



CSE373: Data Structures and Algorithms

Lecture 2+: Induction Supplemental

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The problem

- Find the sum of the integers from 1 to n
- $1 + 2 + 3 + 4 + \dots + (n-1) + n$

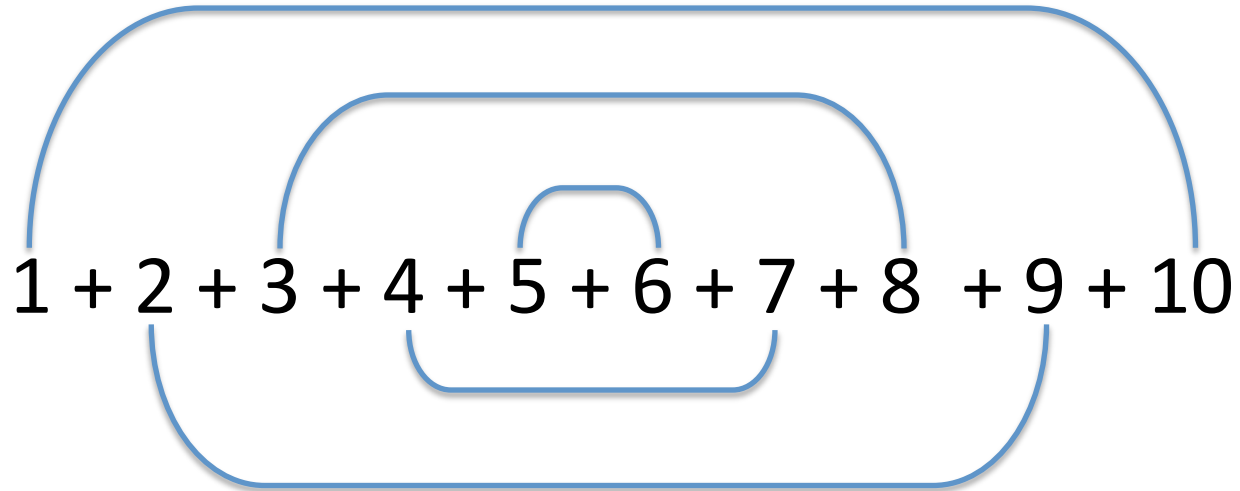
$$\sum_{i=1}^n i$$

- For any $n \geq 1$
- Could use brute force, but would be slow
- There's probably a clever **shortcut**

Finding the formula

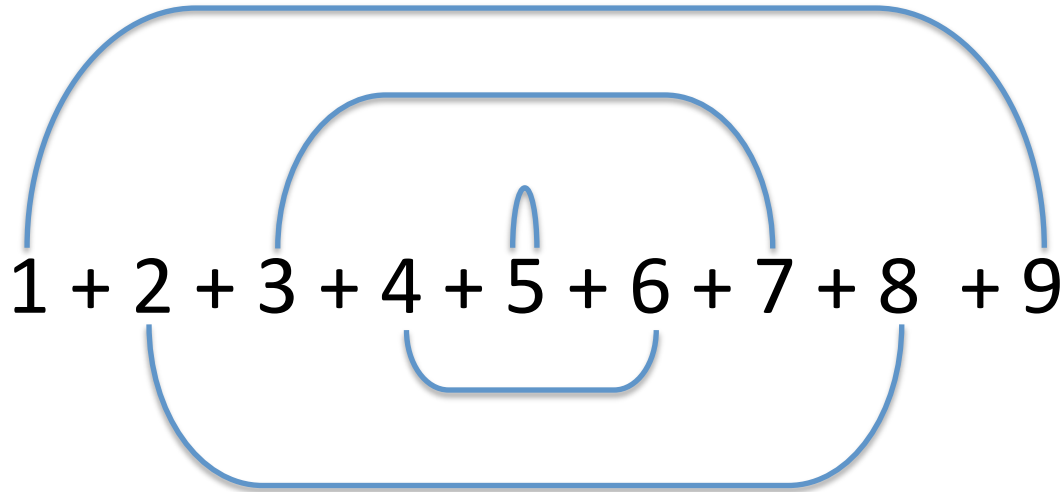
- Shortcut will be some **formula** involving n
- Compare examples and look for patterns
 - Not something I will ask you to do!
- Start with $n = 10$:
 $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$
 - Large enough to be a pain to add up
 - Worthwhile to find shortcut

Finding the formula



$$= 5 \times 11$$

Finding the formula



$$= 4 \times 10 + 5$$

Finding the formula

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$$

$$= 4 \times 9$$

Finding the formula

1 + 2 + 3 + 4 + 5 + 6 + 7

$$= 3 \times 8 + 4$$

Finding the formula

n=7	$3 \times 8 + 4$
n=8	4×9
n=9	$4 \times 10 + 5$
n=10	5×11

Finding the formula

n=7	$3 \times 8 + 4$	n is odd
n=8	4×9	n is even
n=9	$4 \times 10 + 5$	n is odd
n=10	5×11	n is even

Finding the formula

When n is even

$$1 + 2 + 3 + 4 + \dots + (n-3) + (n-2) + (n-1) + n$$

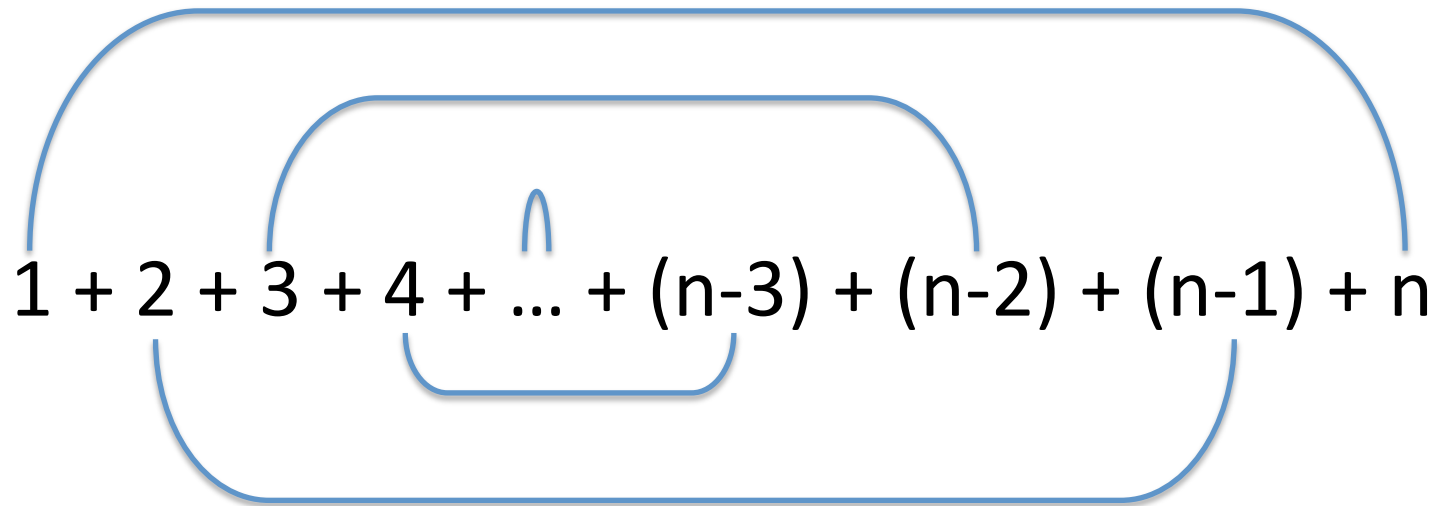
$$= (n/2) \times (n+1)$$

Finding the formula

$3 \times 8 + 4$	
4×9	$n(n+1)/2$
$4 \times 10 + 5$	
5×11	$n(n+1)/2$

Finding the formula

When n is odd



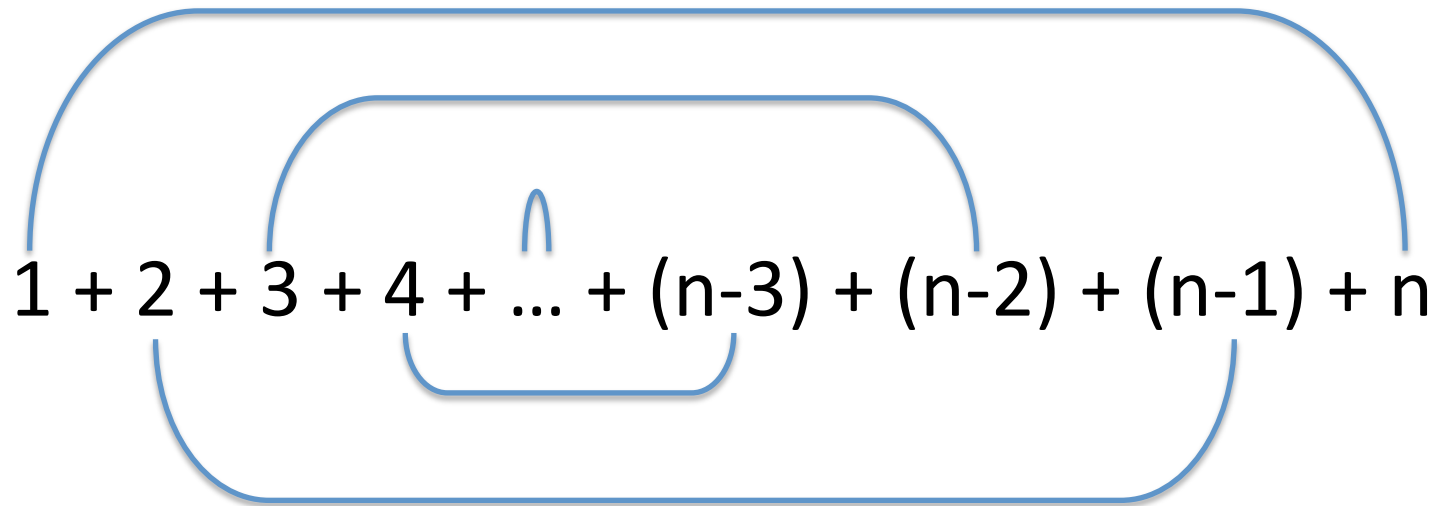
The diagram shows the arithmetic series $1 + 2 + 3 + 4 + \dots + (n-3) + (n-2) + (n-1) + n$. Blue brackets are drawn above and below the terms to illustrate pairing. The top bracket groups the first three terms (1, 2, 3) and the last three terms ((n-2), (n-1), n). The bottom bracket groups the last three terms ((n-2), (n-1), n) and the first three terms (1, 2, 3). A small blue arch is drawn above the ellipsis (...).

$$1 + 2 + 3 + 4 + \dots + (n-3) + (n-2) + (n-1) + n$$

$$= ((n-1)/2) \times (n+1) + (n+1)/2$$

Finding the formula

When n is odd



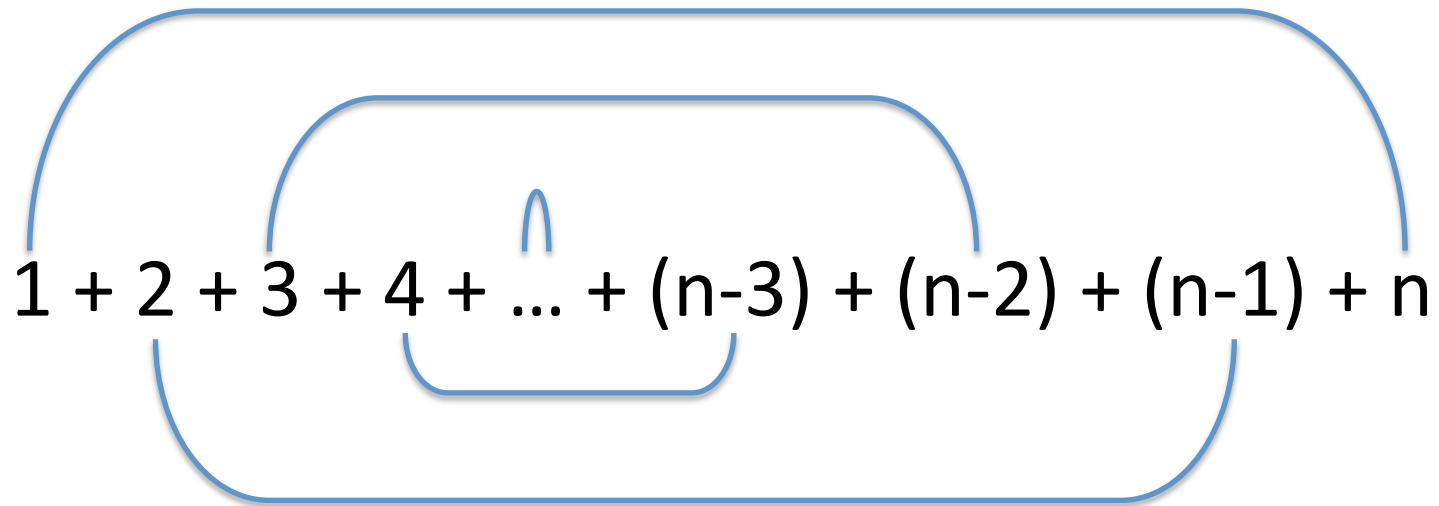
The diagram shows the arithmetic series $1 + 2 + 3 + 4 + \dots + (n-3) + (n-2) + (n-1) + n$. Blue curved brackets are drawn above and below the terms to illustrate pairing. The outermost brackets connect 1 to n and 2 to $(n-1)$. The innermost brackets connect 3 to $(n-2)$. A small blue arch is drawn above the ellipsis \dots . The middle term $(n-3)$ is not paired.

$$1 + 2 + 3 + 4 + \dots + (n-3) + (n-2) + (n-1) + n$$

$$= ((n-1) \times (n+1) + (n+1)) / 2$$

Finding the formula

When n is odd



The diagram illustrates the summation of an arithmetic series for an odd number of terms n . The series is written as $1 + 2 + 3 + 4 + \dots + (n-3) + (n-2) + (n-1) + n$. Blue brackets are drawn above and below the terms, pairing the first and last terms, the second and second-to-last terms, and so on, until the middle term. This visualizes the process of finding a formula by summing pairs of terms that add up to a constant value.

$$1 + 2 + 3 + 4 + \dots + (n-3) + (n-2) + (n-1) + n$$

$$= ((n-1 + 1) \times (n+1)) / 2$$

Finding the formula

When n is odd

$$1 + 2 + 3 + 4 + \dots + (n-3) + (n-2) + (n-1) + n$$

$$= (n \times (n+1)) / 2$$

Finding the formula

$3 \times 8 + 4$	$n(n+1)/2$
4×9	$n(n+1)/2$
$4 \times 10 + 5$	$n(n+1)/2$
5×11	$n(n+1)/2$

Are we done?

- The pattern seems pretty clear
 - Is there any reason to think it changes?
- But we want something for **any** $n \geq 1$
- A mathematical approach is **skeptical**

$$\frac{n(n+1)}{2}$$

Are we done?

- The pattern seems pretty clear
 - Is there any reason to think it changes?
- But we want something for *any* $n \geq 1$
- A mathematical approach is *skeptical*
- All we know is $n(n+1)/2$ works for 7 to 10
- We must *prove* the formula works in all cases
 - A *rigorous* proof

Proof by induction

- Type of mathematical proof
 - Sequence of deductive steps
- $P(n)$ = sum of integers from 1 to n
- Two things we need to do
 - Base case ← *prove for 1*
 - Induction step ← *assume for $P(k)$*
 \downarrow
 $P(k+1)$
- n and k are just *variables!*

Proof by induction

- $P(n)$ = sum of integers from 1 to n (for $n \geq 1$)
- Two things we need to do

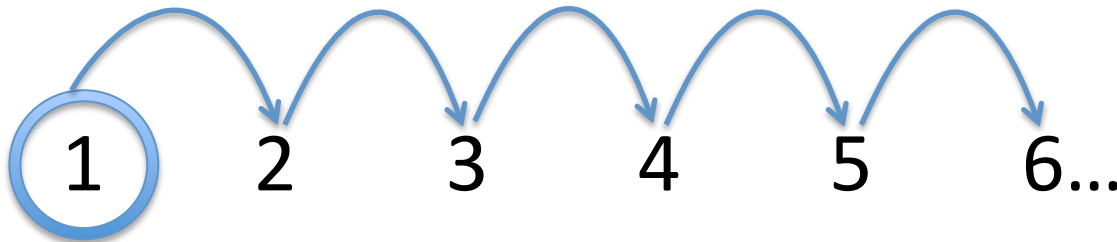
– Base case

prove for 1

– Induction step

assume for $P(k)$

$P(k+1)$



Proof by induction

- What we are trying to prove: $P(n) = n(n+1)/2$
- Base case
 - $P(1) = 1$
 - $1(1+1)/2 = 1(2)/2 = 1(1) = 1$



Proof by induction

- What we are trying to prove: $P(n) = n(n+1)/2$
- Induction step:
 - Assume true for k
 - $P(k) = k(k+1)/2$
 - Now consider $P(k+1)$

Proof by induction

- What we are trying to prove: $P(n) = n(n+1)/2$
- Induction step:
 - Assume true for k
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- Induction step:
 - Assume true for k
 - $P(k) = k(k+1)/2$
 - Now consider $P(k+1) = k(k+1)/2 + (k+1)$
 - $k(k+1)/2 + 2(k+1)/2$

Proof by induction

- What we are trying to prove: $P(n) = n(n+1)/2$
- Induction step:
 - Assume true for k
 - $P(k) = k(k+1)/2$
 - Now consider $P(k+1) = k(k+1)/2 + (k+1)$
 - $k(k+1)/2 + 2(k+1)/2 = (k(k+1) + 2(k+1))/2$
 - $(k+1)(k+2)/2$

Proof by induction

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 - $(k+1)(k+2)/2 = (k+1)((k+1)+1)/2$



Proof by induction

- What we are trying to prove: $P(n) = n(n+1)/2$
- Induction step:
 - Assume true for k
 - $P(k) = k(k+1)/2$
 - Now consider $P(k+1) = k(k+1)/2 + (k+1)$
 - $k(k+1)/2 + 2(k+1)/2 = (k(k+1) + 2(k+1))/2$
 - $(k+1)(k+2)/2 = (k+1)((k+1)+1)/2$



Why you should care

- Induction turns out to be a useful technique
 - AVL trees
 - Heaps
 - Graph algorithms
 - Can also prove things like $3^n > n^3$ for $n \geq 4$
- Exposure to rigorous thinking