CSE 421
Introduction to Algorithms
Winter 2000

NP-Completeness

(Chapter 11)

Easy Problems vs. Hard Problems

Easy - problems whose worst case running time is bounded by some **polynomial** in the size of the input.

Easy = Efficient

Hard - problems that *cannot* be *solved efficiently*.

The class P

Definition: P = set of problems solvable by computers in polynomial time.

i.e. $T(n) = O(n^k)$ for some k.

- These problems are sometimes called **tractable** problems.

Examples: sorting, SCC, matching, max flow, shortest path, MST.

Is P a good definition of efficient?

Is $O(n^{100})$ efficient? Is $O(10^9 n)$ efficient?

Is $O(2^n)$ really so bad?

So we have:

P = "easy" = efficient = tractable

= solvable in polynomial-time.

Decision Problems

- Technically, we will restrict our discussion to **decision problems** - problems that have an answer of either yes or no.
- Most problems can be easily converted to decision problems:
  - **Example**: Instead of looking for the size of the shortest path from s to t in a graph G, we ask:

    "Is there a path from s to t of length $\leq k$?"
  - If we know how to solve the decision problem, then we can usually solve the original problem.

Examples of Decision Problems in P

Big Flow

Given: graph G with edge lengths, vertices s and t, integer k.

Question: Is there an s-t flow of length $\geq k$?

Small Spanning Tree

Given: weighted undirected graph G, integer k.

Question: Is there a spanning tree of weight $\leq k$?

Decision problem as a Language-recognition problem

- Let U be the set of all possible inputs to the decision problem.
- $L \subseteq U$ = the set of all inputs for which the answer to the problem is **yes**.
- We call L the **language** corresponding to the problem. (problem = language)
- The decision problem is thus:
  - to recognize whether or not a given input belongs to L = the language recognition problem.

The class NP

Definition: NP = set of problems solvable by a *nondeterministic* algorithm in polynomial time.

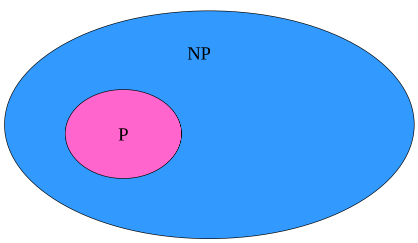
Another way of saying this:

NP = The class of problems whose solution can be **verified** in polynomial time.

NP = "nondeterministic polynomial"

Examples: all of problems in P plus: SAT, TSP, Hamiltonian cycle, bin packing, vertex cover.

Complexity Classes



NP = Polynomial-time **verifiable**

P = Polynomial-time **solvable**

Verifying Solutions

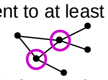
Given a **problem** and a **potential solution**, verify if the solution is correct in polynomial-time.

In general, **guess** a solution, and then **check** if the guess is correct in polynomial time.

Examples of Problems in NP

Vertex Cover

A **vertex cover** of G is a set of vertices such that every edge in G is incident to at least one of these vertices. Example:



Question: Given a graph G, integer k, determine whether G has a vertex cover containing $\leq k$ vertices?

Verify: Given a set of $\leq k$ vertices, does it cover every edge? (Guess and check in polynomial time.)

Examples of Problems in NP

Satisfiability (SAT)

A Boolean formula in conjunctive normal form (CNF) is **satisfiable** if there exists a truth assignment of 0's and 1's to its variables such that the value of the expression is 1. Example:

$$S=(x+y+\neg z)\bullet(\neg x+y+z)\bullet(\neg x+\neg y+\neg z)$$

Question: Given a Boolean formula in CNF, is it satisfiable?

Verify: Given a truth assignment, does it satisfy the formula? (Guess and check in polynomial time.)

Problems in P can also be verified in polynomial-time

<u>**Shortest Path**</u>: Given a graph *G* with edge lengths, is there a path from *s* to *t* of length ≤ *k*?

Verify: Given a path from *s* to *t*, is its length ≤ *k*?

<u>**Small Spanning Tree**</u>: Given a weighted undirected graph *G*, is there a spanning tree of weight ≤ *k*?

Verify: Given a spanning tree, is its weight ≤ *k*?

Nondeterminism

- A **nondeterministic algorithm** has all the "regular" operations of any other algorithm available to it.
- *In addition*, it has a powerful primitive, the **nd-choice primitive**.
- The **nd-choice primitive** is associated with a fixed number of choices, such that each choice causes the algorithm to follow a different computation path.

Nondeterminism (cont.)

- A **nondeterministic algorithm** consists of an interleaving of regular deterministic steps and uses of the **nd-choice primitive**.
- We require that:
  - The algorithm have at least one "good" path sequence of choices for every $x \in L$.
  - For all $x \notin L$, we reach a reject outcome in all paths.

Nondeterminism (cont.)

We say that a nondeterministic algorithm recognizes a language L if:

Given an input *x*, it is possible to convert each nd-choice encountered during the execution of the algorithm into a real choice such that the outcome of the algorithm will be to accept *x*, iff $x \in L$.

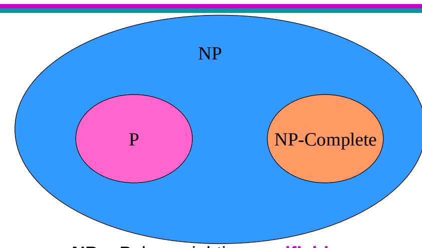
The class NP-complete

Definition: NP-complete = set of problems in NP that (we are pretty sure) ***cannot*** be solved in polynomial time.

These are thought of as the **hardest** problems in the class NP.

Interesting fact: If any one NP-complete problem could be solved in polynomial time, then ***all*** NP-complete problems could be solved in polynomial time.

Complexity Classes



NP = Polynomial-time **verifiable**

P = Polynomial-time **solvable**

NP-Complete = **"Hardest"** problems in **NP**

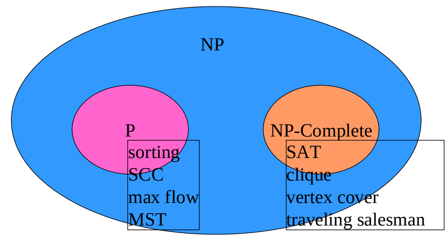
The class NP-complete (cont.)

- Hundreds of important problems have been shown to be NP-complete.

Interesting Fact: The general belief is that there is no efficient algorithm for any **NP-complete** problem, but no proof of that belief is known.

Examples: SAT, clique, vertex cover, Hamiltonian cycle, TSP, bin packing.

Complexity Classes of Problems



Does P = NP?

- This is an open question.
- To show that **P = NP**, we have to show that *every* problem that belongs to NP can be solved by a polynomial time deterministic algorithm.
- No one has shown this yet.
- (It seems unlikely to be true.)

Is all of this useful for anything??!?

Earlier in this class we learned techniques for solving problems in **P**.

Question: Do we just throw up our hands if we come across a problem we suspect **not to be in P**?

Dealing with NP-complete Problems

What if I think my problem is <u>not in P</u>?

<u>Here is what you do</u>:

1) Prove your problem is **NP-complete**.

2) Come up with an algorithm to solve the problem **approximately**.

I will cover (1) this week, Larry will cover (2) next week.

Reductions: a useful tool

Definition: To **reduce** A to B means to figure out how to solve A, given a subroutine solving B.

Example: reduce MEDIAN to SORT
Solution: sort, then select (n/2) th

Example: reduce SORT to FIND_MAX
Solution: FIND_MAX, remove it, repeat

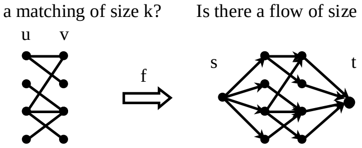
Example: reduce MEDIAN to FIND_MAX
Solution: transitivity: compose solutions above.

More Examples of reductions

Example:

reduce BIPARTITE_MATCHING to MAX_FLOW

Is there a matching of size k? Is there a flow of size k?



All capacities = 1

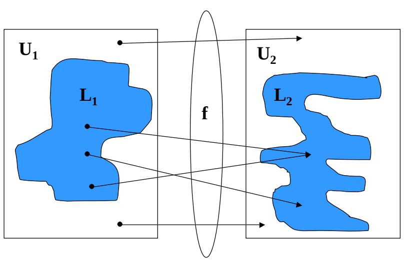
Polynomial-Time Reductions

Definition: Let L_1 and L_2 be two languages from the input spaces U_1 and U_2.

We say that L_1 is **polynomially reducible** to L_2 if there exists a polynomial-time algorithm f that converts each input $u_1 \in U_1$ to another input $u_2 \in U_2$ such that $u_1 \in L_1$ iff $u_2 \in L_2$.

$$u_1 \in L_1 \Leftrightarrow f(u_1) \in L_2$$

Polynomial-time Reduction from language L_1 to language L_2 via reduction function f.



$$u_1 \in L_1 \Leftrightarrow f(u_1) \in L_2$$

Polynomial-Time Reductions (cont.)

Define: $A \le_p B$ *"A is polynomial-time reducible to B", iff there is a polynomial-time computable function f such that:* $x \in A \Leftrightarrow f(x) \in B$

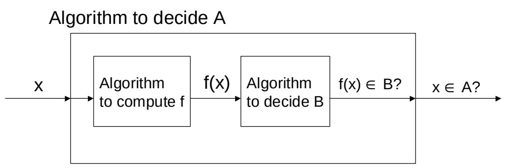
"complexity of A" $\le$ *"complexity of B" + "complexity of f"*

(1) $A \le_p B$ and $B \in P \Rightarrow A \in P$

(2) $A \le_p B$ and $A \notin P \Rightarrow B \notin P$

(3) $A \le_p B$ and $B \le_p C \Rightarrow A \le_p C$ (transitivity)

Using an Algorithm for *B* to Decide *A*



"If A $\le_p$ B, and we can solve B in polynomial time, then we can solve A in polynomial time also."

More Definitions

Definition: Problem *B* is **NP-hard** if *every* problem in NP is polynomially reducible to *B*.

Definition: Problem *B* is **NP-complete** if:

(1) *B* belongs to NP, and

(2) *B* is NP-hard.

Proving a problem is NP-complete

- Technically,for condition (2) We have to show that **every** problem in NP is reducible to B. (yikes!) This sounds like a lot of work.
- For the **very first NP-complete problem** (SAT) this had to be proved directly.
- However, once we have one NP-complete problem, then we don't have to do this every time.
- Why? Transitivity.

Re-stated Definition

Lemma 11.3: Problem *B* is **NP-complete** if:

(1) *B* belongs to NP, and

(2') *A* is polynomial-time reducible to *B*, for some problem *A* that is NP-complete.

That is, to show (2') given a new problem *B*, it is sufficient to show that SAT or any other NP-complete problem is polynomial-time reducible to *B*.

Usefulness of Transitivity

Now we only have to show $L' \leq_p L$, for some problem $L' \in$ ***NP-complete***, in order to show that **L** is NP-hard. Why is this equivalent?

1) Since $L' \in$ ***NP-complete**, we know that* L' is *NP-hard.* That is:

$$\forall L'' \in NP, \text{ we have } L'' \leq_p L'$$

2) If we show $L' \leq_p L$, then by transitivity we know that: $\forall L'' \in NP$, we have $L'' \leq_p L$.

Thus L is NP-hard.

The growth of the number of NP-complete problems

- Steve Cook (1971) showed that SAT was NP-complete.
- Richard Karp (1972) found 24 more NP-complete problems.
- Today there are hundreds of known NP-complete problems.
  - Garey and Johnson (1979) is a good source of NP-complete problems.

SAT is NP-complete

Cook's theorem: **SAT is NP-complete**

Satisfiability (SAT)

A Boolean formula in conjunctive normal form (CNF) is **satisfiable** if there exists a truth assignment of 0's and 1's to its variables such that the value of the expression is 1. Example:

$$S=(x+y+\neg z)\bullet(\neg x+y+z)\bullet(\neg x+\neg y+\neg z)$$

Example above is satisfiable. (We an see this by setting x=1, y=1 and z=0.)

SAT is NP-complete

Rough idea of proof:

(1) **SAT is in NP** because we can guess a truth assignment and check that it satisfies the expression in polynomial time.

(2) **SAT is NP-hard** because …..

SAT is NP-hard

- A Turing machine (even a nondeterministic one) and all of its operations on a given input can be "described" by a Boolean expression.
- That is, the expression will be **satisfiable** iff the Turing machine will terminate in an accepting state for the given input.
- Therefore, any NP algorithm can be described by an instance of a SAT problem.
- Thus: **Cook's theorem**: SAT is NP-complete.

How do you prove problem *A* is NP-complete?

1) **Prove *A* is in NP:** show that given a solution, it can be verified in polynomial time.

2) **Prove that *A* is NP-hard:**

a) Select a **known** NP-complete problem *B*.

b) Describe a polynomial time computable algorithm that computes a function *f*, **mapping *every* instance of *B* to an instance of *A*.** (that is: $\boldsymbol{B \leq_p A}$)

c) Prove that every ***yes***-instance of *B* maps to a ***yes***-instance of *A*, and every ***no***-instance of *B* maps to a ***no***-instance of *A*.

d) Prove that the algorithm computing ***f* runs in polynomial time.**

Proof that problem *A* is NP-complete

1) **Prove *A* is in NP: "**Given a possible solution to *A*, I can verify its correctness in polynomial-time."

2) **Prove that *A* is NP-hard:**

a) "I will reduce known NP-complete problem *B* to *A.*"

your function f

b) "Let ***b*** be an arbitrary instance of problem *B*. Here is how you convert ***b*** to an instance ***a*** of problem *A*."
Note: this method must work for ANY instance of *B*.

c) "If ***a*** is a "yes"-instance, then this implies that ***b*** is also a "yes"-instance. **Conversely**, if ***b*** is a "yes"-instance, then this implies that ***a*** is also a "yes"-instance."

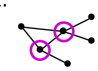
d) "The conversion from *B* to *A* runs in polynomial time because…."

NP-complete problem: **Vertex Cover**

Input: Undirected graph *G = (V, E)*, integer *k*.

Output: True iff there is a subset *C* of *V* of size ≤ *k* such that every edge in *E* is incident to at least one vertex in *C*.

Example: Vertex cover of size ≤ 2.

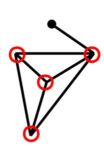


NP-complete problem: **Clique**

Input: Undirected graph *G = (V, E)*, integer *k*.

Output: True iff there is a subset *C* of *V* of size ≥ *k* such that all vertices in *C* are connected to all other vertices in *C*.

Example: Clique of size ≥ 4



NP-complete problem: **Satisfiability (SAT)**

Input: A Boolean formula in CNF form.

Output: True iff there is a truth assignment of 0's and 1's to the variables such that the value of the expression is 1.

Example: Formula *S* is satisfiable with the truth assignment *x=1, y=1* and *z=0*.

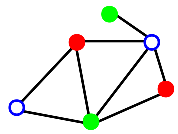
$$S=(x+y+\neg z)\bullet(\neg x+y+z)\bullet(\neg x+\neg y+\neg z)$$

NP-complete problem: **3-Coloring**

Input: An undirected graph G=(V,E).

Output: True iff there is an assignment of colors to the vertices in G such that no two adjacent vertices have the same color. (using only 3 colors)

Example:



NP-complete problem: **Knapsack**

Input: set of objects with weights and values, a maximum weight that can be carried and a desired value. (see p. 357 in Manber)

Output: True iff there is a subset of the objects with (total weight ≤ *allowable weight)* and (total value ≥ *desired value)*.

Example: Items: {a, b, c},size(a)=3,size(b)=6,size(c)=4

value(a)=\$30, value(b)=\$24, value(c)=\$18

Max weight = 10, Desired value = \$50.

Answer: yes, {a,b}

NP-complete problem: **Partition**

Input: Set of items *S*, each with an associated size. The sum of the items' sizes is *2k*.

Output: True iff there is a subset of the items whose sizes add up to *k*.

Example: *S* = (2,3,1,10,4,6). Is there a subset of items that sums to 13? (yes)