CSci 421

Introduction to Algorithms

Homework Assignment 6

Due: Wednesday, 1 Mar 2000

In probs. 2 and 3 below, as in lecture, I use the terms *alternating* and *augmenting* path slightly differently from the book. A path is *alternating* with respect to a given matching M if its edges alternate between M and $E - M$. An *augmenting* path is an alternating path whose end points are both unmatched. Compare to the book's definition on page 236.

1. Draw the residual graph corresponding to the flow in figure 7.41, pg241. Is this flow maximum? Why or why not? If maximum, what is the corresponding min cut?

2. Let G be the bipartite graph shown in figure 7.37, page 236. Let M be the (non-maximum) matching $\{\{3, A\}, \{4, E\}, \{6, F\}\}$.

   (a) List 3 alternating paths that that are *not* augmenting paths.

   (b) List *all* augmenting paths in G (with respect to M).

   (c) What is the smallest set of pairwise vertex-disjoint augmenting paths? What is the largest?

   (d) Let P be the augmenting path of length 3 containing $\{4, E\}$. Considering M and P to be sets of edges, $M \oplus P$ is their set theoretic *symmetric difference*: $(M \cup P) - (M \cap P)$. What set of edges is $M' = M \oplus P$? Is it a matching?

3. Let G be any bipartite graph, M any matching in G, and P any augmenting path (with respect to M).

   (a) Prove that $M' = M \oplus P$ is a matching.

   (b) Show $|M'| = |M| + 1$. How is the set of matched vertices in M' related to the set of matched vertices in M and the set of vertices (incident to edges) in P?

   (c) Give a counterexample to 3a if P is an arbitrary path, i.e. show that there is a graph G, matching M and path P such that $M \oplus P$ is not a matching. Is it true or false if P is an alternating path that is not an augmenting path? Prove or give a counterexample.

   (d) Now suppose that there are *two* augmenting paths P and P' with respect to M, and that P and P' are vertex-disjoint. Show that P' also is an augmenting path with respect to the *augmented* matching $(M \oplus P)$, and similarly that P is augmenting with respect to $(M \oplus P')$. What could you say about a case where there were, say, 17 pairwise disjoint paths $P_1, \ldots, P_{17}$, all augmenting paths with respect to M? What, and how big, is $M \oplus P_1 \oplus \cdots \oplus P_{17}$?

4. The Hopcroft-Karp bipartite matching algorithm discussed in class and sketched in the book needs a subroutine to solve the following problem: Given a directed acyclic graph G with a designated set U of vertices having indegree 0 (the *source* vertices) and a designated set V of vertices having outdegree 0 (the *sink* vertices), find a maximal set of pairwise vertex disjoint paths that go from some source to some sink. Give a linear time algorithm for this problem.

   [Note that in the matching example the graph G has the additional property that, since it is produced by breadth-first search, it is nicely *layered* — each vertex has been assigned a layer number with all sources on layer 0, all sinks on layer k for some fixed k, and all edges going from a layer i to the next layer $i + 1$. Although I confess I haven't given it much thought, I don't think this extra information is either necessary or particularly useful in solving the problem, BUT you may assume it if you find it helpful.]