# CSE373: Data Structures & Algorithms Lecture 10: Implementing Union-Find

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# *The plan*

Last lecture:

- What are *disjoint sets*
  - And how are they “the same thing” as *equivalence relations*
- The union-find ADT for disjoint sets
- Applications of union-find

Now:

- **Basic implementation of the ADT with “up trees”**
- Optimizations that make the implementation much faster

# *Our goal*

- Start with an initial partition of $n$ subsets
  - Often 1-element sets, e.g., {1}, {2}, {3}, …, {$n$}

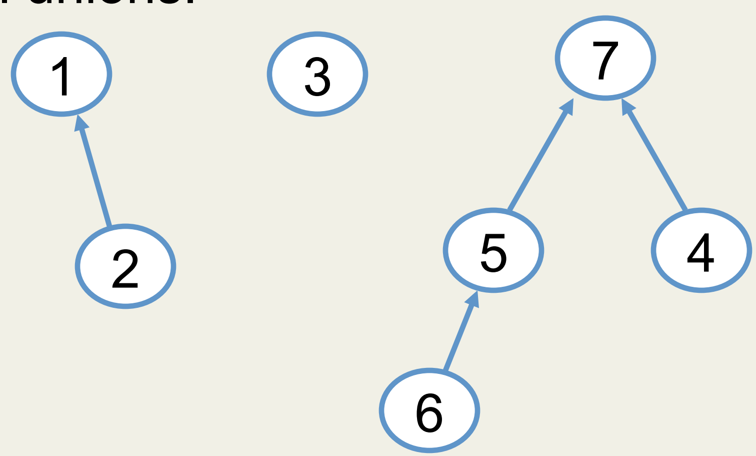
- May have $m$ **`find`** operations and up to $n$-1 **`union`** operations in any order
  - After $n$-1 **`union`** operations, every **`find`** returns same 1 set

- If total for all these operations is $O(m+n)$, then amortized $O(1)$
  - We will get very, very close to this
  - $O(1)$ worst-case is impossible for **`find`** *and* **`union`**
    - Trivial for one *or* the other

# Up-tree data structure

- Tree with:
  - No limit on branching factor
  - References from children to parent
- Start with *forest* of 1-node trees


- Possible forest after several unions:
  - Will use roots for set names

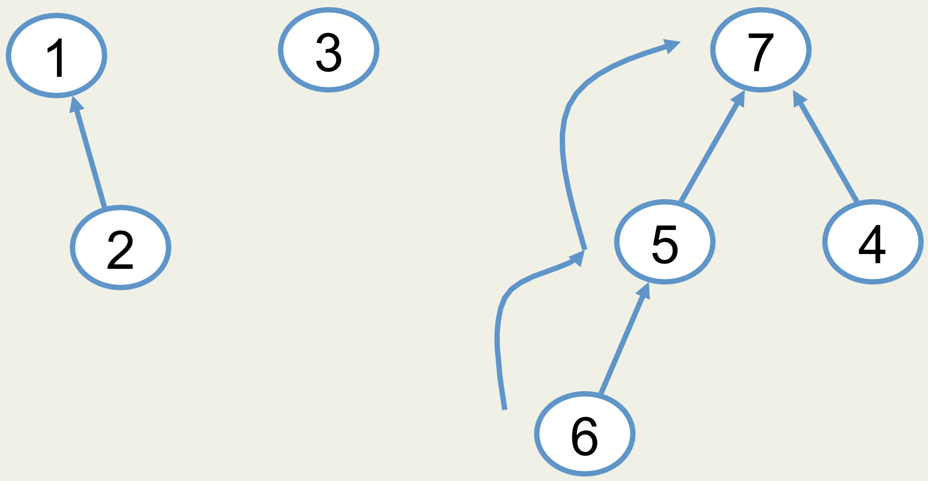


# *Find*

**`find`**(**`x`**):

- *Assume* we have *O*(1) access to each node
  - Will use an array where index **`i`** holds node **`i`**
- Start at **`x`** and follow parent pointers to root
- Return the root

**`find`**(6) = 7

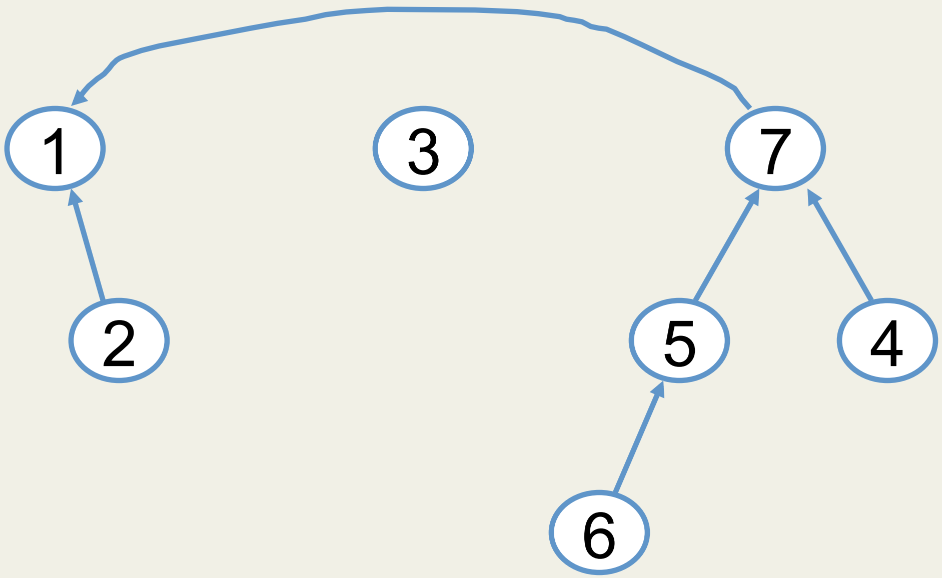


# Union

**`union(x,y)`**:

- Assume **`x`** and **`y`** are roots
  - If they are not, just find the roots of their trees
- Assume distinct trees (else do nothing)
- Change root of one to have parent be the root of the other
  - Notice no limit on branching factor

**`union`**(1,7)

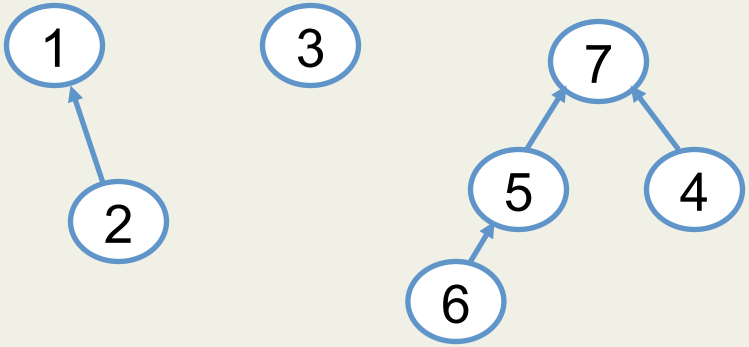


# *Simple implementation*

- If set elements are contiguous numbers (e.g., 1,2,…,*n*), use an array of length *n* called **`up`**
  - Starting at index 1 on slides
  - Put in array index of parent, with 0 (or -1, etc.) for a root
- Example:


| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| up | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

- Example:




| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| up | 0 | 1 | 0 | 7 | 7 | 5 | 0 |

- If set elements are not contiguous numbers, could have a separate dictionary to map elements (keys) to numbers (values)

# Implement operations

```
// assumes x in range 1,n
int find(int x) {
  while(up[x] != 0) {
     x = up[x];
  }
  return x;
}
```

```
// assumes x,y are roots
void union(int x, int y){
  // y = find(y)
  // x = find(x)
  up[y] = x;
}
```

- Worst-case run-time for **`union`**?
- Worst-case run-time for **`find`**?
- Worst-case run-time for *m* **`find`**s and *n*-1 **`union`**s?

# Implement operations

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  // x = find(x)
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}
```

- Worst-case run-time for **`union`**? *O*(1) (with our assumption…)
- Worst-case run-time for **`find`**? *O(n)*
- Worst-case run-time for *m* **`finds`** and *n*-1 **`unions`**? *O(m *n)*

# *The plan*

Last lecture:

- What are *disjoint sets*
  - And how are they “the same thing” as *equivalence relations*
- The union-find ADT for disjoint sets
- Applications of union-find

Now:

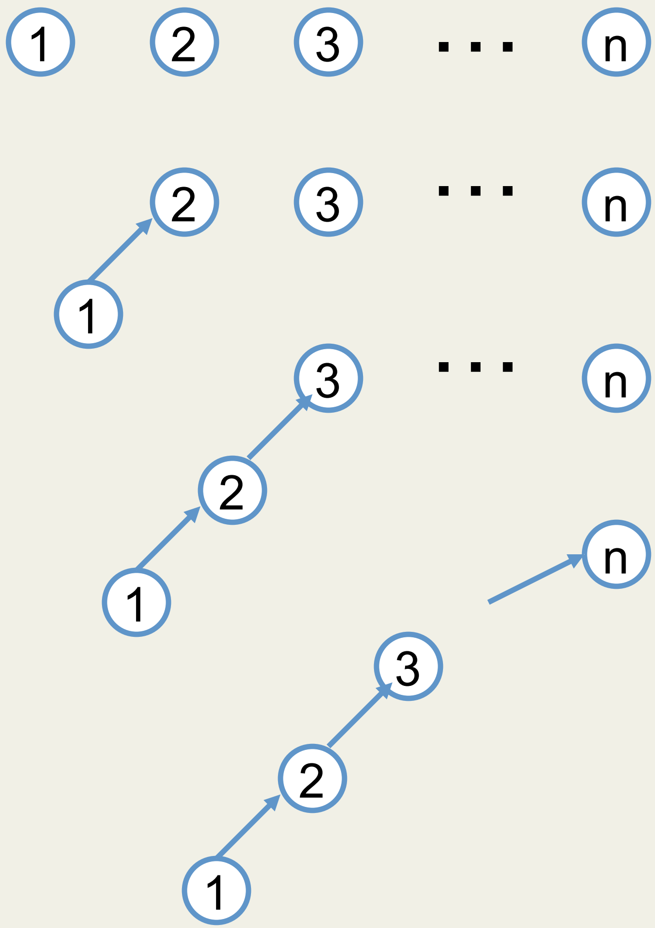
- Basic implementation of the ADT with “up trees”
- Optimizations that make the implementation much faster

# *Two key optimizations*

1. Improve **`union`** so it stays *O(1)* but makes **`find`** $O(\mathbf{log}\ n)$
   - So *m* **`find`**s and *n*-1 **`union`**s is $O(m\ \mathbf{log}\ n + n)$
   - *Union-by-size:* connect smaller tree to larger tree

2. Improve **`find`** so it becomes even faster
   - Make *m* **`find`**s and *n*-1 **`union`**s ***almost*** $O(m + n)$
   - *Path-compression:* connect directly to root during finds

# *The bad case to avoid*




`union`(2,1)

`union`(3,2)

:

.

`union`(*n*,*n*-1)

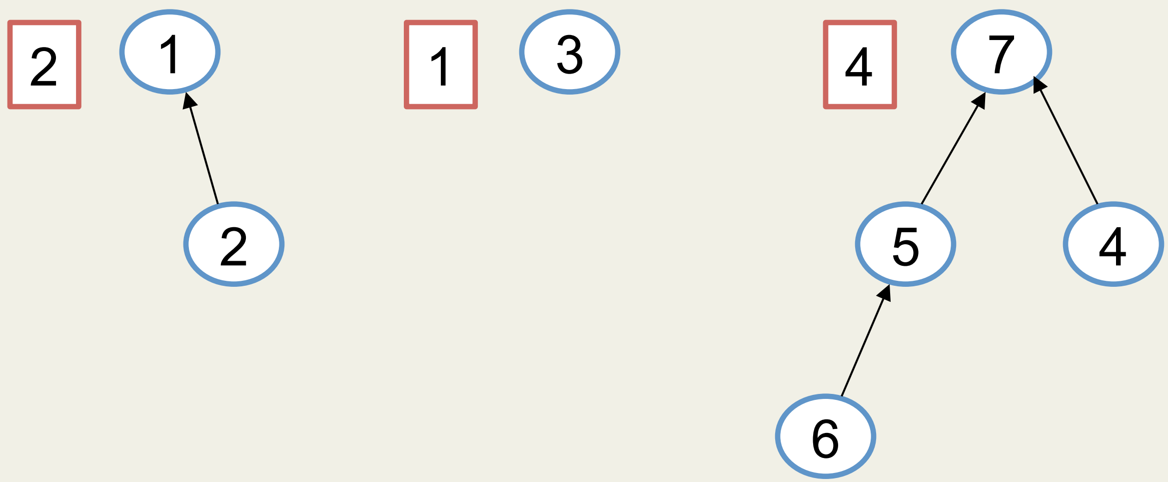
`find`(1) *n* steps!!

# *Weighted union*

Weighted union:

- Always point the *smaller* (total # of nodes) tree to the root of the larger tree

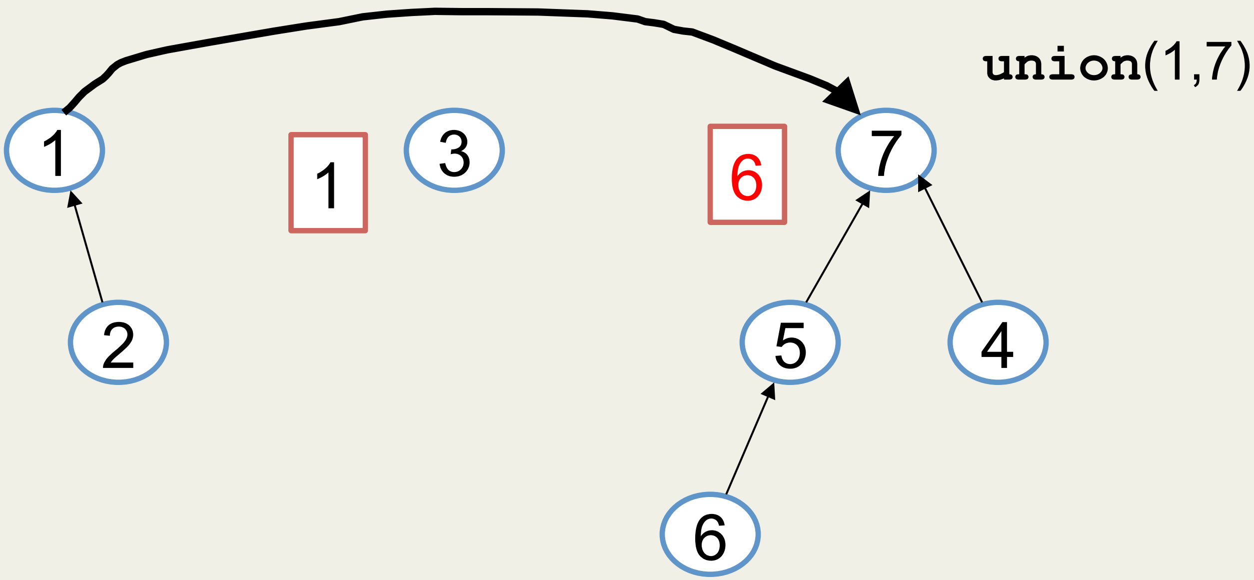
**union**(1,7)



# *Weighted union*

Weighted union:

– Always point the *smaller* (total # of nodes) tree to the root of the larger tree



# *Weighted union*

Weighted union:

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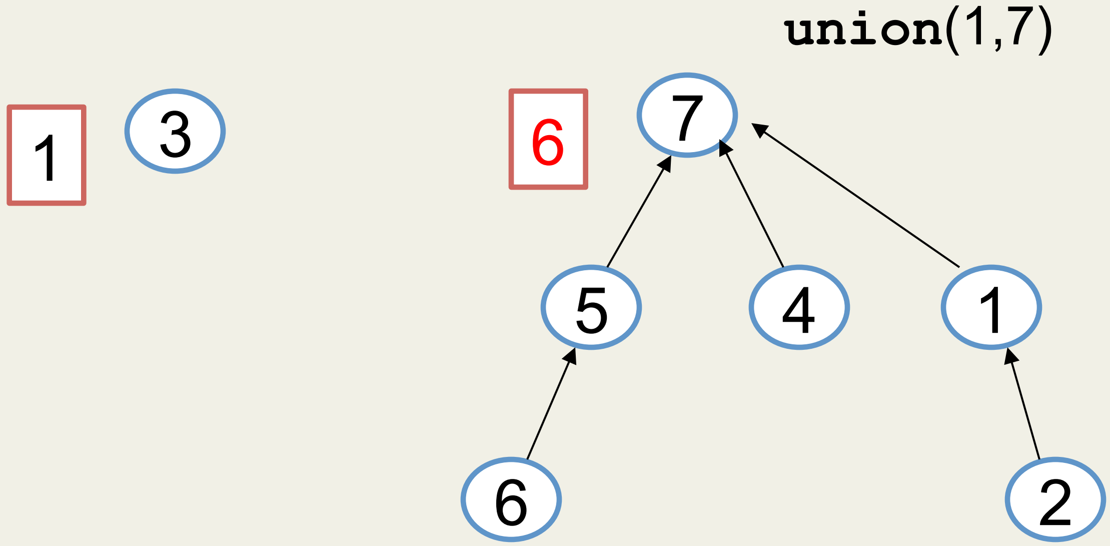
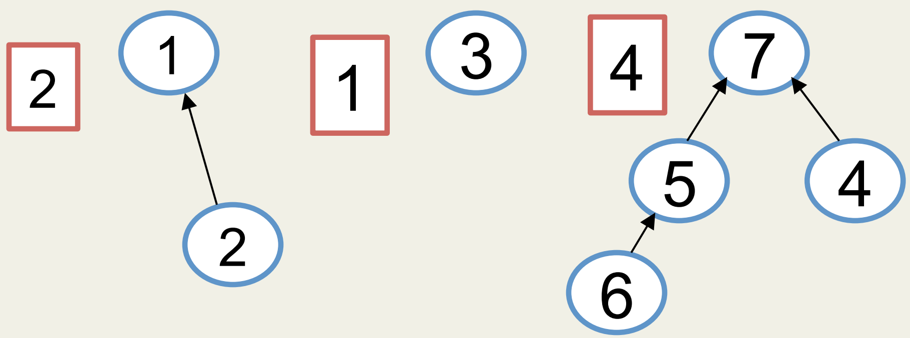


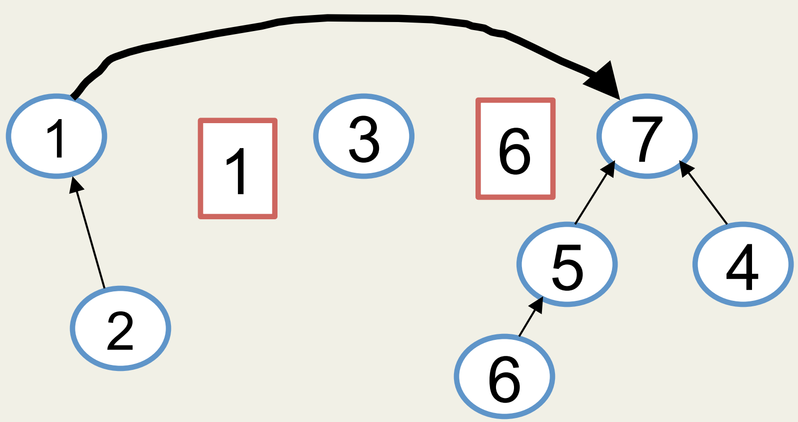
# *Array implementation*

Keep the weight (number of nodes in a second array)

- Or have one array of objects with two fields




| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| up | 0 | 1 | 0 | 7 | 7 | 5 | 0 |
| weight | 2 | | 1 | | | | 4 |

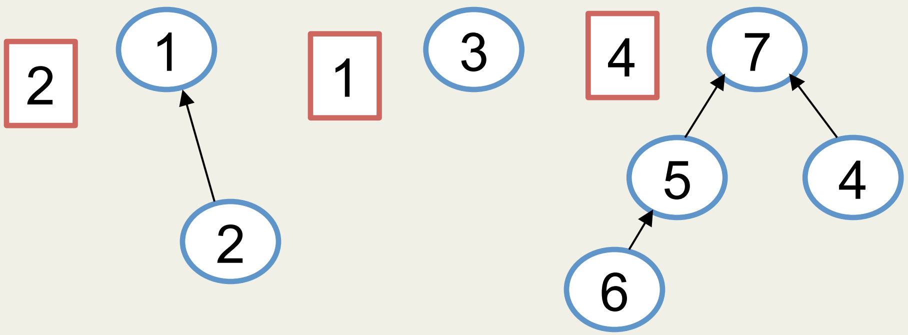



| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| up | 7 | 1 | 0 | 7 | 7 | 5 | 0 |
| weight | 2 | | 1 | | | | 6 |

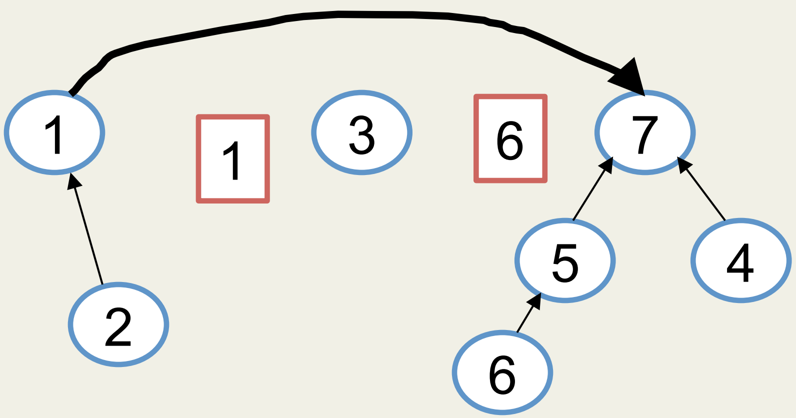
# *Nifty trick*

Actually we do not need a second array…

- Instead of storing 0 for a root, store negation of weight
- So up value < 0 means a root

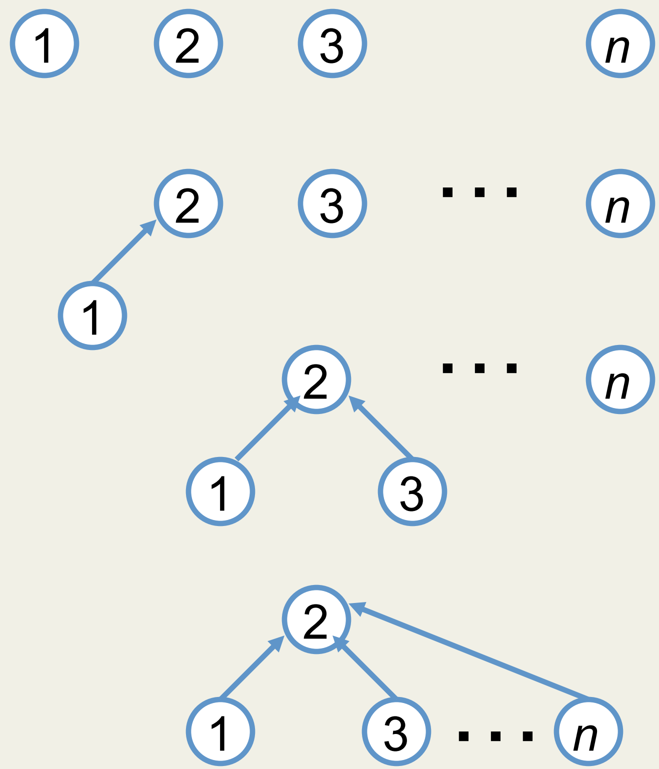



| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| up | -2 | 1 | -1 | 7 | 7 | 5 | -4 |




| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| up | 7 | 1 | -1 | 7 | 7 | 5 | -6 |

# *Bad example? Great example…*




**`union`**(2,1)

**`union`**(3,2)

⋮

**`union`**($n$,$n$-1)

**`find`**(1) *constant here*

# *General analysis*

- Showing that one worst-case example is now good is *not* a proof that the worst-case has improved

- So let's prove:
  - **`union`** is still $O(1)$ – this is fairly easy to show
  - **`find`** is now $O($**`log`** $n)$

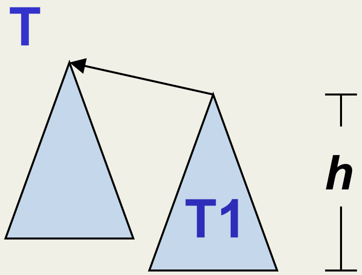
- Claim: If we use weighted-union, an up-tree of height $h$ has at least $2^h$ nodes
  - Proof by induction on $h$…

# *Exponential number of nodes*

P(*h*)= With weighted-union, up-tree of height *h* has at least $2^h$ nodes

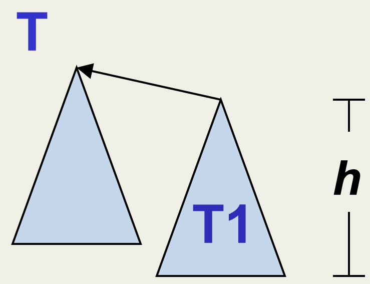
Proof by induction on *h*…

- Base case: *h* = 0: The up-tree has 1 node and $2^0$= 1
- Inductive case: Assume P(*h*) and show P(*h*+1)
  - A height *h*+1 tree T has at least one height *h* child T1
  - T1 has at least $2^h$ nodes by induction
  - And T has *at least* as many nodes not in T1 than in T1
    - Else weighted-union would have had T point to T1, not T1 point to T (!!)
  - So total number of nodes is *at least* $2^h + 2^h = 2^{h+1}$



# *The key idea*

Intuition behind the proof: No one child can have more than half the nodes

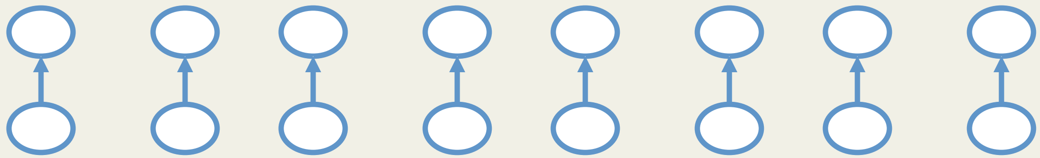



So, as usual, if number of nodes is exponential in height, then height is logarithmic in number of nodes

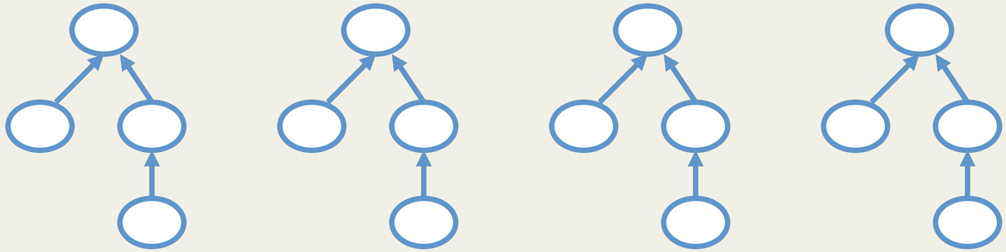
So **`find`** is $O($**`log`** $n)$

# The new worst case

n/2 Weighted Unions

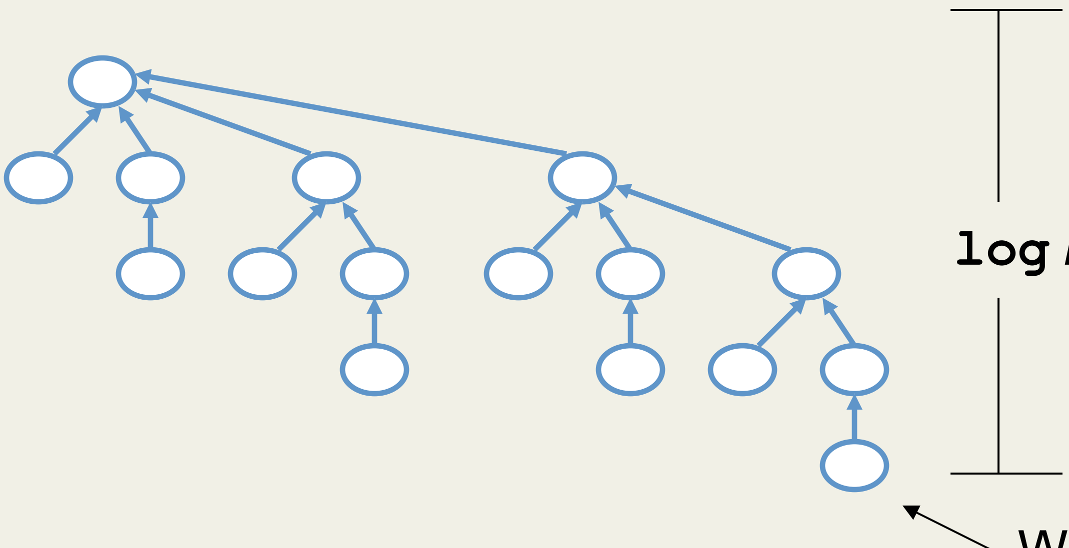


n/4 Weighted Unions



# *The new worst case (continued)*

After n/2 + n/4 + …+ 1 Weighted Unions:




Worst find

Height grows by 1 a total of **`log`** *n* times

# *What about union-by-height*

We could store the height of each root rather than number of descendants (weight)

- Still guarantees logarithmic worst-case find
  - Proof left as an exercise if interested

- But does not work well with our next optimization
  - Maintaining height becomes inefficient, but maintaining weight still easy

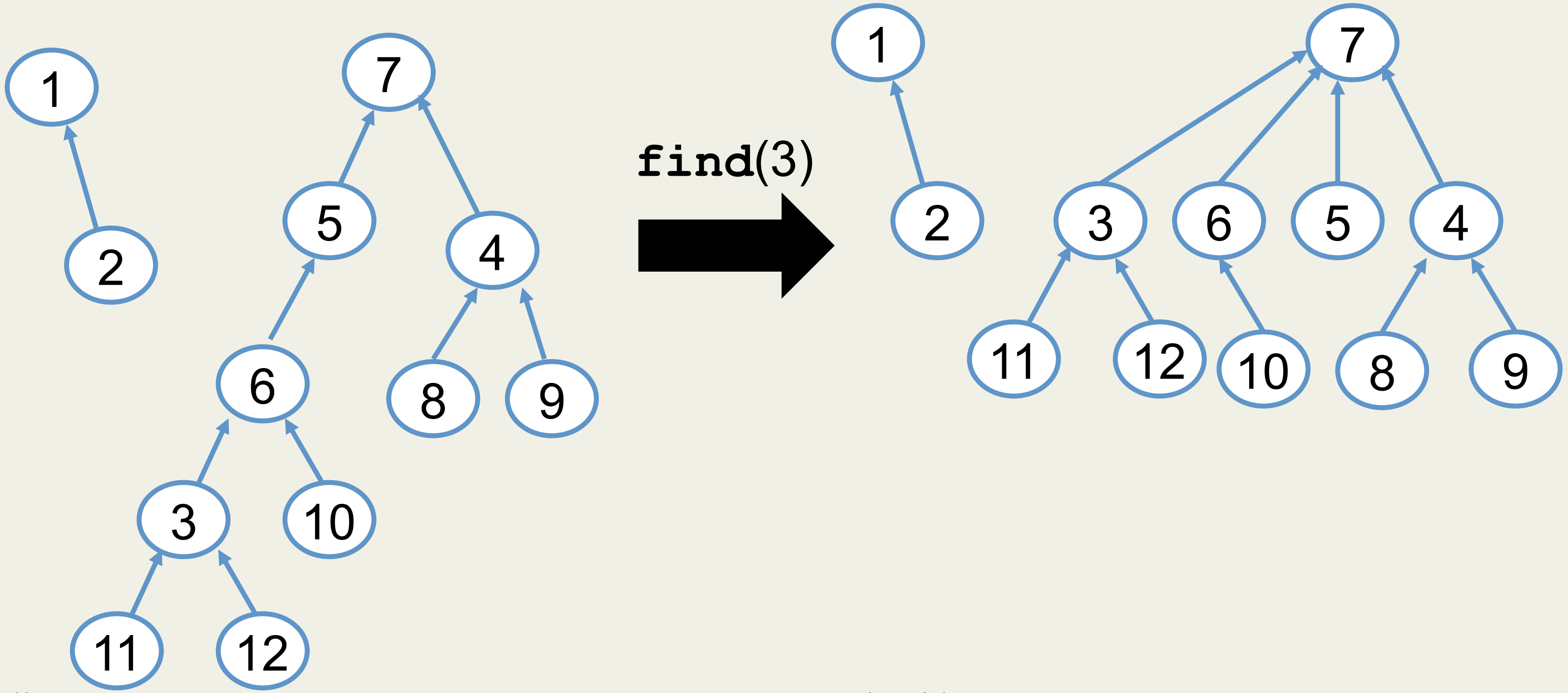
# *Two key optimizations*

1. Improve **`union`** so it stays *O(1)* but makes **`find`** *O*(**`log`** *n*)
   - So *m* **`find`**s and *n*-1 **`union`**s is *O*(*m* **`log`** *n* + *n*)
   - *Union-by-size:* connect smaller tree to larger tree

2. Improve **`find`** so it becomes even faster
   - Make *m* **`find`**s and *n*-1 **`union`**s ***almost*** *O*(*m* + *n*)
   - *Path-compression:* connect directly to root during finds

# *Path compression*

- Simple idea: As part of a **`find`**, change each encountered node's parent to point directly to root
  - Faster future **`find`**s for everything on the path (and their descendants)



# Solution

*(good exampleof psuedocode!)*

```
// performs path compression
find(i)
    // find root
    r = i
    while up[r] > 0
        r = up[r]

    // compress path
    if i == r
        return r

    old_parent = up[i]
    while (old_parent != r)
        up[i] = r
        i = old_parent
        old_parent = up[i]

    return r
```

# *So, how fast is it?*

A single worst-case **`find`** could be $O(\texttt{log}\ n)$

- But only if we did a lot of worst-case unions beforehand
- And path compression will make future finds faster

Turns out the amortized worst-case bound is much better than $O(\texttt{log}\ n)$

- We won't *prove* it – see text if curious
- But we will *understand* it:
  - How it is *almost* $O(1)$
  - Because total for $m$ **`find`**s and $n$-1 **`union`**s is *almost* $O(m+n)$

# A really slow-growing function

`log*(`*x*) is the minimum number of times you need to apply "`log` of `log` of `log` of" to go from *x* to a number <= 1

For just about every number we care about, `log*`(*x*) is *5* (!)

If $x <= 2^{65536}$ then `log*` $x <= 5$

- log* 2 = 1
- log* 4 = log* $2^2$ = 2
- log* 16 = log* $2^{(2^2)}$ = 3 (log(log(log(16))) = 1)
- log* 65536 = log* $2^{((2^2)^2)}$ = 4 (log(log(log(log(65536)))) = 1)
- log* $2^{65536}$ = …………… = 5

# *Wait…. how big?*

Just how big is $2^{65536}$

Well $2^{10} = 1024$

$2^{20} = 1048576$

$2^{30} = 1073741824$

$2^{100} = 1.125 \times 10^{15}$

$2^{65536} =$ ... pretty big

But its still not technically constant

# *Almost linear*

- Turns out total time for $m$ **`finds`** and $n$-1 **`unions`** is:
  $O((m+n)*($**`log*`** $(m+n))$
  - Remember, if $m+n < 2^{65536}$ then **`log*`** $(m+n) < 5$

- At this point, it feels almost silly to mention it, but even that bound is not tight…
  - "Inverse Ackerman's function" grows even more slowly than **`log*`**
    - Inverse because Ackerman's function grows really fast
    - Function also appears in combinatorics and geometry
    - For any number you can possibly imagine, it is < 4
  - Can replace **`log*`** with "Inverse Ackerman's" in bound

# *Theory and terminology*

- Because **`log*`** or Inverse Ackerman's grows so incredibly slowly
  - For all practical purposes, amortized bound is constant, i.e., total cost is linear
  - We say "near linear" or "effectively linear"
- Need weighted-union and path-compression for this bound
  - Path-compression changes height but not weight, so they interact well
- As always, asymptotic analysis is separate from "coding it up"