CSE373: Data Structures & Algorithms

Lecture 6: Binary Search Trees continued

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Announcements

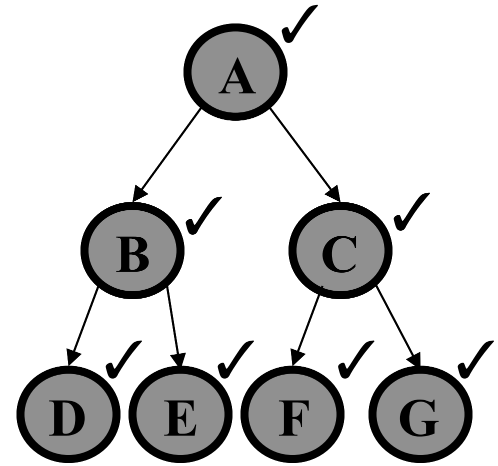
- HW2 out, due beginning of class Wednesday
- Two TA sessions next week
  - Asymptotic analysis on Tuesday
  - AVL Trees on Thursday
- MLK day Monday

Previously on CSE 373

- Dictionary ADT
  - stores (key, value) pairs
  - **`find`**, **`insert`**, **`delete`**
- Trees
  - terminology
  - traversals

More on traversals

```
void inOrderTraversal(Node t){
  if(t != null) {
    inOrderTraversal(t.left);
    process(t.element);
    inOrderTraversal(t.right);
  }
}
```



Sometimes order doesn't matter

- Example: sum all elements

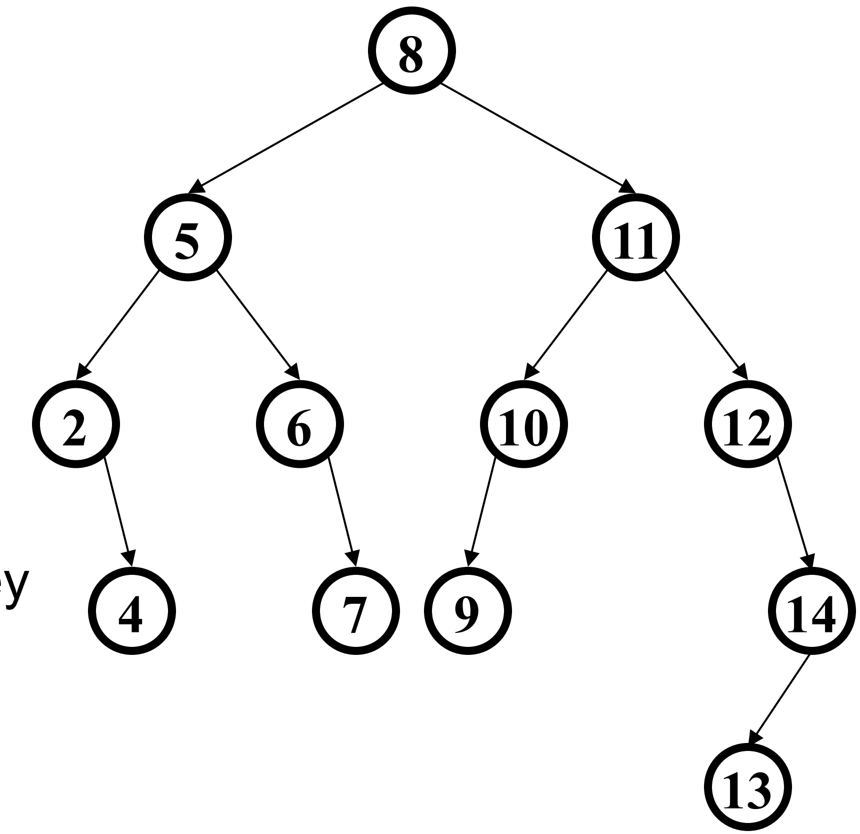
Sometimes order matters

- Example: print tree with parent above indented children (pre-order)
- Example: evaluate an expression tree (post-order)

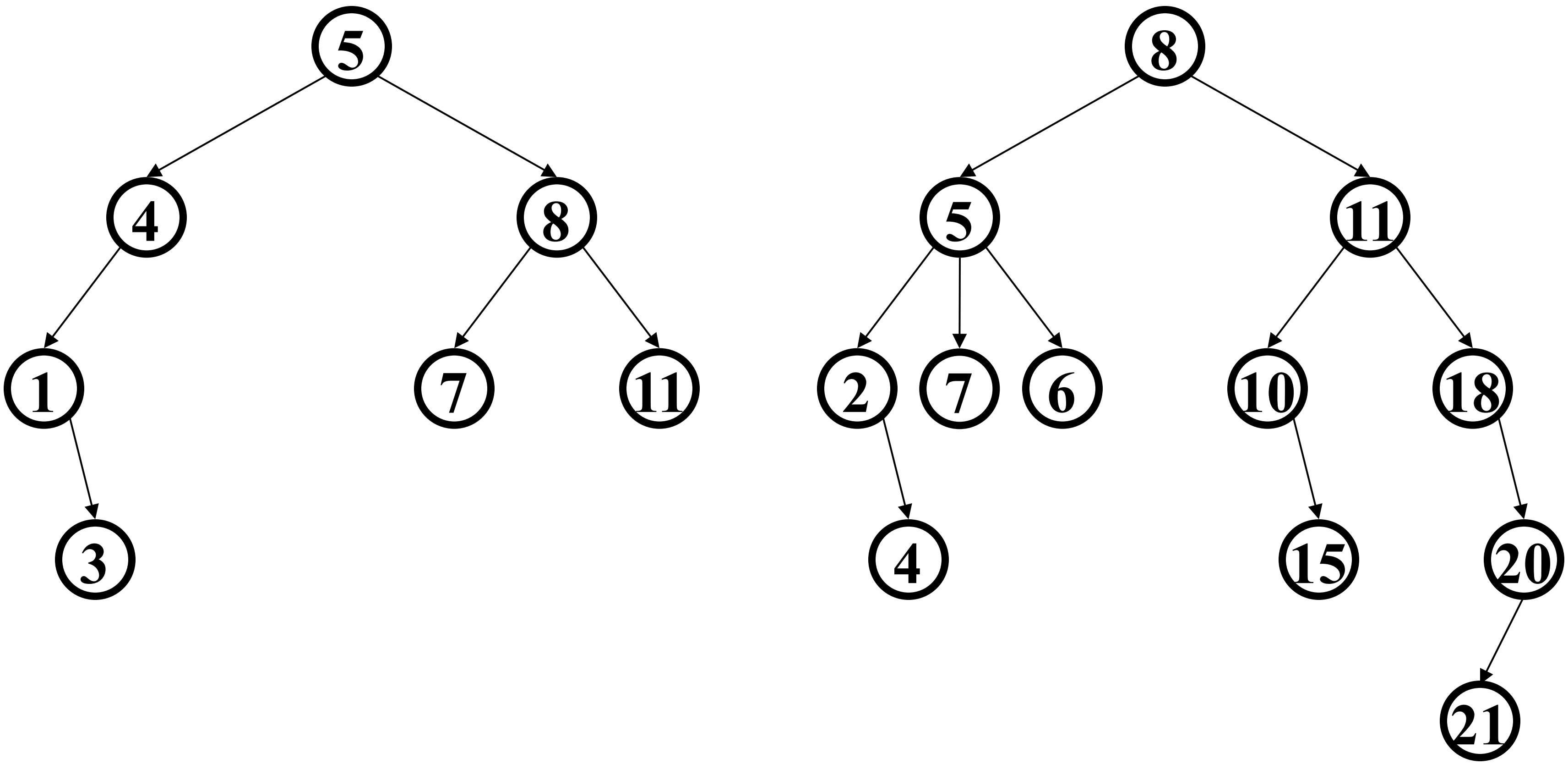
```
A
  B
    D
    E
  C
    F
    G
```

Binary Search Tree

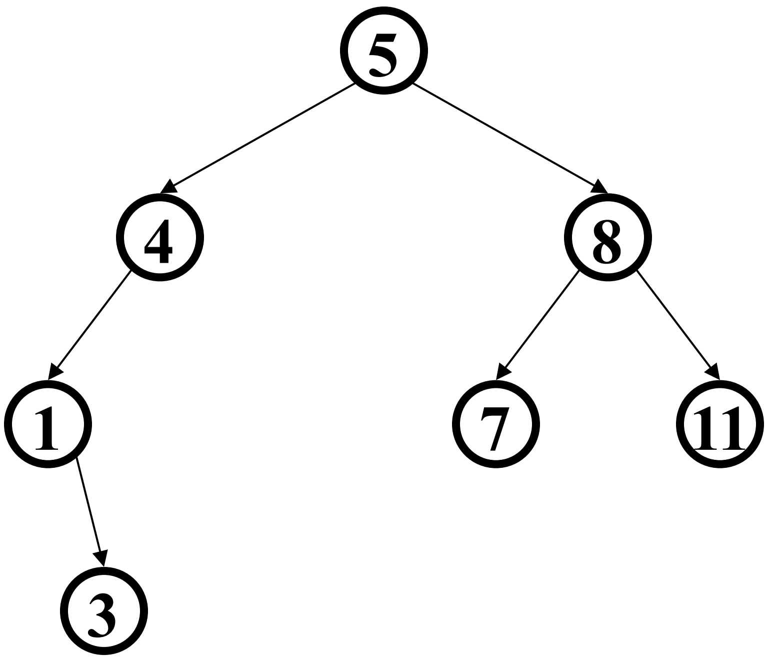
- Structure property ("binary")
  - Each node has ≤ 2 children
  - Result: keeps operations simple
- Order property
  - All keys in left subtree smaller than node's key
  - All keys in right subtree larger than node's key
  - Result: easy to find any given key

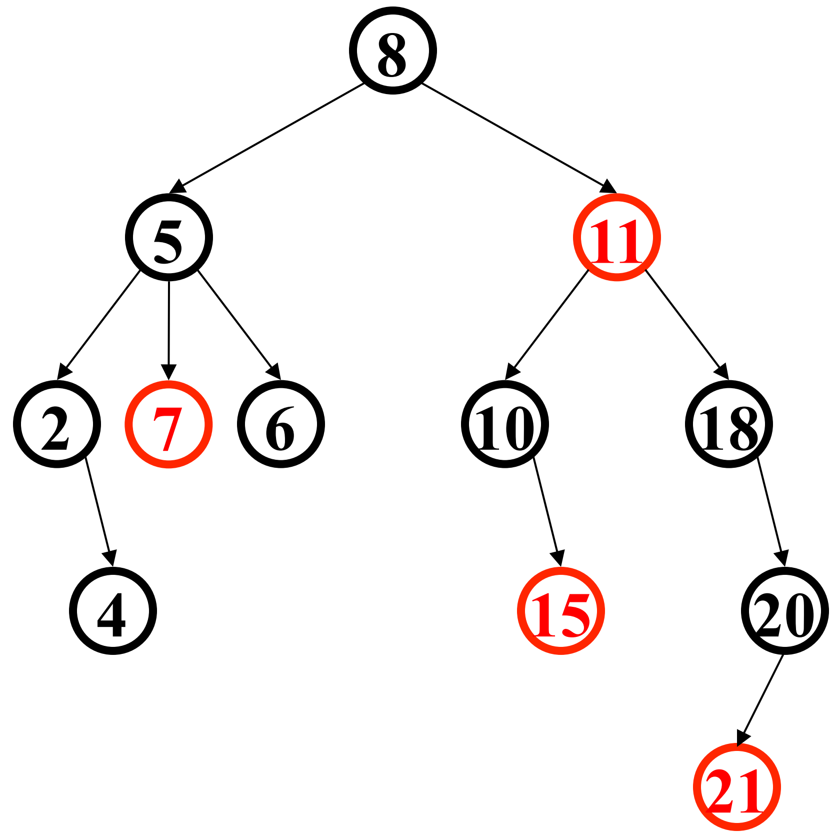


Are these BSTs?

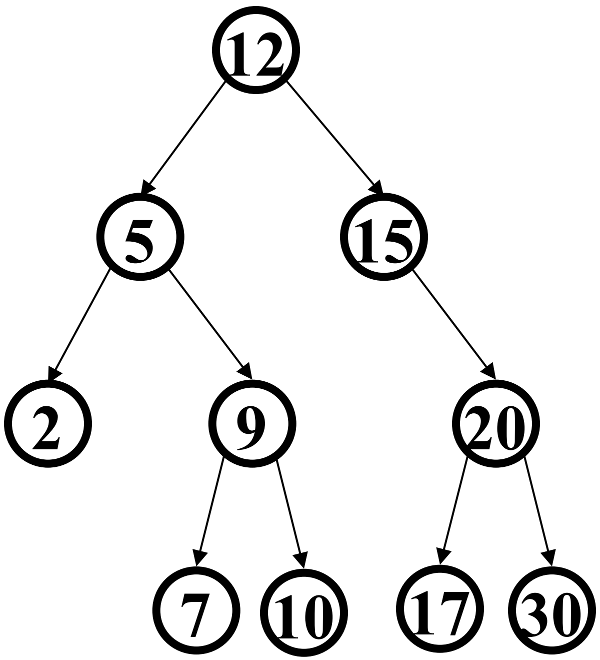


Are these BSTs?



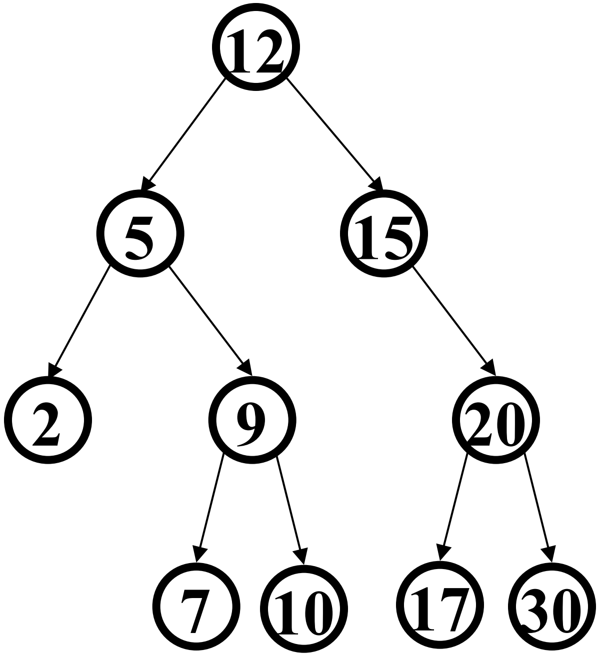


Find in BST, Recursive




```
Data find(Key key, Node root){
 if(root == null)
   return null;
 if(key < root.key)
   return find(key,root.left);
 if(key > root.key)
   return find(key,root.right);
 return root.data;
}
```

Find in BST, Iterative

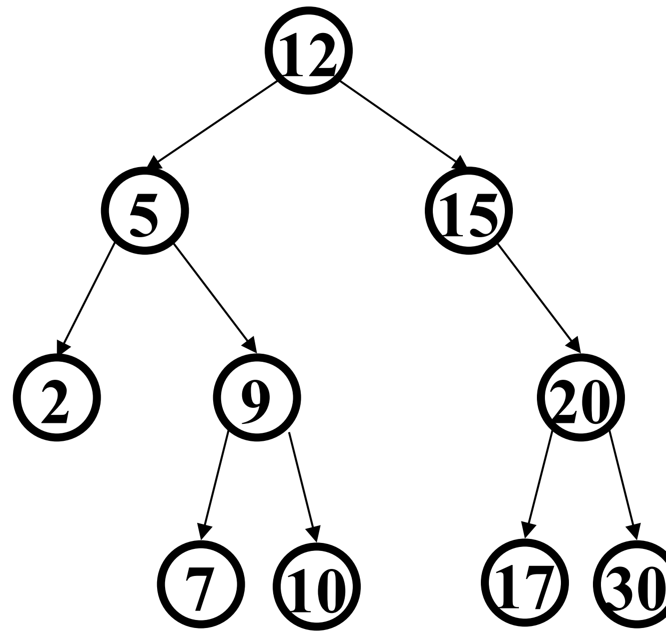



```
Data find(Key key, Node root){
 while(root != null
       && root.key != key) {
  if(key < root.key)
    root = root.left;
  else(key > root.key)
    root = root.right;
 }
 if(root == null)
    return null;
 return root.data;
}
```

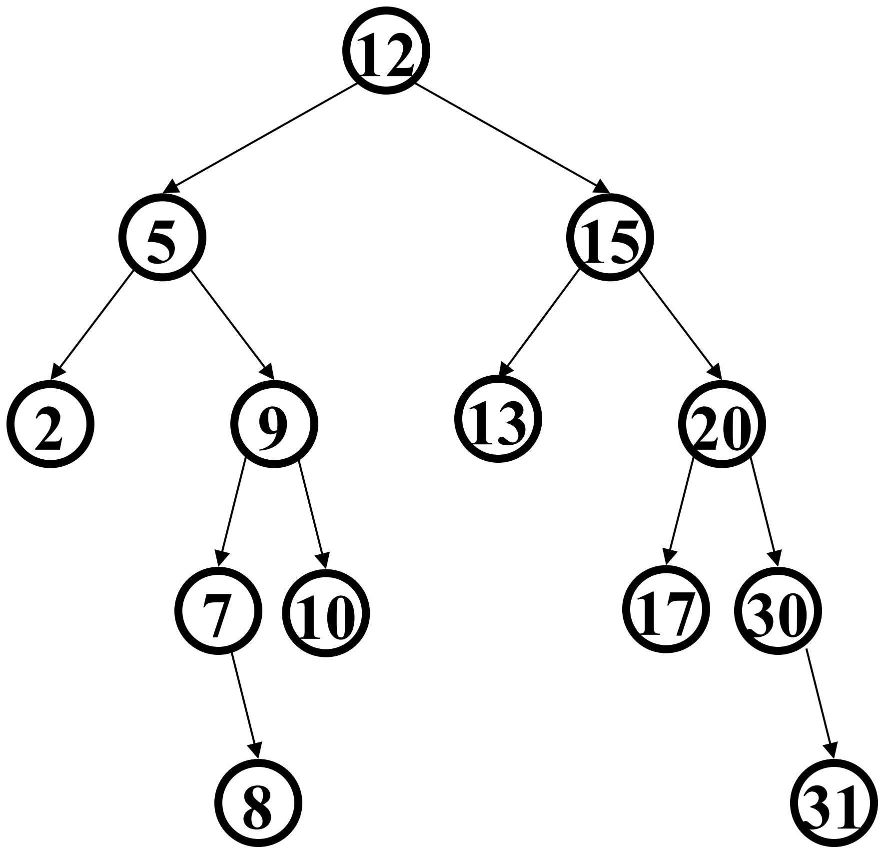
Other "Finding" Operations

- Find *minimum* node
  - "the liberal algorithm"
- Find *maximum* node
  - "the conservative algorithm"

- Find *predecessor*
- Find *successor*



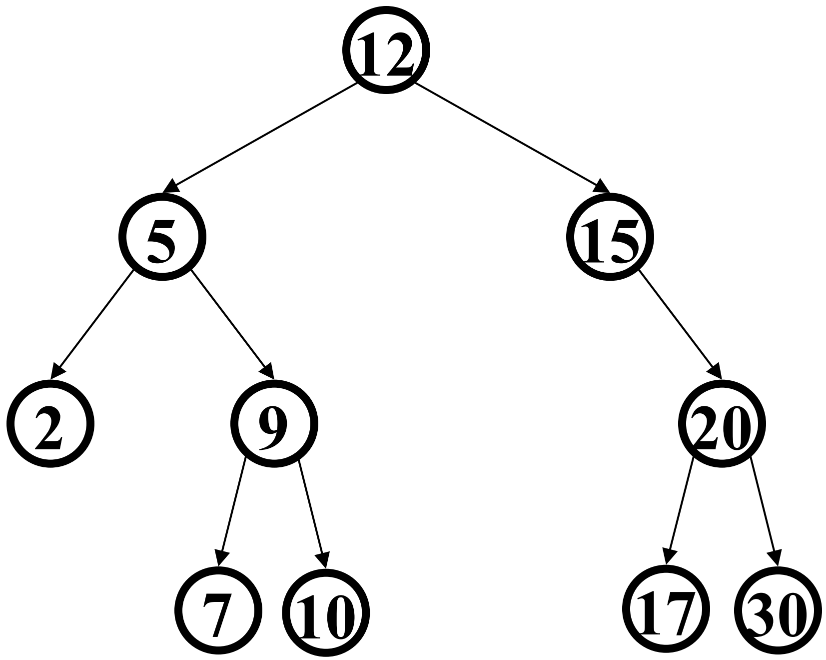
Insert in BST




```
insert(13)
insert(8)
insert(31)
```

(New) insertions happen only at leaves – easy!

Deletion in BST



Why might deletion be harder than insertion?

Deletion

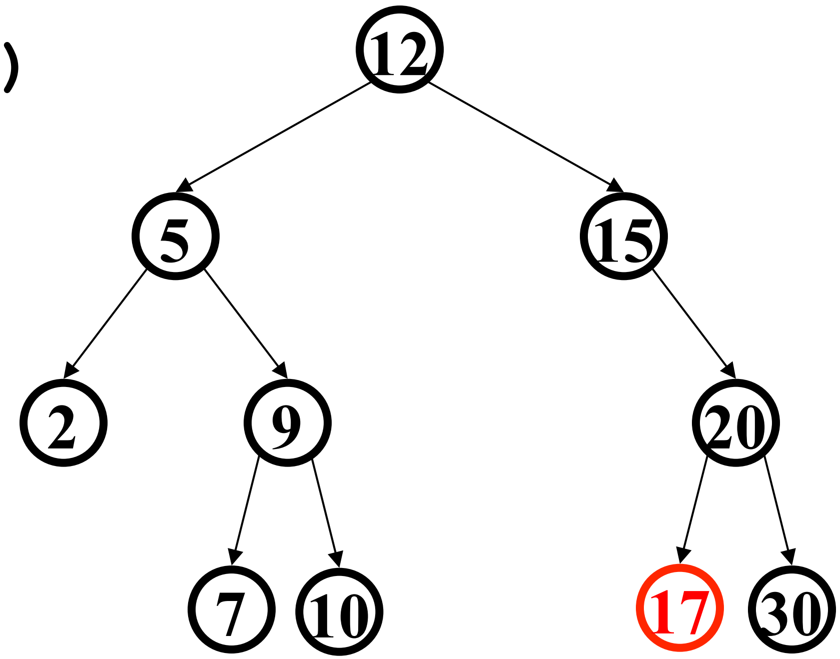
- Removing an item disrupts the tree structure

- Basic idea: **`find`** the node to be removed, then "fix" the tree so that it is still a binary search tree

- Three cases:
  - Node has no children (leaf)
  - Node has one child
  - Node has two children

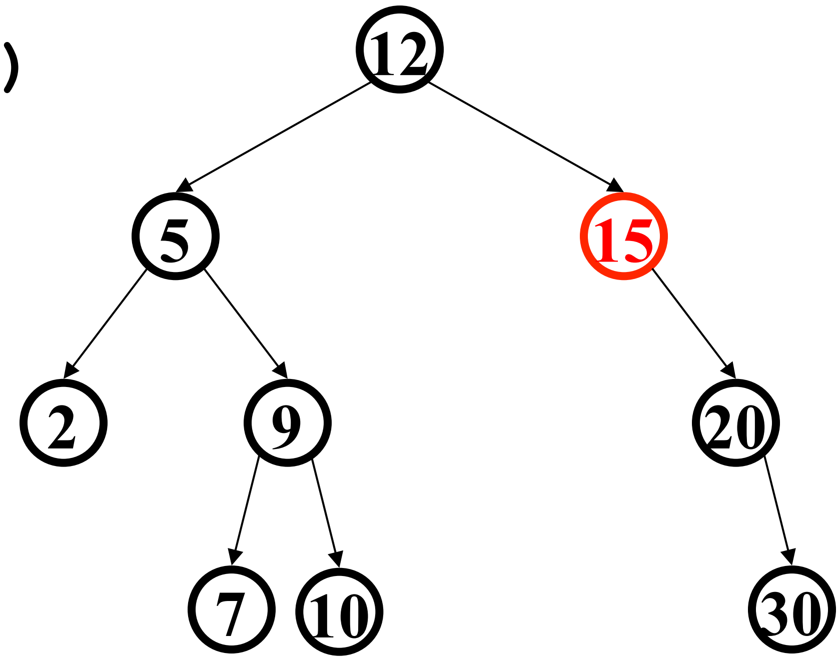
Deletion – The Leaf Case

`delete(17)`



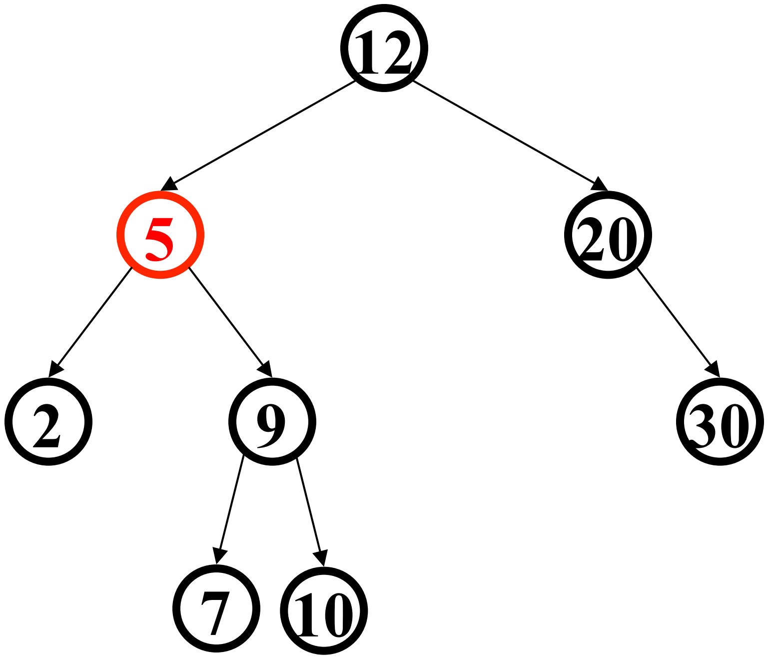
Deletion – The One Child Case

`delete(15)`



Deletion – The Two Child Case

`delete(5)`



What can we replace 5 with?

Deletion – The Two Child Case

Idea: Replace the deleted node with a value guaranteed to be between the two child subtrees

Options:

- *successor* from right subtree: **`findMin(node.right)`**
- *predecessor* from left subtree: **`findMax(node.left)`**
  - These are the easy cases of predecessor/successor

Now delete the original node containing *successor* or *predecessor*

- Leaf or one child case – easy cases of delete!

Lazy Deletion

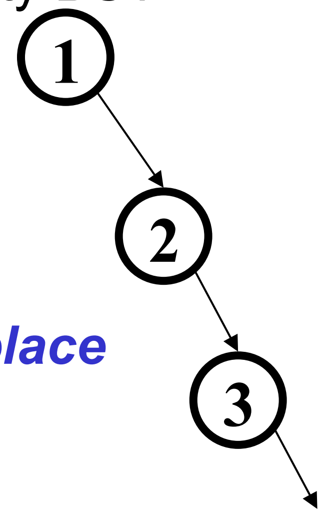
- Lazy deletion can work well for a BST
  - Simpler
  - Can do “real deletions” later as a batch
  - Some inserts can just “undelete” a tree node

- But
  - Can waste space and slow down find operations
  - Make some operations more complicated:
    - How would you change **`findMin`** and **`findMax`**?

BuildTree for BST

- Let's consider **`buildTree`**
  - Insert all, starting from an empty tree

- Insert keys 1, 2, 3, 4, 5, 6, 7, 8, 9 into an empty BST
  - If inserted in given order, what is the tree?
  - What big-O runtime for this kind of sorted input?
  - Is inserting in the reverse order any better?

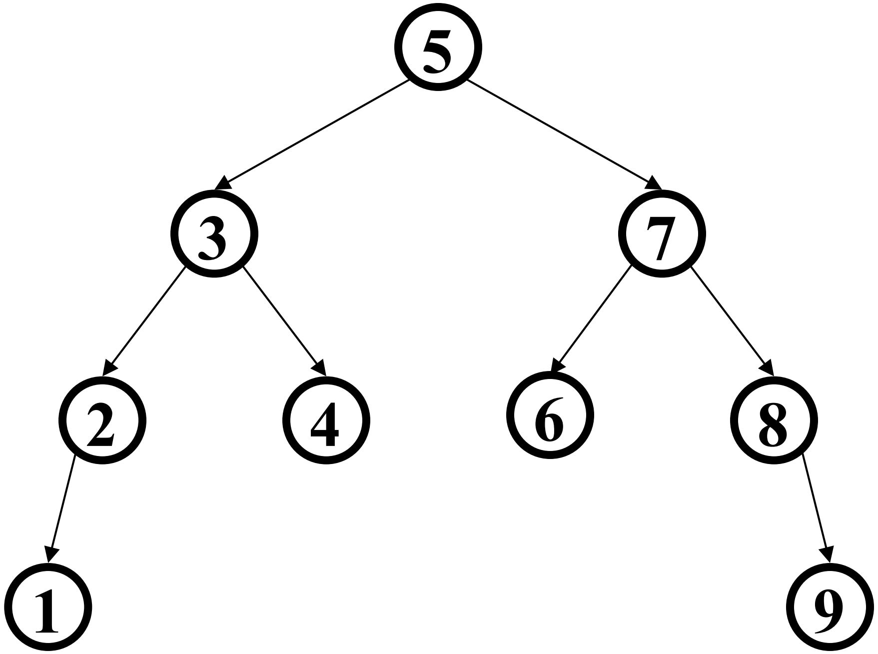


$O(n^2)$
Not a happy place

BuildTree for BST

- Insert keys 1, 2, 3, 4, 5, 6, 7, 8, 9 into an empty BST
- What we if could somehow re-arrange them
  - median first, then left median, right median, etc.
  - 5, 3, 7, 2, 1, 4, 8, 6, 9
  - What tree does that give us?
  - What big-O runtime?

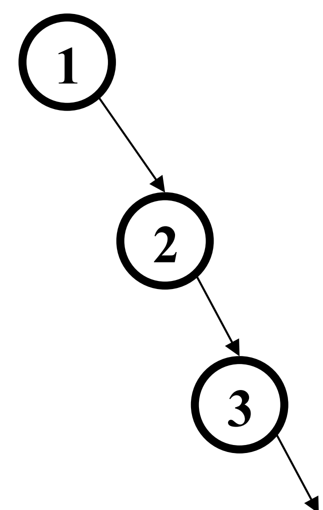
O(n* log *n), definitely better



Unbalanced BST

- Balancing a tree at build time is insufficient, as sequences of operations can eventually transform that carefully balanced tree into the dreaded list

- At that point, everything is *O(n)* and nobody is happy
  - **`find`**
  - **`insert`**
  - **`delete`**



Balanced BST

Observation

- BST: the shallower the better!
- For a BST with n nodes inserted in arbitrary order
  - Average height is $O(\texttt{log}\ n)$ – see text for proof
  - Worst case height is $O(n)$
- Simple cases, such as inserting in key order, lead to the worst-case scenario

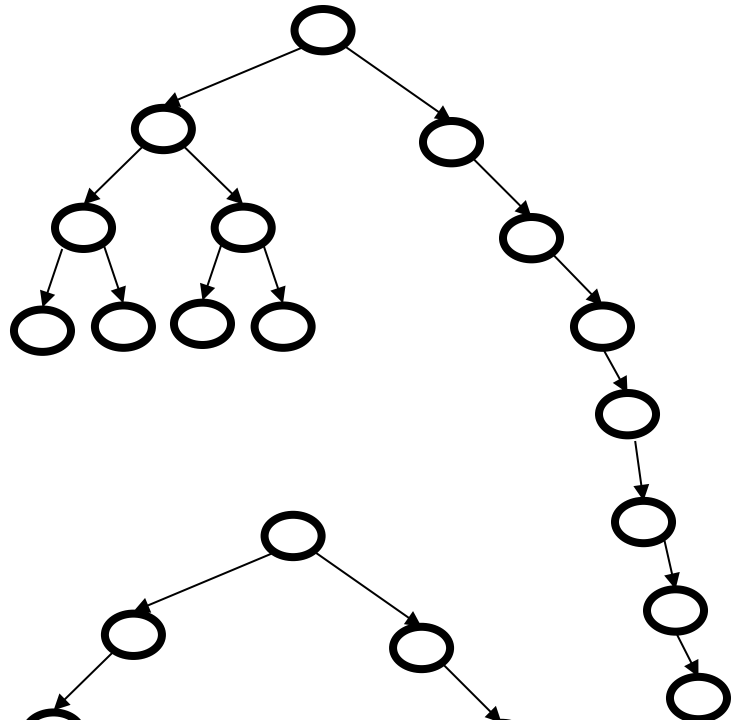
Solution: Require a **Balance Condition** that

1. Ensures depth is always $O(\texttt{log}\ n)$ – strong enough!
2. Is efficient to maintain – not too strong!

Potential Balance Conditions

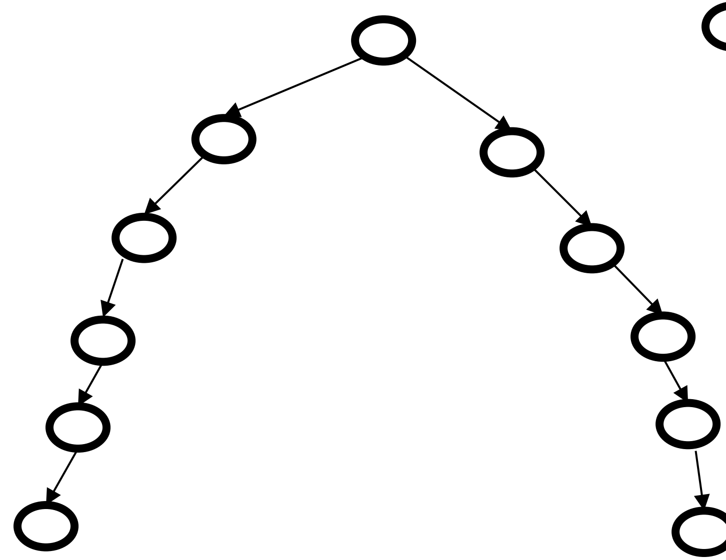
1. Left and right subtrees of the *root* have equal number of nodes

Too weak!
Height mismatch example:

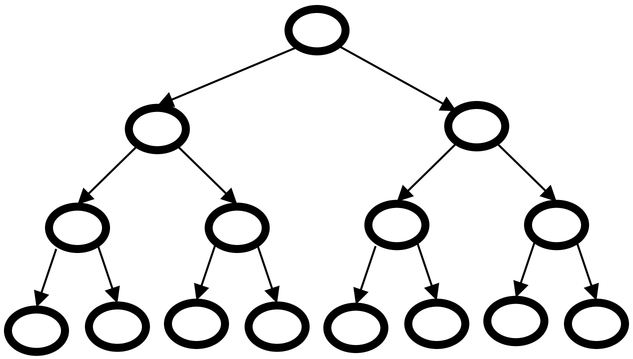


2. Left and right subtrees of the *root* have equal *height*

Too weak!
Double chain example:



Potential Balance Conditions



3. Left and right subtrees of every node have equal number of nodes

Too strong!
Only perfect trees (2^n – 1 nodes)

4. Left and right subtrees of every node have equal *height*

Too strong!
Only perfect trees (2^n – 1 nodes)

The AVL Balance Condition

Left and right subtrees of *every node*
have *heights* **differing by at most 1**

Definition: **balance**(*node*) = height(*node*.left) – height(*node*.right)

AVL *property*: **for every node *x*, –1 ≤ balance(*x*) ≤ 1**

- Ensures small depth
  - Will prove this by showing that an AVL tree of height *h* must have a number of nodes *exponential* in *h*
- Efficient to maintain
  - Using single and double rotations

The AVL Tree Data Structure

Structural properties

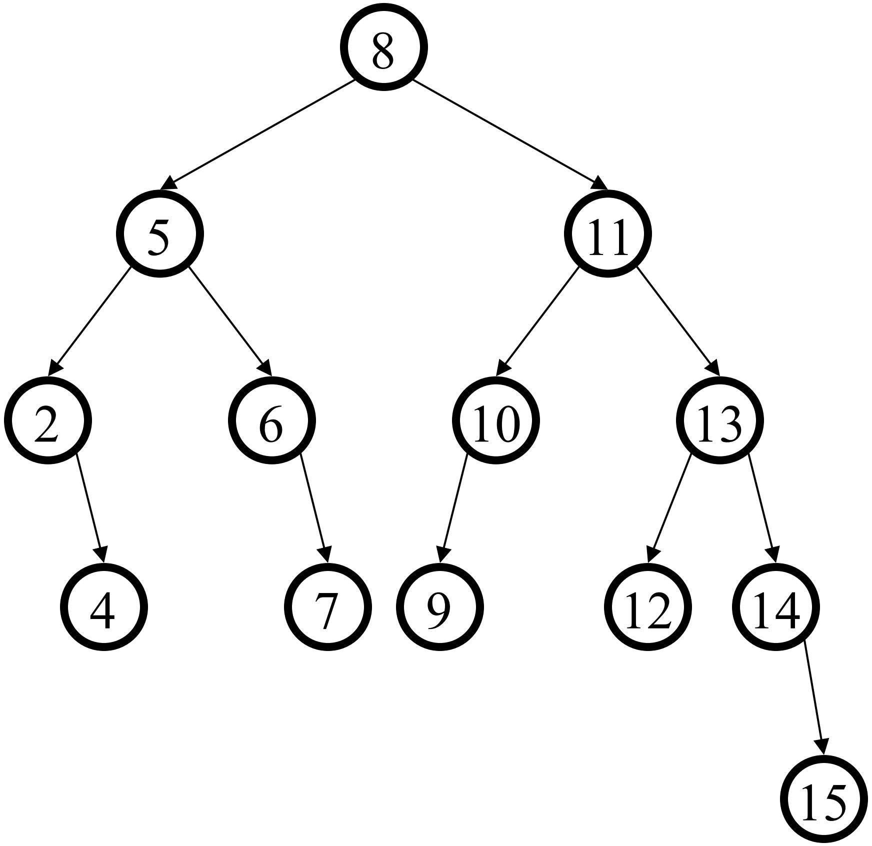
1. Binary tree property
2. Balance property:
   balance of every node is
   between -1 and 1

Result:

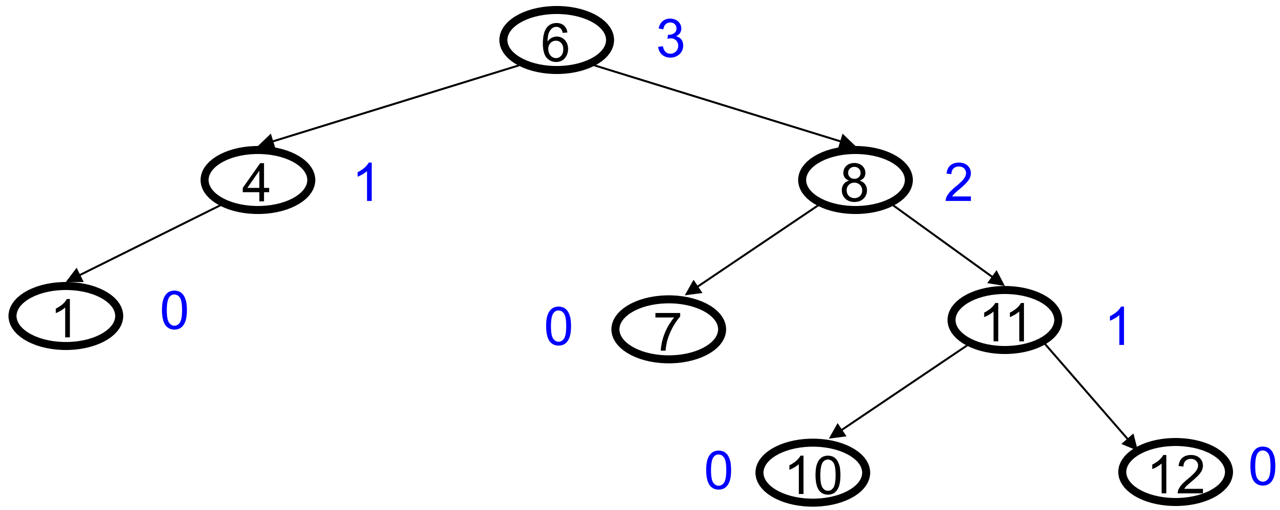
Worst-case depth is
O(log *n*)

Ordering property

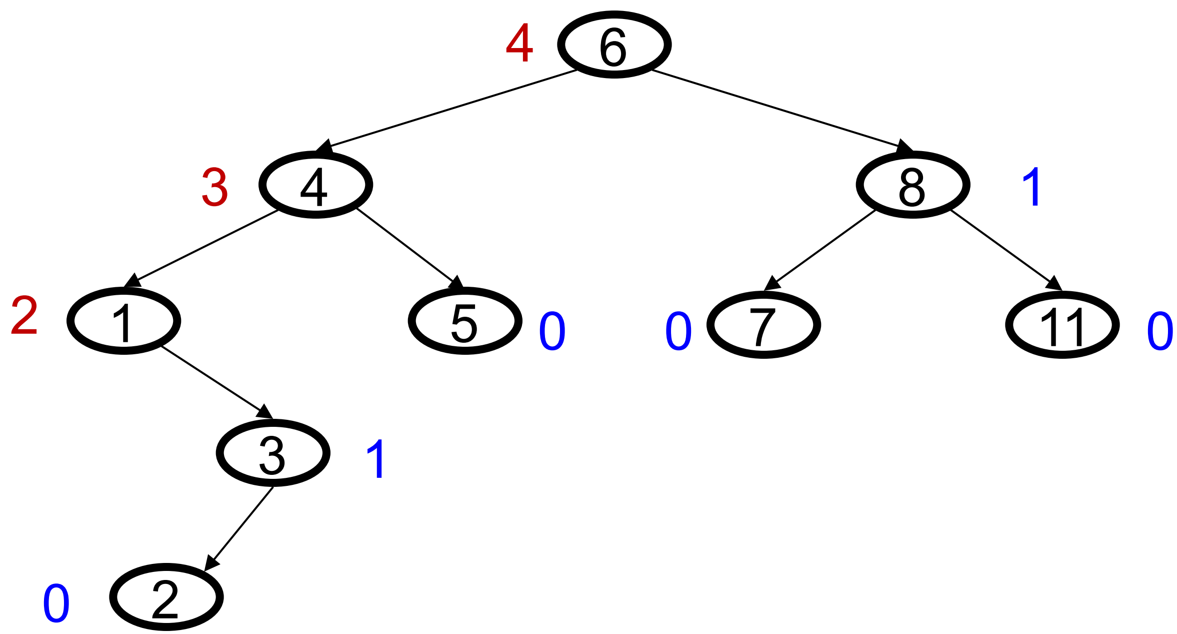
– Same as for BST



An AVL tree?



An AVL tree?

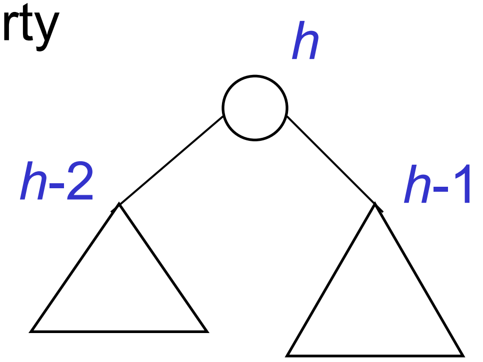


The shallowness bound

Let $S(h)$ = the minimum number of nodes in an AVL tree of height h

– If we can prove that $S(h)$ grows exponentially in h, then a tree with n nodes has a logarithmic height

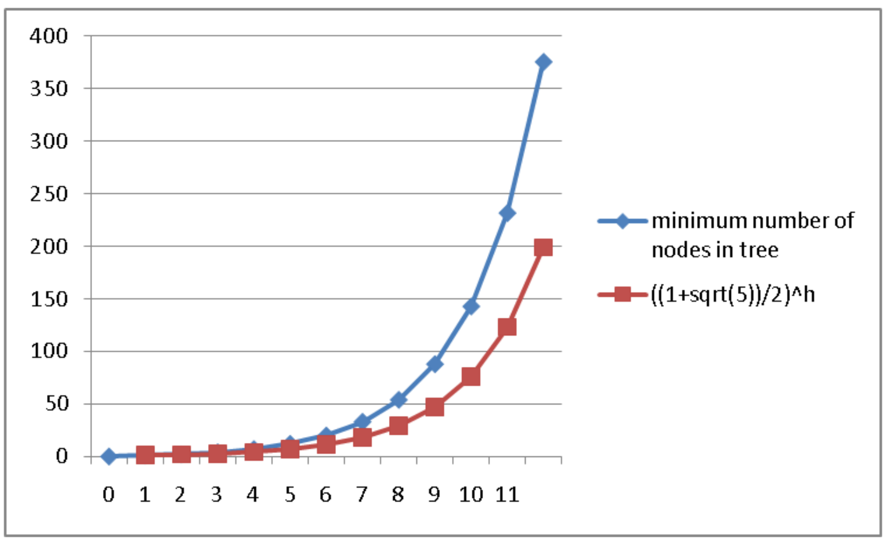
- Step 1: Define $S(h)$ inductively using AVL property
  - $S(-1)=0$, $S(0)=1$, $S(1)=2$
  - *For* $h \geq 1$, $S(h) = 1+S(h-1)+S(h-2)$

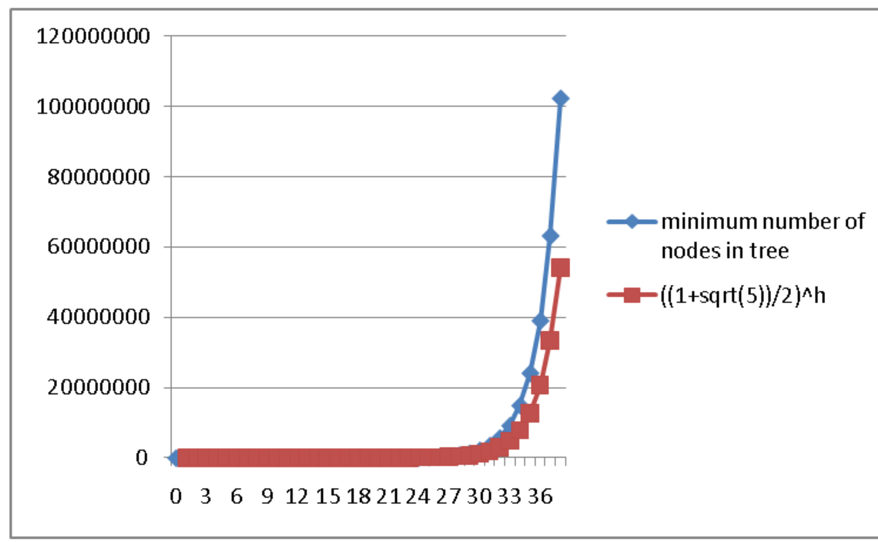


- Step 2: Show this recurrence grows really fast
  - Can prove for all h, $S(h) > \phi^{h} - 1$ where ϕ is the golden ratio, $(1+\sqrt{5})/2$, about 1.62
  - Growing faster than 1.6^{h} is "plenty exponential"
    - It does not grow faster than 2^{h}

Before we prove it

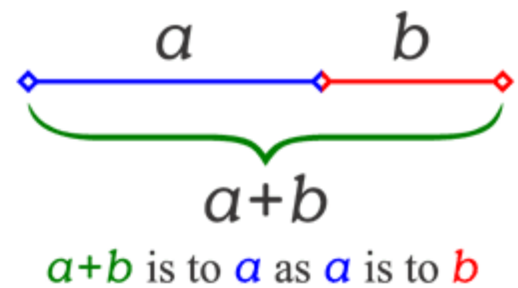
- Good intuition from plots comparing:
  - $S(h)$ computed directly from the definition
  - $((1+\sqrt{5})/2)^{h}$
- $S(h)$ is always bigger, up to trees with huge numbers of nodes
  - Graphs aren't proofs, so let's prove it





The Golden Ratio

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.62$$



This is a special number

- Aside: Since the Renaissance, many artists and architects have proportioned their work (e.g., length:height) to approximate the *golden ratio*: If `(a+b)/a = a/b`, then `a = ϕb`

- We will need one special arithmetic fact about ϕ :

$$
\begin{aligned}
\phi^2 &= ((1+5^{1/2})/2)^2 \\
&= (1 + 2*5^{1/2} + 5)/4 \\
&= (6 + 2*5^{1/2})/4 \\
&= (3 + 5^{1/2})/2 \\
&= 1 + (1 + 5^{1/2})/2 \\
&= 1 + \phi
\end{aligned}
$$

The proof

$S(-1)=0$, $S(0)=1$, $S(1)=2$
For $h \geq 1$, $S(h) = 1+S(h-1)+S(h-2)$

Theorem: For all $h \geq 0$, $S(h) > \phi^h - 1$

Proof: By induction on h

Base cases:

$S(0) = 1 > \phi^0 - 1 = 0$ $\qquad$ $S(1) = 2 > \phi^1 - 1 \approx 0.62$

Inductive case ($k > 1$):

Show $S(k+1) > \phi^{k+1} - 1$ assuming $S(k) > \phi^k - 1$ and $S(k-1) > \phi^{k-1} - 1$

$\mathbf{S(k+1)} = 1 + S(k) + S(k-1)$	by definition of S
$> 1 + \phi^k - 1 + \phi^{k-1} - 1$	by induction
$= \phi^k + \phi^{k-1} - 1$	by arithmetic (1-1=0)
$= \phi^{k-1} (\phi + 1) - 1$	by arithmetic (factor ϕ^{k-1})
$= \phi^{k-1} \phi^2 - 1$	by special property of ϕ
$= \boldsymbol{\phi^{k+1} - 1}$	by arithmetic (add exponents)

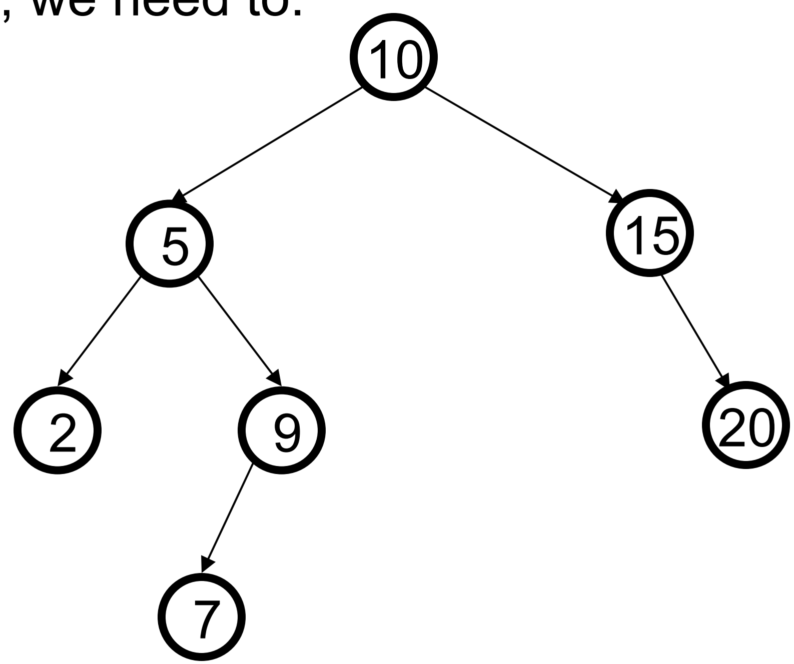
Good news

Proof means that if we have an AVL tree, then **`find`** is $O($**`log`** $n)$

- Recall logarithms of different bases > 1 differ by only a constant factor

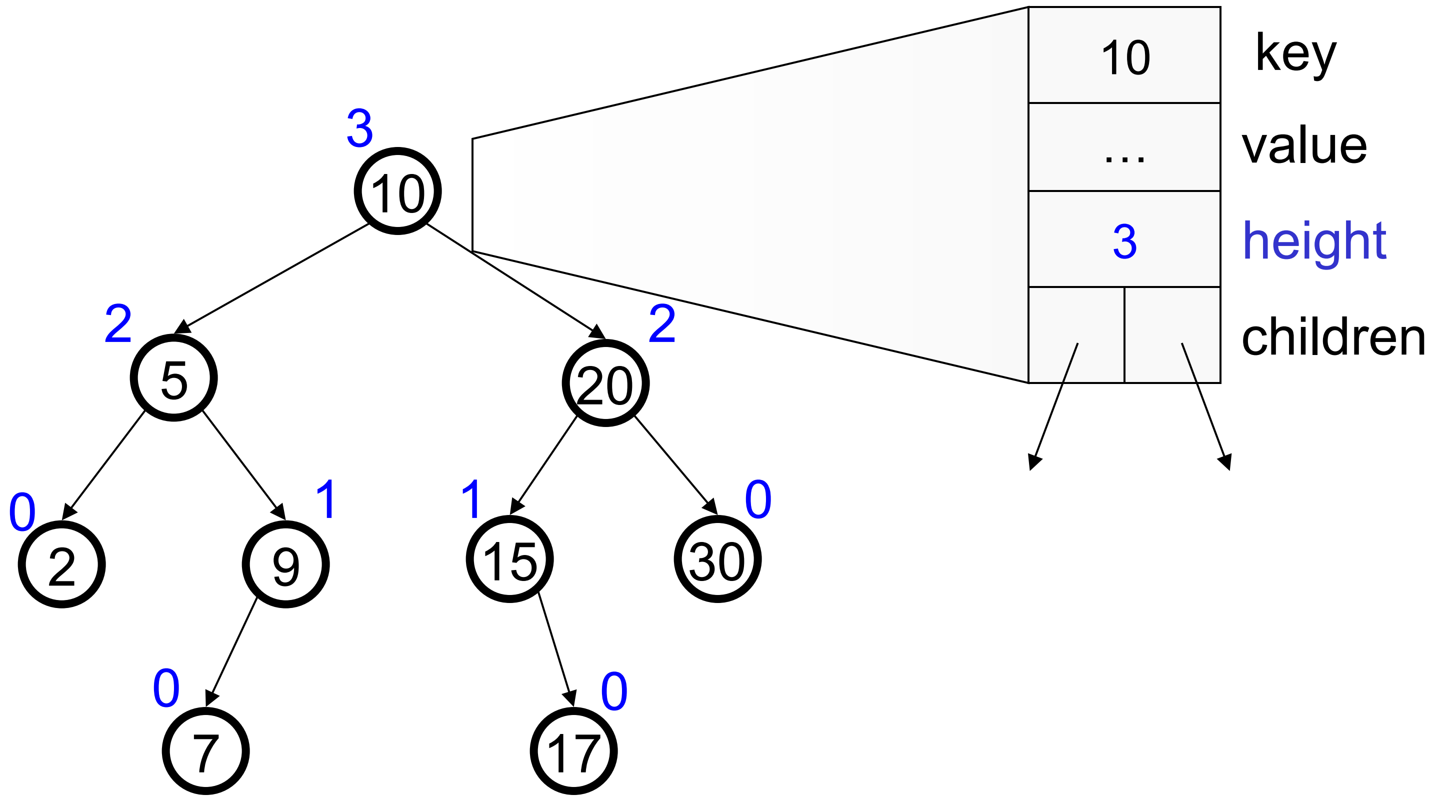
But as we insert and delete elements, we need to:

1. Track balance
2. Detect imbalance
3. Restore balance



Is this AVL tree balanced?
How about after `insert(30)`?

An AVL Tree



Track height at all times!

AVL tree operations

- **AVL `find`**:
  - Same as BST **`find`**

- **AVL `insert`**:
  - First BST **`insert`**, *then* check balance and potentially "fix" the AVL tree
  - Four different imbalance cases

- **AVL `delete`**:
  - The "easy way" is lazy deletion
  - Otherwise, do the deletion and then have several imbalance cases (we will likely skip this but post slides for those interested)