CSE373: Data Structures and Algorithms

Lecture 2+: Induction Supplemental

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The problem

- Find the sum of the integers from 1 to *n*
- 1 + 2 + 3 + 4 + … + (n-1) + n

$$\sum_{i=1}^{n} i$$

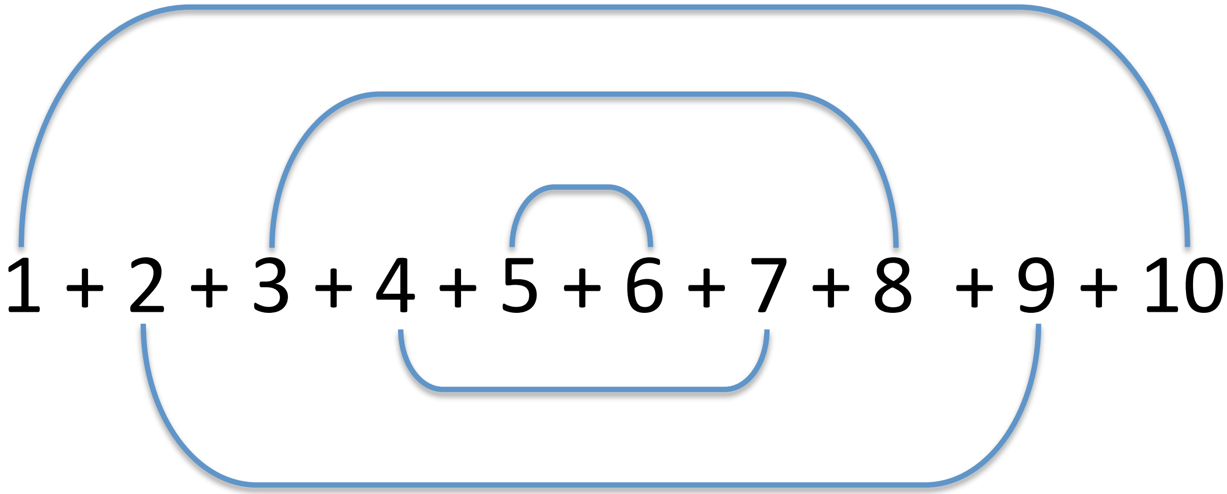
- For any $n \geq 1$
- Could use brute force, but would be slow
- There's probably a clever shortcut

Finding the formula

- Shortcut will be some formula involving *n*
- Compare examples and look for patterns
  - Not something I will ask you to do!
- Start with n = 10:

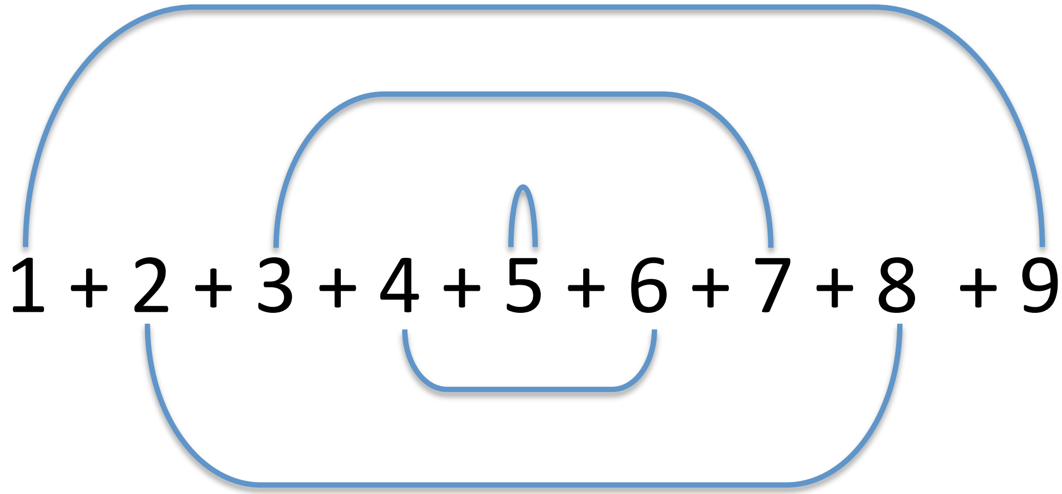
  1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10
  - Large enough to be a pain to add up
  - Worthwhile to find shortcut

Finding the formula



$$= 5 \times 11$$

Finding the formula



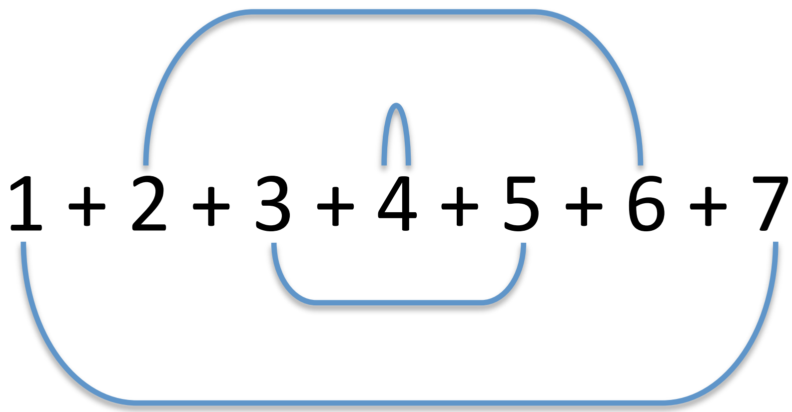
$$= 4 \times 10 + 5$$

Finding the formula

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$$

$$= 4 \times 9$$

Finding the formula



$$= 3 \times 8 + 4$$

Finding the formula

n=7	3×8 + 4
n=8	4×9
n=9	4×10 + 5
n=10	5×11

Finding the formula

n=7	3×8 + 4	n is odd
n=8	4×9	n is even
n=9	4×10 + 5	n is odd
n=10	5×11	n is even

Finding the formula

When n is even

$$1 + 2 + 3 + 4 + \ldots + (n\text{-}3) + (n\text{-}2) + (n\text{-}1) + n$$

$$= (n/2)\times(n+1)$$

Finding the formula

3×8 + 4	
4×9	n(n+1)/2
4×10 + 5	
5×11	n(n+1)/2

Finding the formula

When n is odd

$$1 + 2 + 3 + 4 + \ldots + (n-3) + (n-2) + (n-1) + n$$

$$= ((n-1)/2) \times (n+1) + (n+1)/2$$

Finding the formula

When n is odd

$$1 + 2 + 3 + 4 + \ldots + (n-3) + (n-2) + (n-1) + n$$

$$= ((n-1)\times(n+1) + (n+1))/2$$

Finding the formula

When n is odd

$$1 + 2 + 3 + 4 + \ldots + (n-3) + (n-2) + (n-1) + n$$

$$= ((n-1 + 1)\times(n+1))/2$$

Finding the formula

When n is odd

$$1 + 2 + 3 + 4 + \ldots + (n-3) + (n-2) + (n-1) + n$$

$$= (n\times(n+1))/2$$

Finding the formula

$3 \times 8 + 4$	$n(n+1)/2$
4×9	$n(n+1)/2$
$4 \times 10 + 5$	$n(n+1)/2$
5×11	$n(n+1)/2$

Are we done?

- The pattern seems pretty clear
  - Is there any reason to think it changes?
- But we want something for any *n ≥ 1*
- A mathematical approach is skeptical

$$\frac{n(n+1)}{2}$$

Are we done?

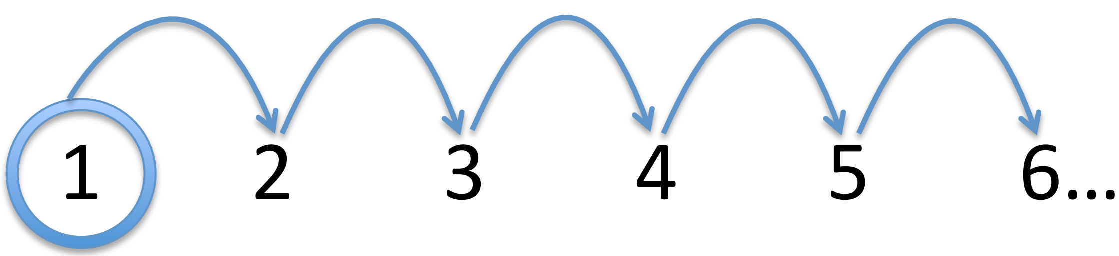
- The pattern seems pretty clear
  - Is there any reason to think it changes?
- But we want something for *any* $n \geq 1$
- A mathematical approach is *skeptical*
- All we know is $n(n+1)/2$ works for 7 to 10
- We must *prove* the formula works in all cases
  - A *rigorous* proof

Proof by induction

- Type of mathematical proof
  - Sequence of deductive steps
- P(n) = sum of integers from 1 to n
- Two things we need to do
  - Base case ← *prove for 1*
  - Induction step ← *assume for P(k)* ⇓ *P(k+1)*
- *n* and *k* are just *variables*!

Proof by induction

- $P(n)$ = sum of integers from 1 to n (for $n \geq 1$)
- Two things we need to do
  - Base case — *prove for 1*
  - Induction step — *assume for P(k)* ⇓ *P(k+1)*



Proof by induction

- What we are trying to prove: $P(n) = n(n+1)/2$
- Base case
  - P(1) = 1
  - 1(1+1)/2 = 1(2)/2 = 1(1) = 1 ✓

Proof by induction

- What we are trying to prove: $P(n) = n(n+1)/2$
- Induction step:
  - Assume true for k
  - $P(k) = k(k+1)/2$
  - Now consider $P(k+1)$

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  - Now consider $P(k+1) = k(k+1)/2 + (k+1)$
  - $k(k+1)/2 + 2(k+1)/2$

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  - $k(k+1)/2 + 2(k+1)/2 = (k(k+1) + 2(k+1))/2$
  - $(k+1)(k+2)/2$

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  - $P(k) = k(k+1)/2$
  - Now consider $P(k+1) = k(k+1)/2 + (k+1)$
  - $k(k+1)/2 + 2(k+1)/2 = (k(k+1) + 2(k+1))/2$
  - $(k+1)(k+2)/2 = (k+1)((k+1)+1)/2$

✓

Why you should care

- Induction turns out to be a useful technique
  - AVL trees
  - Heaps
  - Graph algorithms
  - Can also prove things like $3^n > n^3$ *for* $n \geq 4$
- Exposure to rigorous thinking