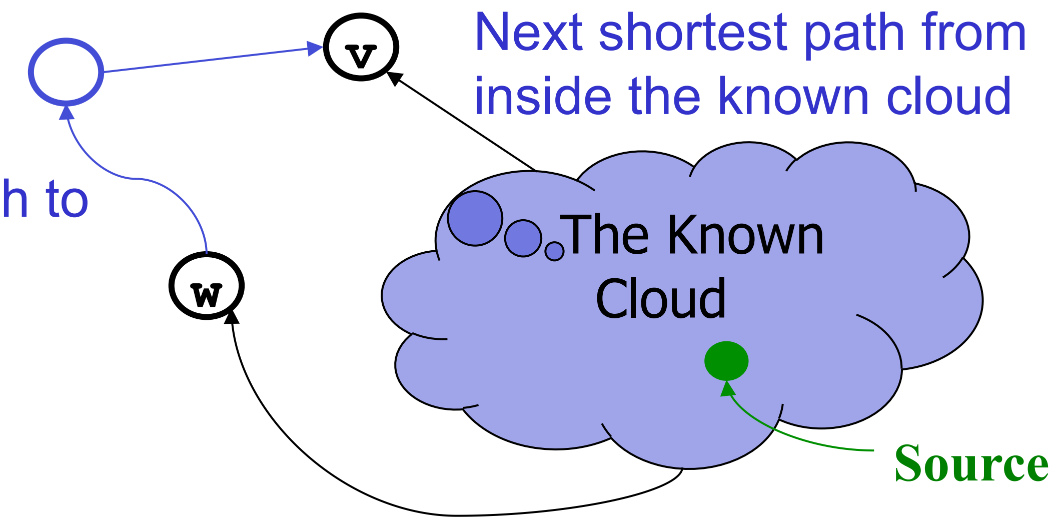
All the “known” vertices have the correct shortest path

- True initially: shortest path to start node has cost 0
- If it stays true every time we mark a node “known”, then by induction this holds and eventually everything is “known”

Key fact we need: When we mark a vertex “known” we won’t discover a shorter path later!

- This holds only because Dijkstra’s algorithm picks the node with the next shortest path-so-far
- The proof is by contradiction…

Correctness: The Cloud (Rough Sketch)



Suppose **v** is the next node to be marked known ("added to the cloud")

- The best-known path to **v** must have only nodes "in the cloud"
  - Else we would have picked a node closer to the cloud than **v**
- Suppose the actual shortest path to **v** is different
  - It won't use only cloud nodes, or we would know about it
  - So it must use non-cloud nodes. Let **w** be the *first* non-cloud node on this path. The part of the path up to **w** is already known and must be shorter than the best-known path to **v**. So **v** would not have been picked. Contradiction.

Efficiency, first approach

Use pseudocode to determine asymptotic run-time

- Notice each edge is processed only once

```
dijkstra(Graph G, Node start) {
  for each node: x.cost=infinity, x.known=false
  start.cost = 0
  while(not all nodes are known) {
    b = find unknown node with smallest cost
    b.known = true
    for each edge (b,a) in G
     if(!a.known)
       if(b.cost + weight((b,a)) < a.cost){
         a.cost = b.cost + weight((b,a))
         a.path = b
       }
}
```

Efficiency, first approach

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```

O(|V|)

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}
```

$O(|V|)$

$O(|V|^2)$

Efficiency, first approach

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```

$O(|V|)$

$O(|V|^2)$

$O(|E|)$

Efficiency, first approach

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}
```

$O(|V|)$

$O(|V|^2)$

$O(|E|)$

$O(|V|^2)$

Improving asymptotic running time

- So far: $O(|V|^2)$
- We had a similar "problem" with topological sort being $O(|V|^2)$ due to each iteration looking for the node to process next
  – We solved it with a queue of zero-degree nodes
  – But here we need the lowest-cost node and costs can change as we process edges
- Solution?

Improving (?) asymptotic running time

- So far: $O(|V|^2)$
- We had a similar "problem" with topological sort being $O(|V|^2)$ due to each iteration looking for the node to process next
  - We solved it with a queue of zero-degree nodes
  - But here we need the lowest-cost node and costs can change as we process edges
- Solution?
  - A priority queue holding all unknown nodes, sorted by cost
  - But must support **`decreaseKey`** operation
    - Must maintain a reference from each node to its current position in the priority queue
    - Conceptually simple, but can be a pain to code up

Efficiency, second approach

Use pseudocode to determine asymptotic run-time

```
dijkstra(Graph G, Node start) {
  for each node: x.cost=infinity, x.known=false
  start.cost = 0
  build-heap with all nodes
  while(heap is not empty) {
    b = deleteMin()
    b.known = true
    for each edge (b,a) in G
     if(!a.known)
      if(b.cost + weight((b,a)) < a.cost){
        decreaseKey(a,"new cost – old cost")
        a.path = b
      }
}
```

Efficiency, second approach

Use pseudocode to determine asymptotic run-time

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dijkstra(Graph G, Node start) {
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O(|V|)

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```

O(|V|)

O(|V|log|V|)

Efficiency, second approach

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    b = deleteMin()
    b.known = true
    for each edge (b,a) in G
     if(!a.known)
      if(b.cost + weight((b,a)) < a.cost){
        decreaseKey(a,"new cost – old cost")
        a.path = b
      }
}
```

O(|V|)

O(|V|log|V|)

O(|E|log|V|)

Efficiency, second approach

Use pseudocode to determine asymptotic run-time

```
dijkstra(Graph G, Node start) {
  for each node: x.cost=infinity, x.known=false
  start.cost = 0
  build-heap with all nodes
  while(heap is not empty) {
    b = deleteMin()
    b.known = true
    for each edge (b,a) in G
     if(!a.known)
      if(b.cost + weight((b,a)) < a.cost){
        decreaseKey(a,"new cost – old cost")
        a.path = b
      }
}
```

O(|V|)

O(|V|log|V|)

O(|E|log|V|)

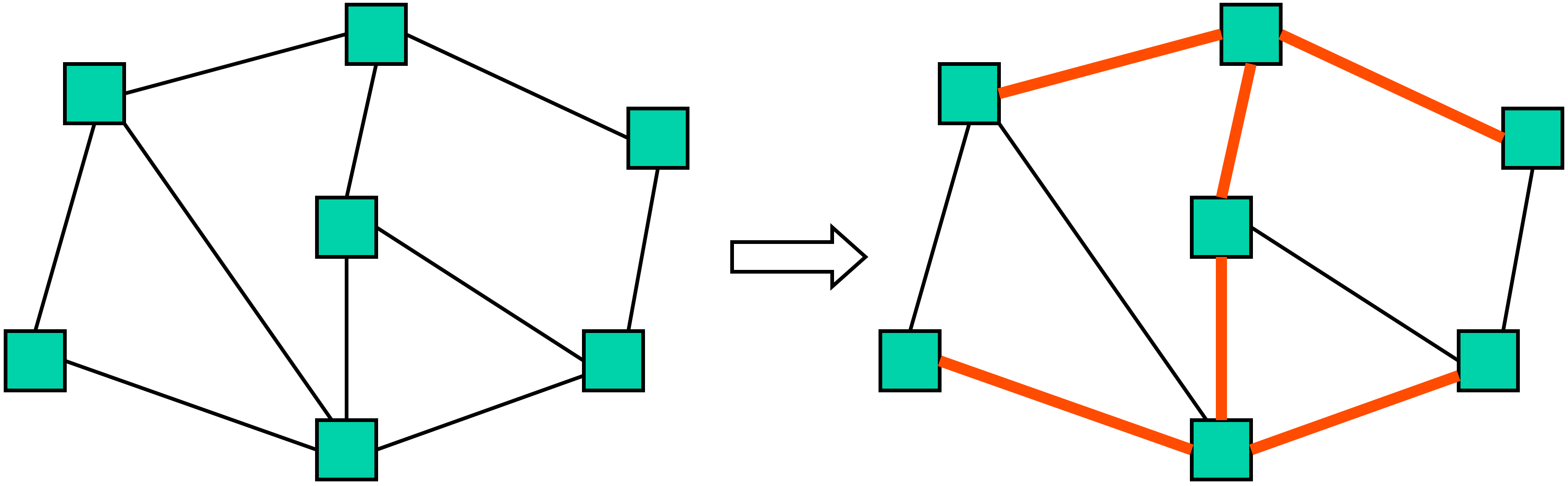
O(|V|log|V|+|E|log|V|)

Dense vs. sparse again

- First approach: $O(|V|^2)$
- Second approach: $O(|V|\log|V|+|E|\log|V|)$
- So which is better?
  - Sparse: $O(|V|\log|V|+|E|\log|V|)$ (if $|E| > |V|$, then $O(|E|\log|V|)$)
  - Dense: $O(|V|^2)$
- But, remember these are worst-case and asymptotic
  - Priority queue might have slightly worse constant factors
  - On the other hand, for “normal graphs”, we might call **`decreaseKey`** rarely (or not percolate far), making $|E|\log|V|$ more like $|E|$

Spanning Trees

- A simple problem: Given a *connected* undirected graph **G**=(**V**,**E**), find a minimal subset of edges such that **G** is still connected
  - A graph **G2**=(**V**,**E2**) such that **G2** is connected and removing any edge from **E2** makes **G2** disconnected



Observations

1. Any solution to this problem is a tree
   – Recall a tree does not need a root; just means acyclic
   – For any cycle, could remove an edge and still be connected

2. Solution not unique unless original graph was already a tree

3. Problem ill-defined if original graph not connected
   – So **|E| ≥ |V|-1**

4. A tree with **|V|** nodes has **|V|-1** edges
   – So every solution to the spanning tree problem has **|V|-1** edges

Motivation

A spanning tree connects all the nodes with as few edges as possible

- Example: A "phone tree" so everybody gets the message and no unnecessary calls get made
  - Bad example since would prefer a balanced tree

In most compelling uses, we have a *weighted* undirected graph and we want a tree of least total cost

- Example: Electrical wiring for a house or clock wires on a chip
- Example: A road network if you cared about asphalt cost rather than travel time

This is the minimum spanning tree problem

- Will do that next, after intuition from the simpler case

Two Approaches

Different algorithmic approaches to the spanning-tree problem:

1. Do a graph traversal (e.g., depth-first search, but any traversal will do), keeping track of edges that form a tree

2. Iterate through edges; add to output any edge that does not create a cycle