CSE 417: Algorithms and Computational Complexity

Winter 2007

Larry Ruzzo

Divide and Conquer Algorithms

The Divide and Conquer Paradigm

Outline:

- General Idea
- Review of Merge Sort
- Why does it work?
  - Importance of balance
  - Importance of super-linear growth
- Two interesting applications
  - Polynomial Multiplication
  - Matrix Multiplication
- Finding & Solving Recurrences

Algorithm Design Techniques

Divide & Conquer

- Reduce problem to one or more sub-problems of the same type
- Typically, each sub-problem is at most a constant fraction of the size of the original problem
  - e.g. Mergesort, Binary Search, Strassen's Algorithm, Quicksort (kind of)

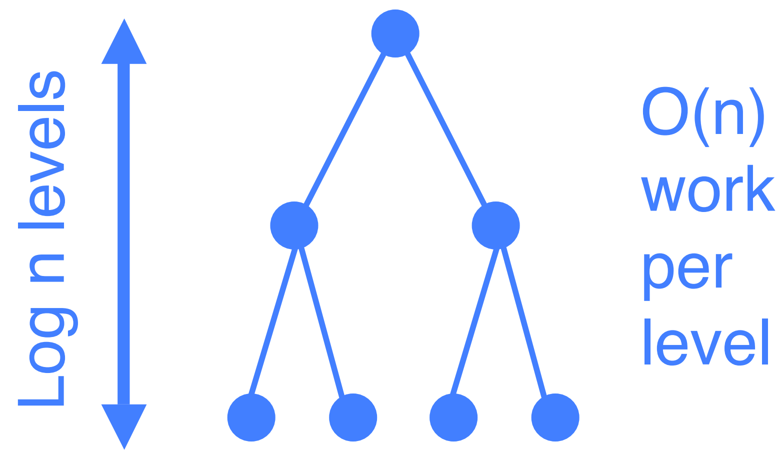
Mergesort (review)

Mergesort: (recursively) sort 2 half-lists, then merge results.

$T(n)=2T(n/2)+cn, \quad n\geq 2$

$T(1)=0$

Solution: $O(n \log n)$
(details later)



Why Balanced Subdivision?

Alternative "divide & conquer" algorithm:

Sort n-1

Sort last 1

Merge them

$T(n)=T(n-1)+T(1)+3n$ for $n \geq 2$

$T(1)=0$

Solution: $3n + 3(n-1) + 3(n-2) \ldots = \Theta(n^2)$

Another D&C Approach

Suppose we've already invented DumbSort, taking time n^2

Try *Just One Level* of divide & conquer:

DumbSort(first n/2 elements)

DumbSort(last n/2 elements)

Merge results

Time: $2\ (n/2)^2 + n = n^2/2 + n << n^2$

Almost twice as fast!

D&C in a nutshell

Another D&C Approach, cont.

Moral 1: "two halves are better than a whole"

Two problems of half size are *better* than one full-size problem, even given the O(n) overhead of recombining, since the base algorithm has *super-linear* complexity.

Moral 2: "If a little's good, then more's better"

two levels of D&C would be almost 4 times faster, 3 levels almost 8, etc., even though overhead is growing. Best is usually full recursion down to some small constant size (balancing "work" vs "overhead").

Another D&C Approach, cont.

Moral 3: unbalanced division less good:

$(.1n)^2 + (.9n)^2 + n = .82n^2 + n$

The 18% savings compounds significantly if you carry recursion to more levels, actually giving O(nlogn), but with a bigger constant. So worth doing if you can't get 50-50 split, but balanced is better if you can.

This is intuitively why Quicksort with random splitter is good – badly unbalanced splits are rare, and not instantly fatal.

$(1)^2 + (n-1)^2 + n = n^2 - 2n + 2 + n$

Little improvement here.

5.4 Closest Pair of Points

Closest Pair of Points

Closest pair. Given n points in the plane, find a pair with smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

↑ fast closest pair inspired fast algorithms for these problems

Brute force. Check all pairs of points p and q with $\Theta(n^2)$ comparisons.

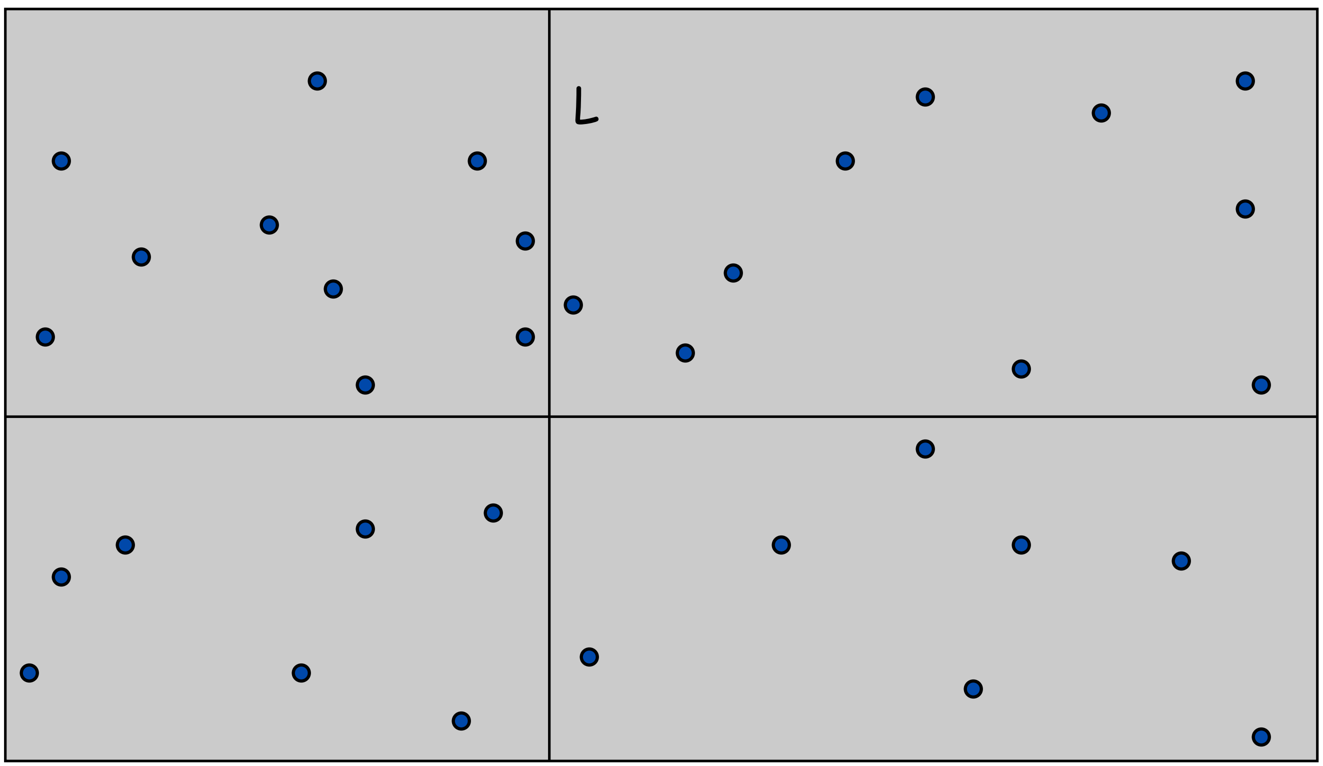
1-D version. O(n log n) easy if points are on a line.

Assumption. No two points have same x coordinate.

↑ to make presentation cleaner

Closest Pair of Points: First Attempt

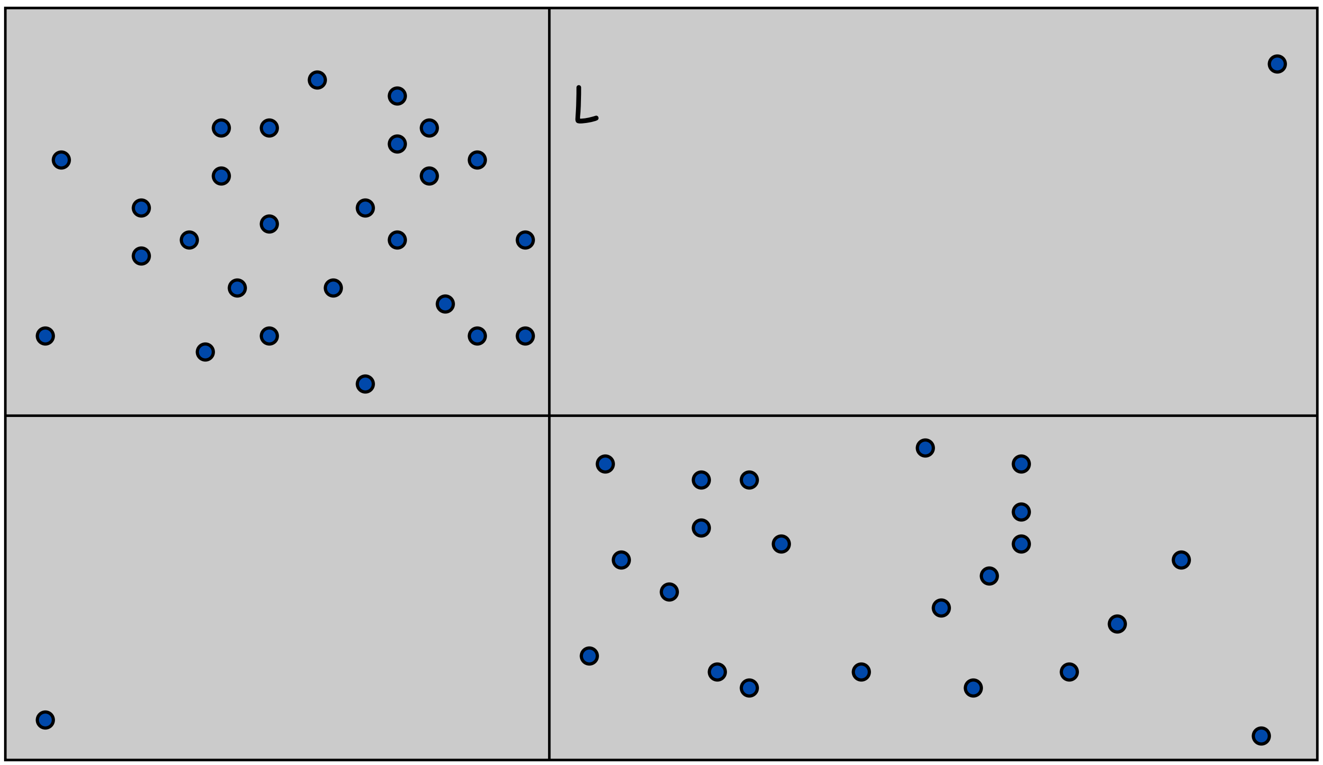
Divide. Sub-divide region into 4 quadrants.



Closest Pair of Points: First Attempt

Divide. Sub-divide region into 4 quadrants.

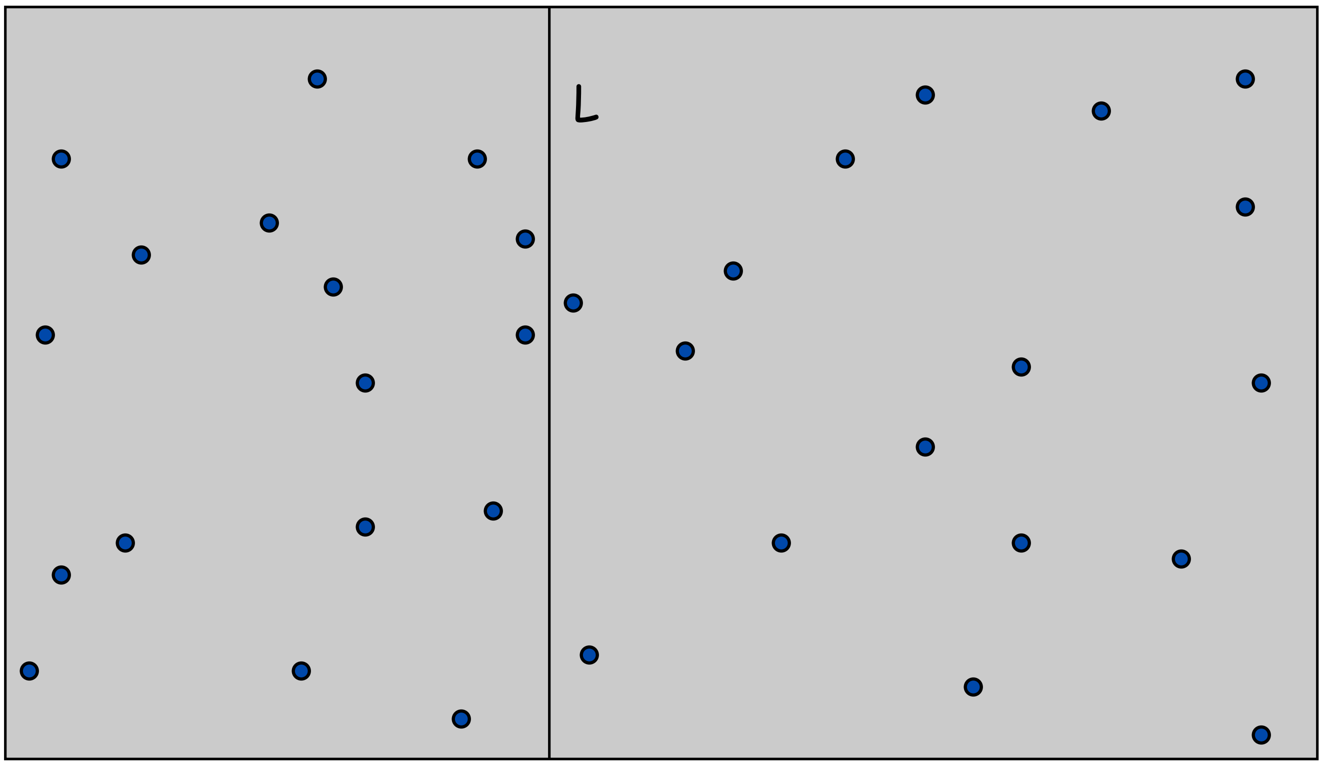
Obstacle. Impossible to ensure n/4 points in each piece.



Closest Pair of Points

Algorithm.

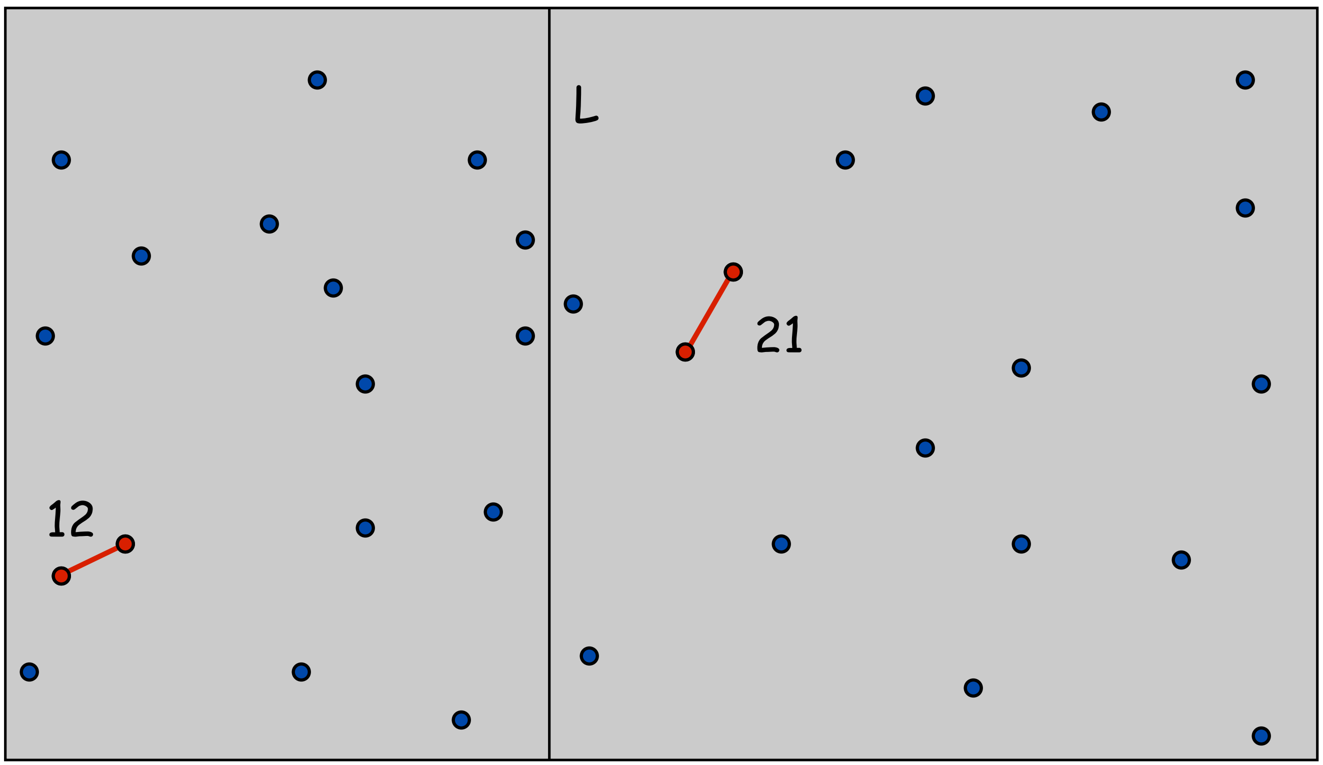
- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.



Closest Pair of Points

Algorithm.

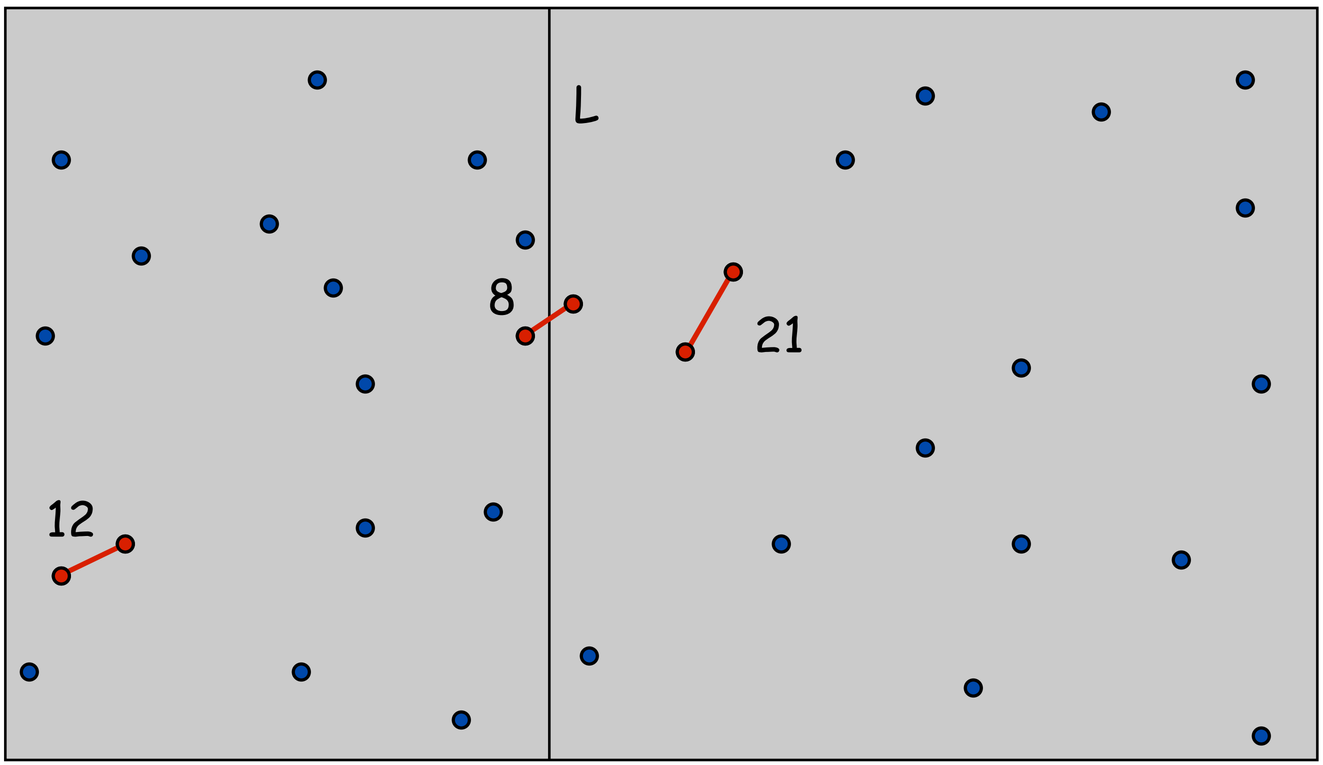
- Divide: draw vertical line L so that roughly $\frac{1}{2}$n points on each side.
- Conquer: find closest pair in each side recursively.



Closest Pair of Points

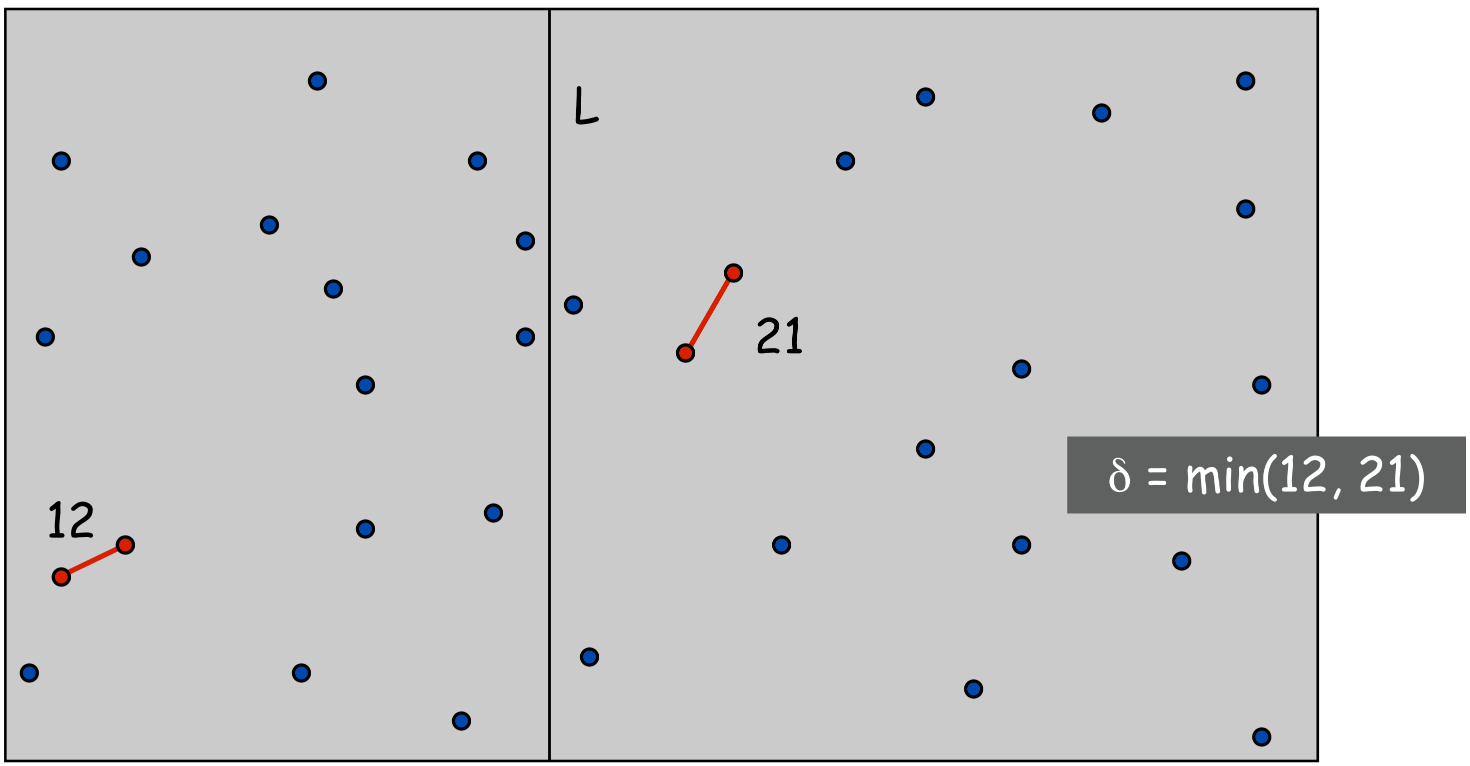
Algorithm.

- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side. ← seems like $\Theta(n^2)$
- Return best of 3 solutions.



Closest Pair of Points

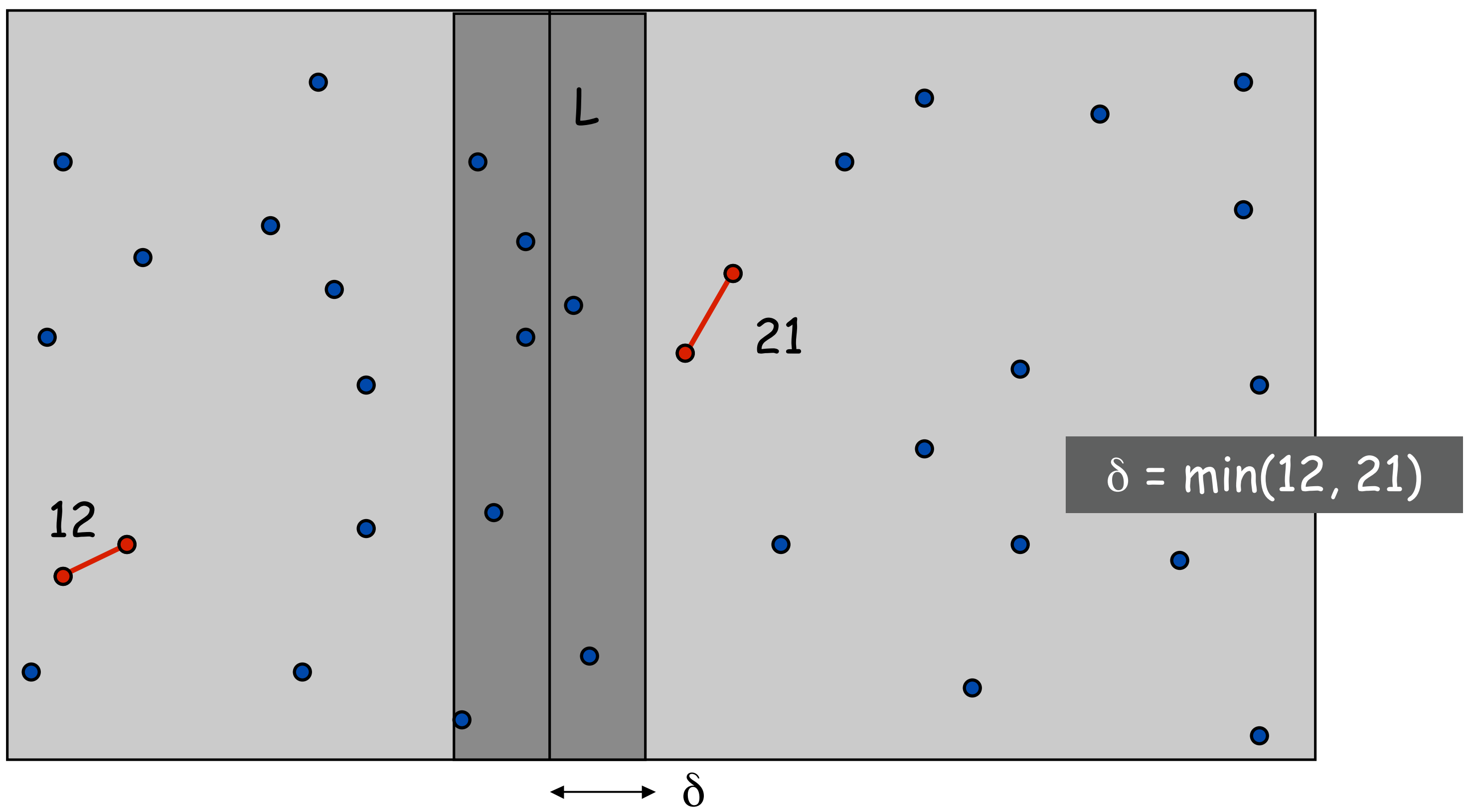
Find closest pair with one point in each side, assuming that distance < δ.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

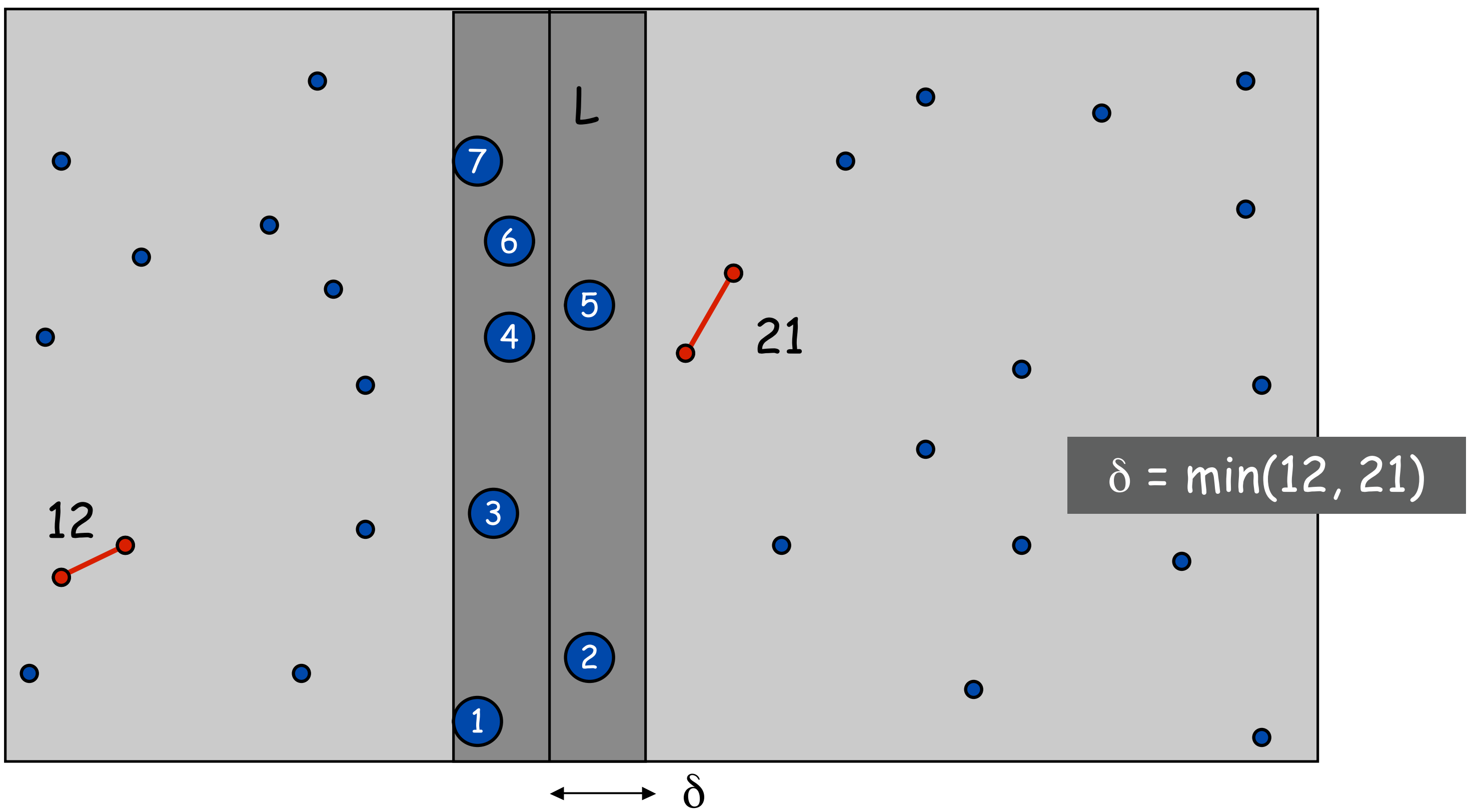
- Observation: only need to consider points within δ of line L.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

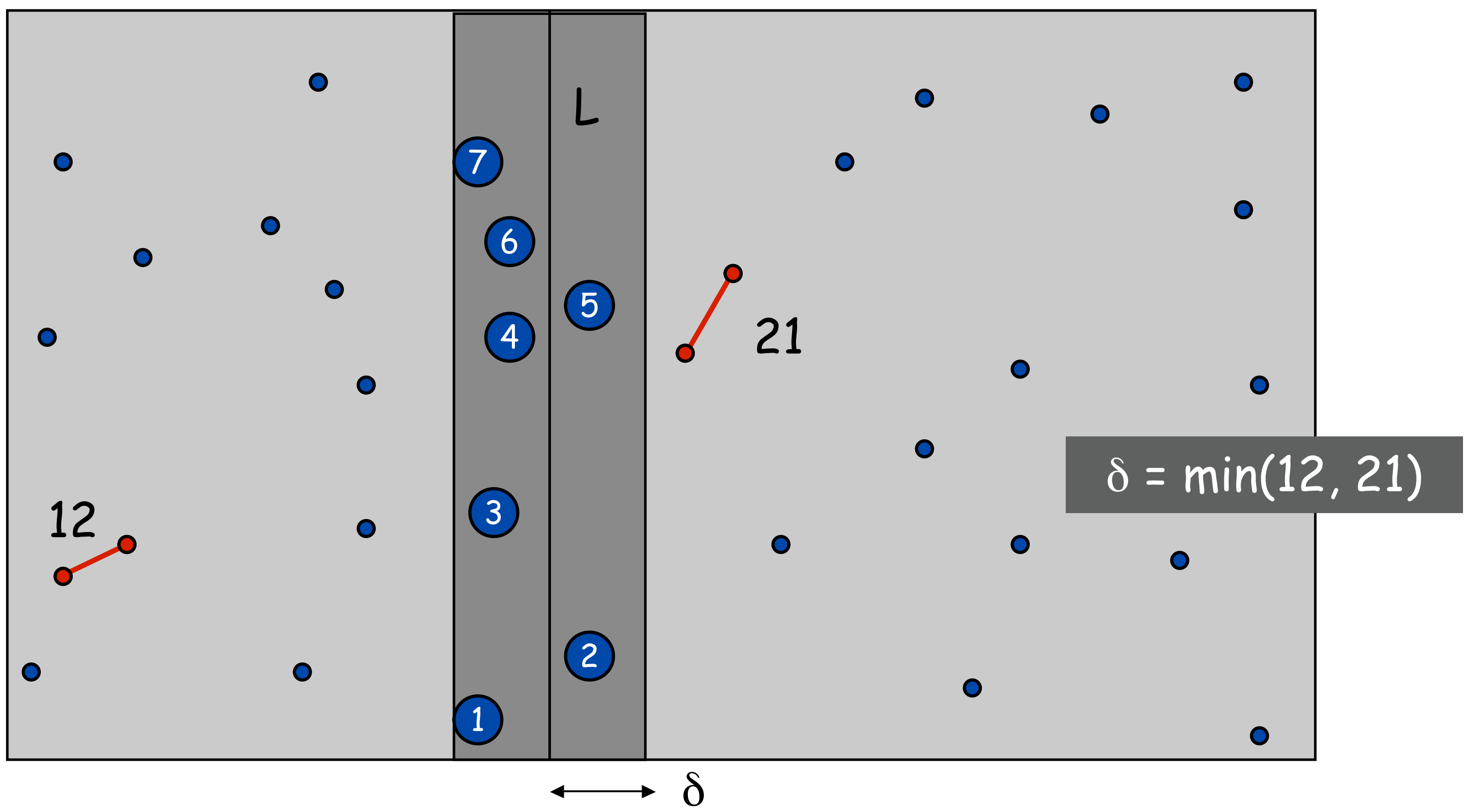
- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y coordinate.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y coordinate.
- Only check distances of those within 11 positions in sorted list!



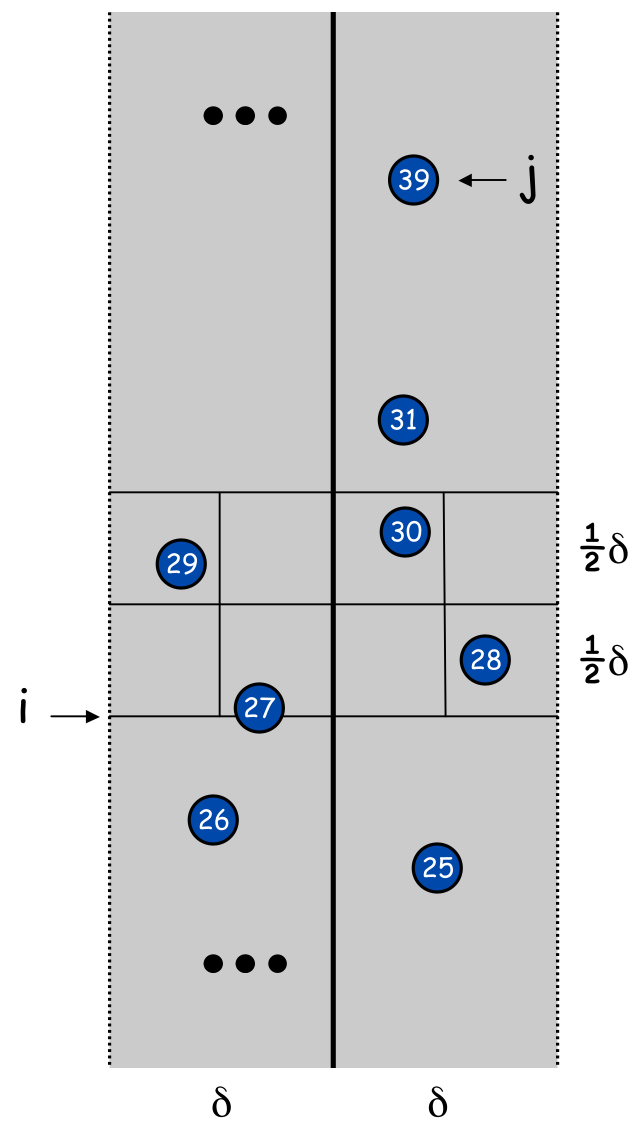
Closest Pair of Points

Def. Let s_i be the point in the 2δ-strip, with the i^{th} smallest y-coordinate.

Claim. If $|i - j| \geq 8$, then the distance between s_i and s_j is at least δ.

Pf.

- No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.
- only 8 boxes



Closest Pair Algorithm

```
Closest-Pair(p₁, …, pₙ) {
   if(n <= ??) return ??

   Compute separation line L such that half the points
   are on one side and half on the other side.

   δ₁ = Closest-Pair(left half)
   δ₂ = Closest-Pair(right half)
   δ  = min(δ₁, δ₂)

   Delete all points further than δ from separation line L

   Sort remaining points p[1]…p[m] by y-coordinate.

   for i = 1..m
      k = 1
      while i+k <= m && p[i+k].y < p[i].y + δ
        δ = min(δ, distance between p[i] and p[i+k]);
        k++;

   return δ.
}
```

Going From Code to Recurrence

Carefully define what you're counting, and write it down!

"Let C(n) be the number of comparisons between sort keys used by MergeSort when sorting a list of length n ≥ 1"

In code, clearly separate **base case** from **recursive case,** highlight **recursive calls,** and **operations being counted.**

Write Recurrence(s)

Closest Pair Algorithm

Base Case

Basic operations: distance calcs

Recursive calls (2)

```
Closest-Pair(p1, …, pn) {
   if(n <= 1) return ∞

   Compute separation line L such that half the points
   are on one side and half on the other side.

   δ1 = Closest-Pair(left half)
   δ2 = Closest-Pair(right half)
   δ  = min(δ1, δ2)

   Delete all points further than δ from separation line L

   Sort remaining points p[1]…p[m] by y-coordinate.

   for i = 1..m
      k = 1
      while i+k <= m && p[i+k].y < p[i].y + δ
        δ = min(δ, distance between p[i] and p[i+k]);
        k++;

   return δ.
}
```

Basic operations at this recursive level

0

2T(n / 2)

O(n)

Closest Pair of Points: Analysis

Running time.

$$T(n) \le \begin{cases} 0 & n = 1 \\ 2T(n/2) + 7n & n > 1 \end{cases} \Rightarrow T(n) = O(n \log n)$$

BUT - that's only the number of distance calculations

Closest Pair Algorithm

Base Case

Basic operations: comparisons

Recursive calls (2)

```
Closest-Pair(p1, …, pn) {
   if(n <= 1) return ∞                                          0

   Compute separation line L such that half the points
   are on one side and half on the other side.                   O(n log n)

   δ1 = Closest-Pair(left half)
   δ2 = Closest-Pair(right half)                                 2T(n / 2)
   δ  = min(δ1, δ2)                                              1

   Delete all points further than δ from separation line L       O(n)

   Sort remaining points p[1]…p[m] by y-coordinate.              O(n log n)

   for i = 1..m
      k = 1
      while i+k <= m && p[i+k].y < p[i].y + δ
         δ = min(δ, distance between p[i] and p[i+k]);           O(n)
         k++;

   return δ.
}
```

Basic operations at this recursive level

Closest Pair of Points: Analysis

Running time.

$$T(n) \le \begin{cases} 0 & n = 1 \\ 2T(n/2) + O(n \log n) & n > 1 \end{cases} \Rightarrow T(n) = O(n \log^2 n)$$

Q. Can we achieve O(n log n)?

A. Yes. Don't sort points from scratch each time.

- Sort by x at top level only.
- Each recursive call returns δ and list of all points sorted by y
- Sort by merging two pre-sorted lists.

$$T(n) \le 2T(n/2) + O(n) \Rightarrow T(n) = O(n \log n)$$

5.5 Integer Multiplication

Integer Arithmetic

Add. Given two n-digit integers a and b, compute a + b.

- O(n) bit operations.

Multiply. Given two n-digit integers a and b, compute a × b.

- Brute force solution: $\Theta(n^2)$ bit operations.

1	1	1	1	1	1	0	1	
	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0

Add

								1	1	0	1	0	1	0	1
							*	0	1	1	1	1	1	0	1
								1	1	0	1	0	1	0	1
							0	0	0	0	0	0	0	0	0
						1	1	0	1	0	1	0	1	0	
					1	1	0	1	0	1	0	1	0		
				1	1	0	1	0	1	0	1	0			
			1	1	0	1	0	1	0	1	0				
		1	1	0	1	0	1	0	1	0					
	0	0	0	0	0	0	0	0	0						
0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	1

Multiply

Divide-and-Conquer Multiplication: Warmup

To multiply two n-digit integers:

- Multiply four ½n-digit integers.
- Add two ½n-digit integers, and shift to obtain result.

$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right) \\
&= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0
\end{aligned}
$$

$$
T(n) = \underbrace{4T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, shift}} \;\Rightarrow\; T(n) = \Theta(n^2)
$$

↑ assumes n is a power of 2

		1 1 0 1	0 1 0 1	y_1 y_0
		* 0 1 1 1	1 1 0 1	x_1 x_0
		0 1 0 0	0 0 0 1	$x_0 \cdot y_0$
	1 0 1 0	1 0 0 1		$x_0 \cdot y_1$
	0 0 1 0	0 0 1 1		$x_1 \cdot y_0$
0 1 0 1	1 0 1 1			$x_1 \cdot y_1$
0 1 1 0	1 0 0 0	0 0 0 0	0 0 0 1	

Key trick: 2 multiplies for the price of 1:

$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right) \\
&= 2^{n} \cdot x_1 y_1 + 2^{n/2} \cdot \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0
\end{aligned}
$$

Well, ok, 4 for 3 is more accurate…

$$
\begin{aligned}
\alpha &= x_1 + x_0 \\
\beta &= y_1 + y_0 \\
\alpha\beta &= \left(x_1 + x_0\right)\left(y_1 + y_0\right) \\
&= x_1 y_1 + \left(x_1 y_0 + x_0 y_1\right) + x_0 y_0 \\
\left(x_1 y_0 + x_0 y_1\right) &= \alpha\beta - x_1 y_1 - x_0 y_0
\end{aligned}
$$

Karatsuba Multiplication

To multiply two n-digit integers:

- Add two ½n digit integers.
- Multiply three ½n-digit integers.
- Add, subtract, and shift ½n-digit integers to obtain result.

$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= 2^{n} \cdot x_1 y_1 + 2^{n/2} \cdot (x_1 y_0 + x_0 y_1) + x_0 y_0 \\
&= 2^{n} \cdot \underset{A}{x_1 y_1} + 2^{n/2} \cdot \big(\underset{B}{(x_1 + x_0)(y_1 + y_0)} - \underset{A}{x_1 y_1} - \underset{C}{x_0 y_0} \big) + \underset{C}{x_0 y_0}
\end{aligned}
$$

Theorem. [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in $O(n^{1.585})$ bit operations.

$$
\mathrm{T}(n) \le \underbrace{T\big(\lfloor n/2 \rfloor\big) + T\big(\lceil n/2 \rceil\big) + T\big(1 + \lceil n/2 \rceil\big)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, subtract, shift}}
$$

$$
\textit{Sloppy version}: \; T(n) \le 3T(n/2) + O(n)
$$

$$
\Rightarrow \mathrm{T}(n) = O(n^{\log_2 3}) = O(n^{1.585})
$$

Multiplication – The Bottom Line

Naïve: $\Theta(n^2)$

Karatsuba: $\Theta(n^{1.59\ldots})$

Amusing exercise: generalize Karatsuba to do 5 size n/3 subproblems => $\Theta(n^{1.46\ldots})$

Best known: $\Theta(n \log n \log\log n)$

"Fast Fourier Transform"

but mostly unused in practice (unless you need really big numbers - a billion digits of π, say)

High precision arithmetic *IS* important for crypto

Recurrences

Where they come from,
how to find them (above)

Next: how to solve them

Mergesort (review)

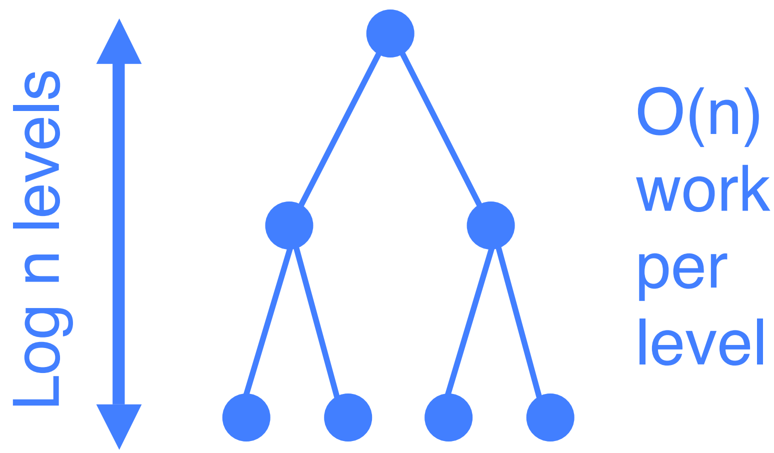
Mergesort: (recursively) sort 2 half-lists, then merge results.

$T(n)=2T(n/2)+cn, \quad n \geq 2$

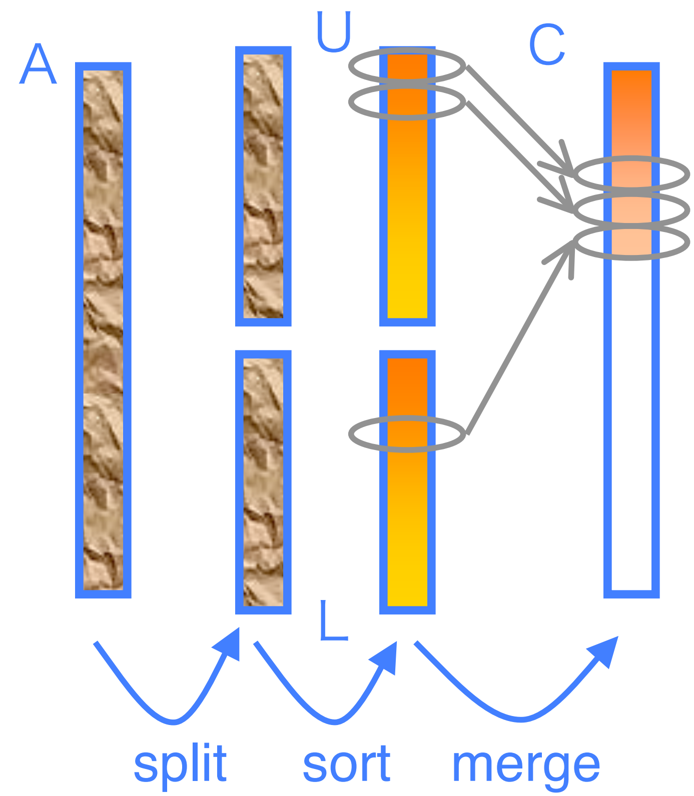
$T(1)=0$

Solution: $\Theta(n \log n)$

(details ~~later~~) **now**



Merge Sort




```
MS(A: array[1..n]) returns array[1..n] {
    If(n=1) return A[1];
    New U:array[1:n/2] = MS(A[1..n/2]);
    New L:array[1:n/2] = MS(A[n/2+1..n]);
    Return(Merge(U,L));
    }
Merge(U,L: array[1..n]) {
    New C: array[1..2n];
    a=1; b=1;
    For i = 1 to 2n
        C[i] = "smaller of U[a], L[b] and correspondingly a++ or b++";
    Return C;
    }
```

Going From Code to Recurrence

Carefully define what you're counting, and write it down!

- "Let $C(n)$ be the number of comparisons between sort keys used by MergeSort when sorting a list of length $n \geq 1$"

In code, clearly separate *base case* from *recursive case,* highlight *recursive calls,* and *operations being counted.*

Write Recurrence(s)

Merge Sort

Base Case

```
MS(A: array[1..n]) returns array[1..n] {
    If(n=1) return A[1];
    New L:array[1:n/2] = MS(A[1..n/2]);
    New R:array[1:n/2] = MS(A[n/2+1..n]);
    Return(Merge(L,R));
    }
Merge(A,B: array[1..n]) {
    New C: array[1..2n];
    a=1; b=1;
    For i = 1 to 2n {
        C[i] = “smaller of A[a], B[b] and a++ or b++”;
    Return C;
    }
```

Recursive calls

Recursive case

Operations being counted

The Recurrence

Base case

$$C(n) = \begin{cases} 0 & \text{if } n = 1 \\ 2C(n/2) + (n-1) & \text{if } n > 1 \end{cases}$$

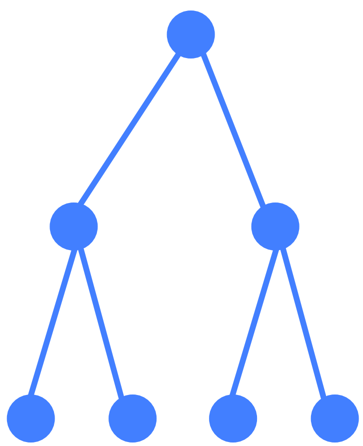
Recursive calls

One compare per element added to merged list, except the last.

Total time: proportional to C(n)

(loops, copying data, parameter passing, etc.)

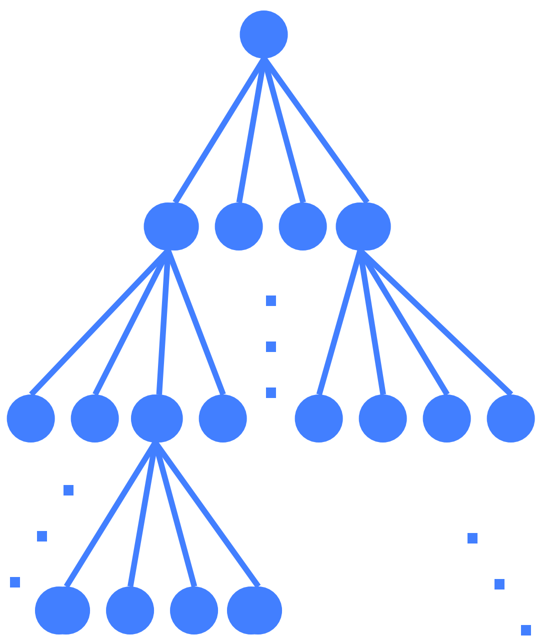
Solve: $T(1) = c$
$T(n) = 2\,T(n/2) + cn$



Level	Num	Size	Work
0	$1=2^0$	n	cn
1	$2=2^1$	$n/2$	$2\ c\ n/2$
2	$4=2^2$	$n/4$	$4\ c\ n/4$
…	…	…	…
i	2^i	$n/2^i$	$2^i\ c\ n/2^i$
…	…	…	…
$k-1$	2^{k-1}	$n/2^{k-1}$	$2^{k-1}\ c\ n/2^{k-1}$
k	2^k	$n/2^k=1$	$2^k\ T(1)$

Total work: add last col

Solve: T(1) = c
T(n) = 4 T(n/2) + cn

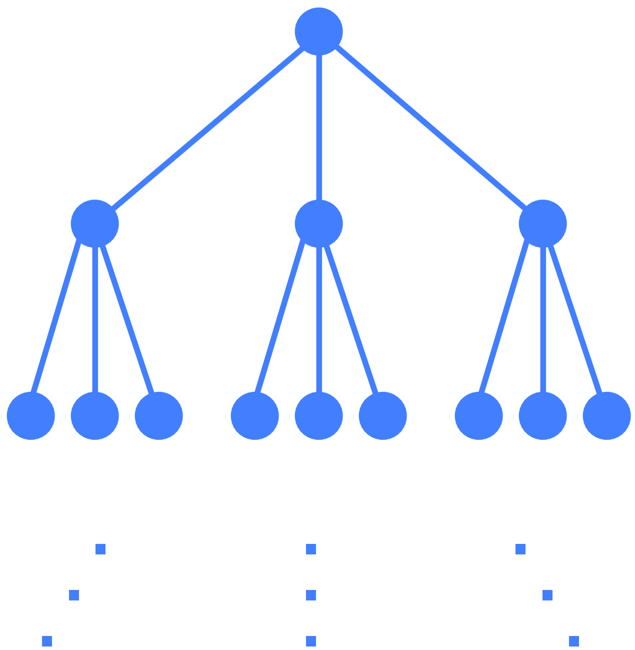


Level	Num	Size	Work
0	$1=4^0$	n	cn
1	$4=4^1$	n/2	4 c n/2
2	$16=4^2$	n/4	16 c n/4
…	…	…	…
i	4^i	$n/2^i$	4^i c $n/2^i$
…	…	…	…
k-1	4^{k-1}	$n/2^{k-1}$	4^{k-1} c $n/2^{k-1}$
k	4^k	$n/2^k=1$	4^k T(1)

$$\sum_{i=0}^{k} 4^i cn/2^i = O(n^2)$$

Solve: $T(1) = c$

$T(n) = 3\,T(n/2) + cn$



Level	Num	Size	Work
0	$1=3^0$	n	cn
1	$3=3^1$	$n/2$	$3\ c\ n/2$
2	$9=3^2$	$n/4$	$9\ c\ n/4$
...	...	...	...
i	3^i	$n/2^i$	$3^i\ c\ n/2^i$
...	...	...	...
k-1	3^{k-1}	$n/2^{k-1}$	$3^{k-1}\ c\ n/2^{k-1}$
k	3^k	$n/2^k=1$	$3^k\ T(1)$

$n = 2^k$; $k = \log_2 n$

Total Work: $T(n) = \sum_{i=0}^{k} 3^i cn / 2^i$

Solve: $T(1) = c$
$T(n) = 3\,T(n/2) + cn$ (cont.)

$$T(n) = \sum_{i=0}^{k} 3^i cn/2^i$$

$$= cn \sum_{i=0}^{k} 3^i/2^i$$

$$= cn \sum_{i=0}^{k} \left(\tfrac{3}{2}\right)^i$$

$$= cn \frac{\left(\frac{3}{2}\right)^{k+1} - 1}{\left(\frac{3}{2}\right) - 1}$$

$$\sum_{i=0}^{k} x^i = \frac{x^{k+1} - 1}{x - 1}$$

$$(x \neq 1)$$

Solve: T(1) = c
T(n) = 3 T(n/2) + cn (cont.)

$$= 2cn\left(\left(\tfrac{3}{2}\right)^{k+1} - 1\right)$$

$$< 2cn\left(\tfrac{3}{2}\right)^{k+1}$$

$$= 3cn\left(\tfrac{3}{2}\right)^{k}$$

$$= 3cn\,\frac{3^k}{2^k}$$

Solve: T(1) = c
T(n) = 3 T(n/2) + cn (cont.)

$$= 3cn \frac{3^{\log_2 n}}{2^{\log_2 n}}$$

$$= 3cn \frac{3^{\log_2 n}}{n}$$

$$= 3c\, 3^{\log_2 n}$$

$$= 3c\left(n^{\log_2 3}\right)$$

$$= O\left(n^{1.59\ldots}\right)$$

$$a^{\log_b n}$$

$$= \left(b^{\log_b a}\right)^{\log_b n}$$

$$= \left(b^{\log_b n}\right)^{\log_b a}$$

$$= n^{\log_b a}$$

Master Divide and Conquer Recurrence

If $T(n) = aT(n/b)+cn^k$ for $n > b$ then

if $a > b^k$ then $T(n)$ is $\Theta(n^{\log_b a})$ [many subproblems => leaves dominate]

if $a < b^k$ then $T(n)$ is $\Theta(n^k)$ [few subproblems => top level dominates]

if $a = b^k$ then $T(n)$ is $\Theta(n^k \log n)$ [balanced => all log n levels contribute]

True even if it is $\lceil n/b \rceil$ instead of n/b.

Another D&C Approach, cont.

Moral 3: unbalanced division less good:

$(.1n)^2 + (.9n)^2 + n = .82n^2 + n$

The 18% savings compounds significantly if you carry recursion to more levels, actually giving O(nlogn), but with a bigger constant. So worth doing if you can't get 50-50 split, but balanced is better if you can.

This is intuitively why Quicksort with random splitter is good – badly unbalanced splits are rare, and not instantly fatal.

In contrast:

$(1)^2 + (n-1)^2 + n = n^2 - 2n + 2 + n$

Little improvement here.

D & C Summary

"two halves are better than a whole"

if the base algorithm has super-linear complexity.

"If a little's good, then more's better"

repeat above, recursively

Analysis: recursion tree or Master Recurrence