CSE 421

Union Find DS
Dijkstra's Algorithm,

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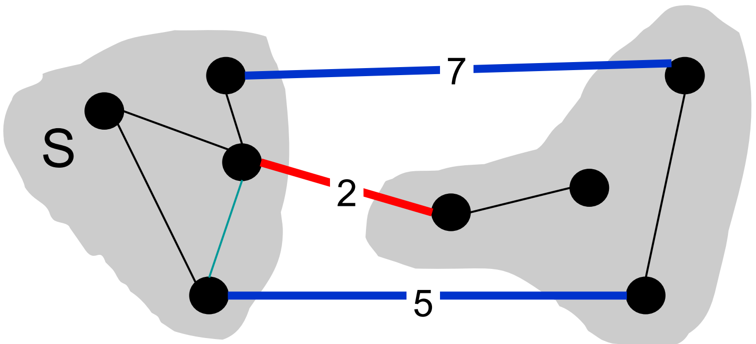
Properties of the OPT

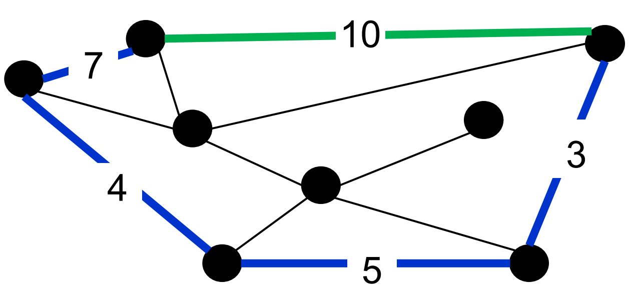
Simplifying assumption: All edge costs c_e are distinct.

Cut property: Let S be any subset of nodes (called a cut), and let e be the min cost edge with exactly one endpoint in S. Then every MST contains e.

Cycle property. Let C be any cycle, and let f be the max cost edge belonging to C. Then no MST contains f.



red edge is in the MST



Green edge is not in the MST

Cut Property: Proof

Simplifying assumption: All edge costs c_e are distinct.

Cut property. Let S be any subset of nodes, and let e be the min cost edge with exactly one endpoint in S. Then T* contains e.

Pf. By contradiction

Suppose e = {u,v} does not belong to T*.

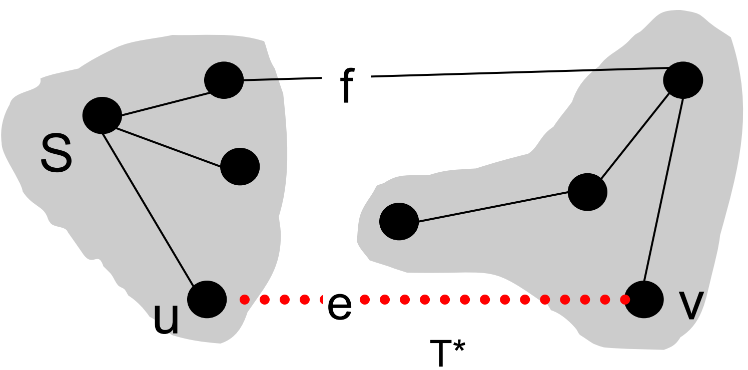
Adding e to T* creates a cycle C in T*.

C crosses S even number of times $\Rightarrow$ there exists another edge, say f, that leaves S.

$T = T^* \cup \{e\} - \{f\}$ is also a spanning tree.

Since $c_e < c_f$, $c(T) < c(T^*)$.

This is a contradiction.



Cycle Property: Proof

Simplifying assumption: All edge costs c_e are distinct.

Cycle property: Let C be any cycle in G, and let f be the max cost edge belonging to C. Then the MST T* does not contain f.

Pf. (By contradiction)

Suppose f belongs to T*.

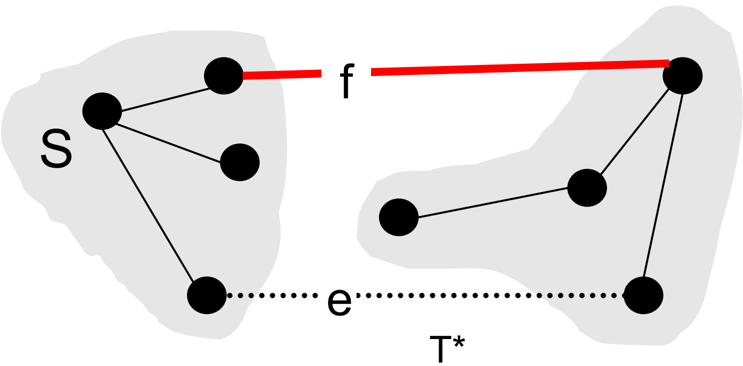
Deleting f from T* cuts T* into two connected components.

There exists another edge, say e, that is in the cycle and connects the components.

$T = T^* \cup \{e\} - \{f\}$ is also a spanning tree.

Since $c_e < c_f$, $c(T) < c(T^*)$.

This is a contradiction.



Kruskal's Algorithm [1956]

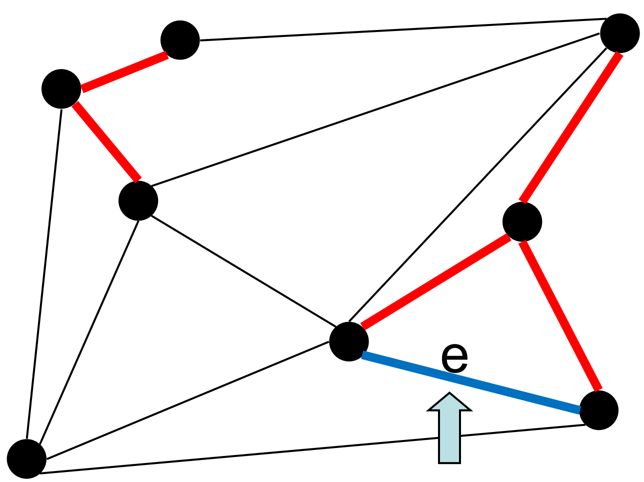
```
Kruskal(G, c) {
   Sort edges weights so that c1 ≤ c2 ≤ ... ≤ cm.
   T ← ∅

   foreach (u ∈ V) make a set containing singleton {u}

   for i = 1 to m
      Let (u,v) = ei
      if (u and v are in different sets) {
         T ← T ∪ {ei}
         merge the sets containing u and v
      }
   return T
}
```

Kruskal's Algorithm: Pf of Correctness

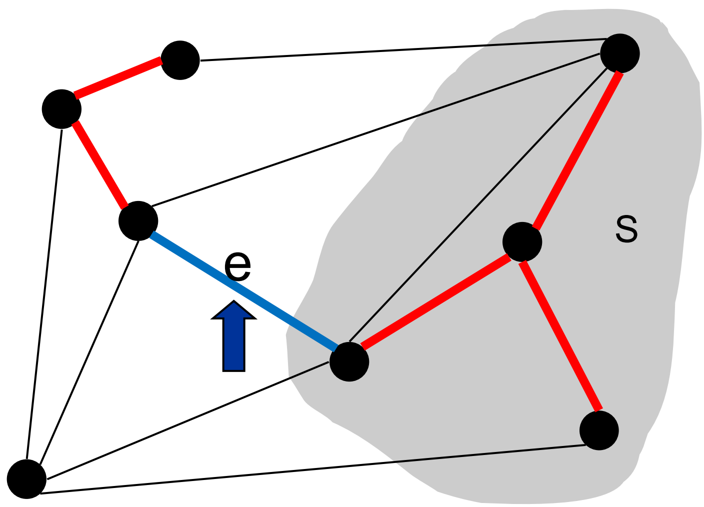
Consider edges in ascending order of weight.

Case 1: If adding e to T creates a cycle, discard e according to cycle property.

Case 2: Otherwise, insert e = (u, v) into T according to cut property where S = set of nodes in u's connected component.



Case 1

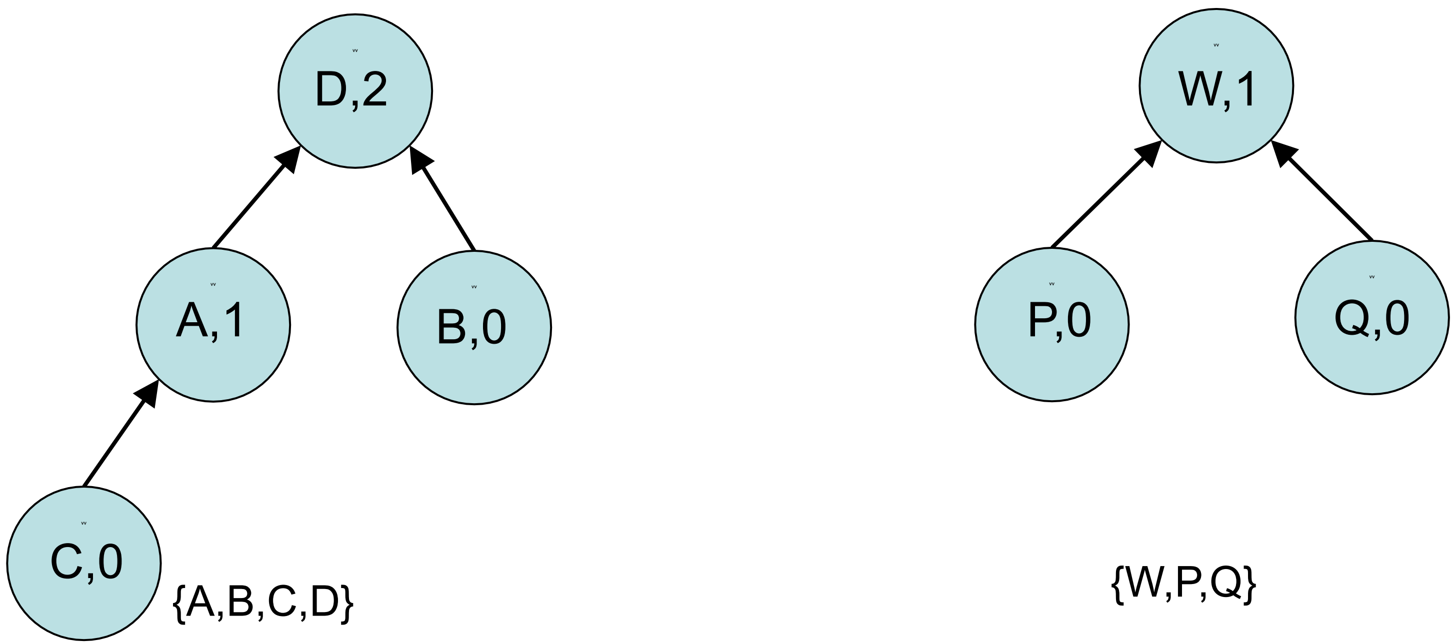


Case 2

Union Find Data Structure

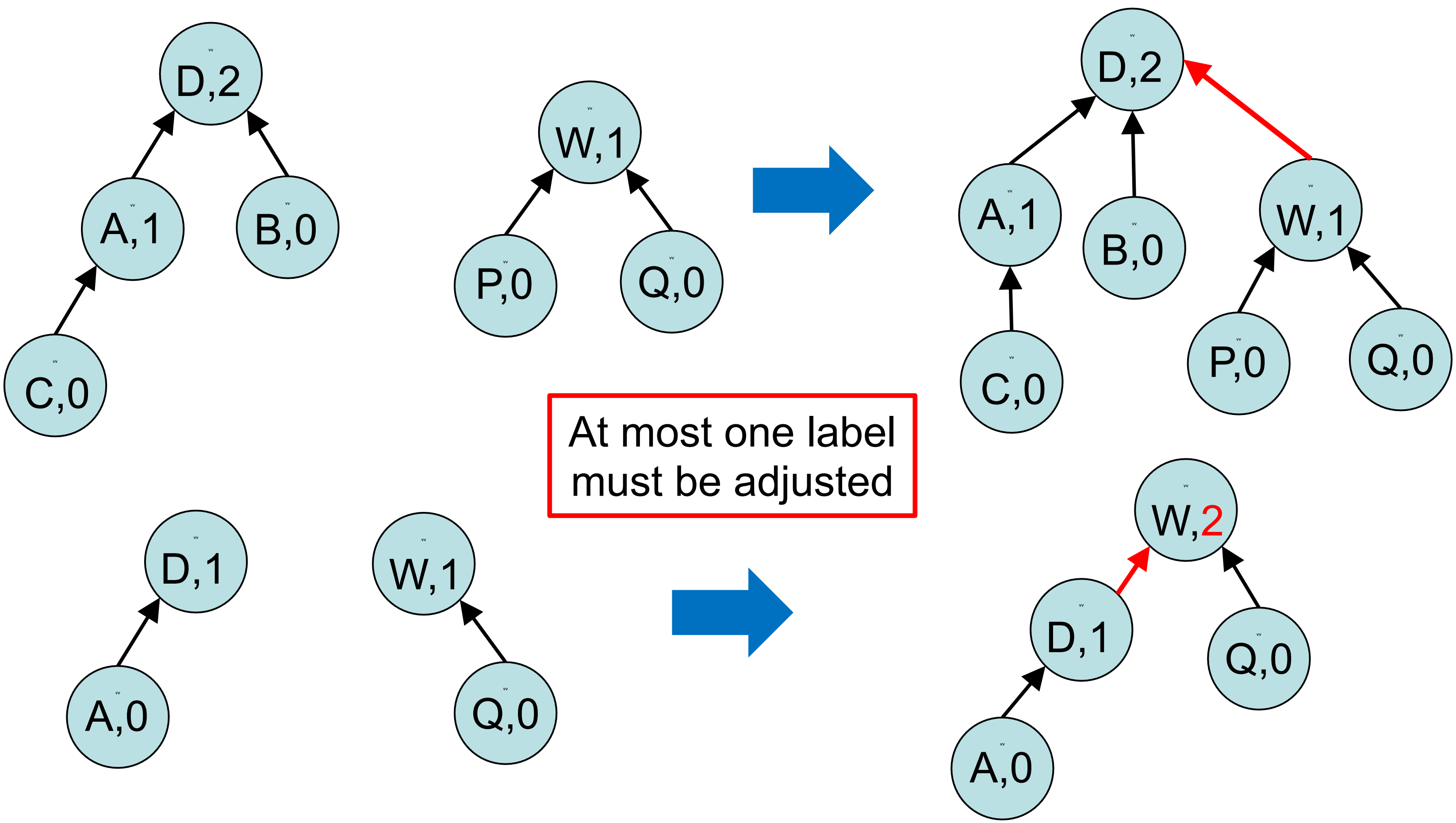
Each set is represented as a tree of pointers, where every vertex is labeled with longest path ending at the vertex

To check whether A,Q are in same connected component, follow pointers and check if root is the same.



Union Find Data Structure

Merge: To merge two connected components, make the root with the smaller label point to the root with the bigger label (adjusting labels if necessary). Runs in O(1) time



Kruskal's Algorithm with Union Find

Implementation. Use the union-find data structure.

- Build set T of edges in the MST.
- Maintain a set for each connected component.
- O(m log n) for sorting and O(m log n) for union-find

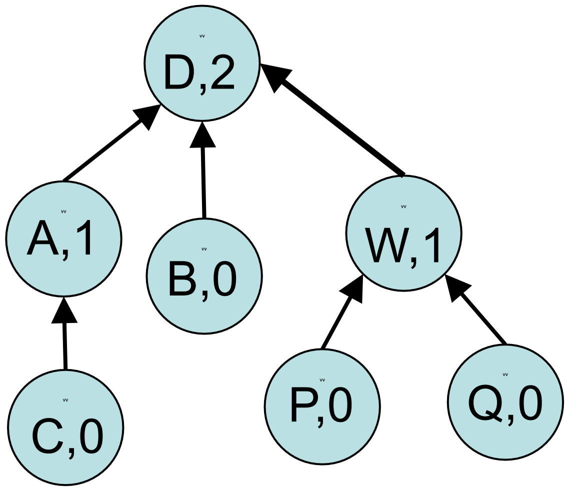
```
Kruskal(G, c) {
   Sort edges weights so that c1 ≤ c2 ≤ ... ≤ cm.
   T ← ∅

   foreach (u ∈ V) make a set containing singleton {u}

   for i = 1 to m                        Find roots and compare
      Let (u,v) = ei
      if (u and v are in different sets) {
         T ← T ∪ {ei}
         merge the sets containing u and v
      }                                  Merge at the roots
   return T
}
```

Depth vs Size

Claim: If the label of a root is k, there are at least 2^k elements in the set.

Therefore the depth of any tree in algorithm is at most log n

So, we can check if u, v are in the same component in time $O(\log n)$



Depth vs Size: Correctness

Claim: If the label of a root is k, there are at least 2^k elements in the set.

Pf: By induction on k.

Base Case (k = 0): this is true. The set has size 1.

IH: Suppose the claim is true until some time t

IS: If we merge roots with labels $k_1 > k_2$, the number of vertices only increases while the label stays the same.

If $k_1 = k_2$, the merged tree has label $k_1 + 1$,

and by induction, it has at least

$$2^{k_1} + 2^{k_2} = 2^{k_1+1}$$

elements.

Removing weight Distinction Assumption

Suppose edge weights are not distinct, and Kruskal's algorithm sorts edges so

$$c_{e_1} \le c_{e_2} \le \cdots \le c_{e_m}$$

Suppose Kruskal finds tree T of weight $c(T)$, but the optimal solution T^* has cost $c(T^*) < c(T)$.

Perturb each of the weights by a very small amount so that

$$c'_{e_1} < c'_{e_2} < \cdots < c'_{e_m}$$

where $c'_{e_i} = c_{e_i} + i.\epsilon$

If ϵ is small enough, $c'(T^*) < c(T)$.

However, this contradicts the correctness of Kruskal's algorithm, since the algorithm will still find T, and Kruskal's algorithm is correct if all weights are distinct. ■

Summary (Greedy Algorithms)

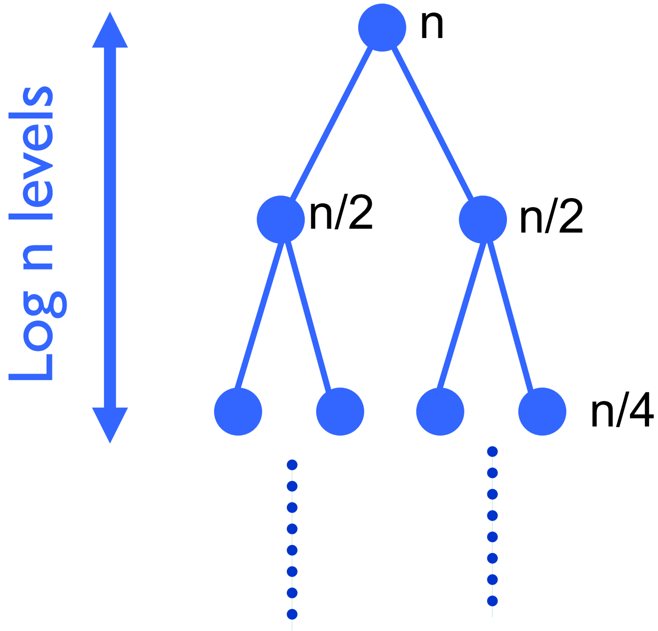
- Greedy Stays Ahead: Interval Scheduling, Dijkstra's algorithm
- Structural: Interval Partitioning
- Exchange Arguments: MST, Kruskal's Algorithm,
- Data Structures: Union Find, Heap

Divide and Conquer Approach

Divide and Conquer

Similar to algorithm design by induction, we reduce a problem to several subproblems.

Typically, each sub-problem is at most a constant fraction of the size of the original problem



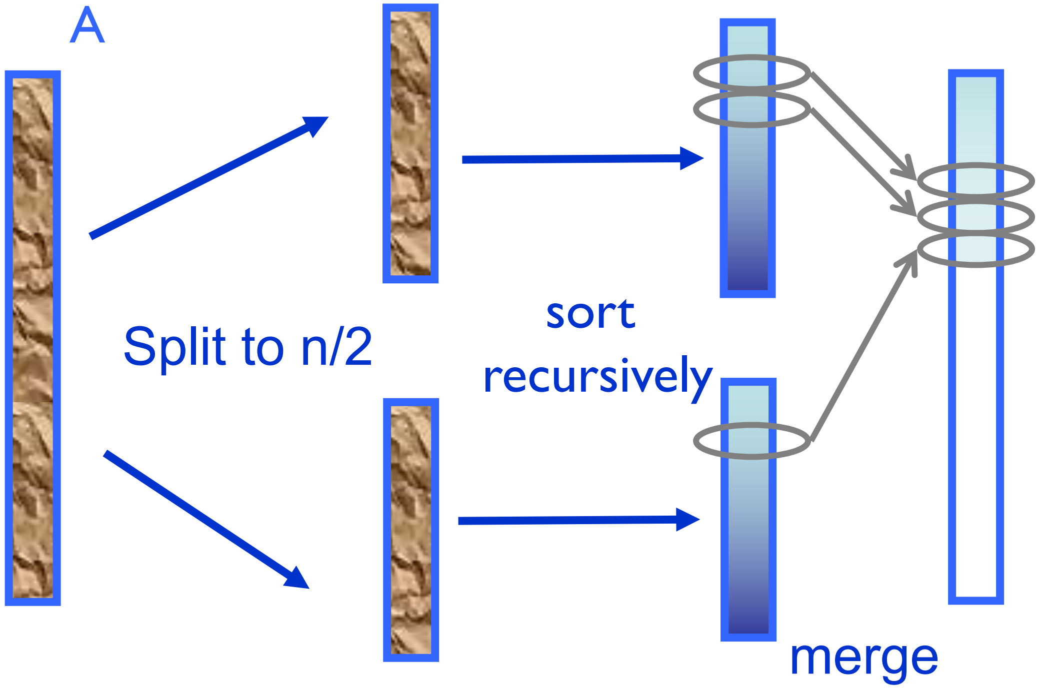
Recursively solve each subproblem

Merge the solutions

Examples:

- Mergesort, Binary Search, Strassen's Algorithm,

A Classical Example: Merge Sort



Why Balanced Partitioning?

An alternative "divide & conquer" algorithm:

- Split into n-1 and 1
- Sort each sub problem
- Merge them

Runtime

$$T(n) = T(n-1) + T(1) + n$$

Solution:

$$T(n) = n + T(n-1) + T(1)$$

$$= n + n - 1 + \ T(n-2)$$

$$= n + n - 1 + n - 2 + T(n-3)$$

$$= n + n - 1 + n - 2 + \cdots + 1 = O(n^2)$$

D&C: The Key Idea

Suppose we've already invented Bubble-Sort, and we know it takes n^2

Try just one level of divide & conquer:

Bubble-Sort(first n/2 elements)

Bubble-Sort(last n/2 elements)

Merge results

Time: $2\,T(n/2) + n = n^2/2 + n \ll n^2$

Almost twice as fast!

D&C in a nutshell

D&C approach

- “the more dividing and conquering, the better”
  - Two levels of D&C would be almost 4 times faster, 3 levels almost 8, etc., even though overhead is growing.
  - Best is usually full recursion down to a small constant size (balancing "work" vs "overhead").

  In the limit: you’ve just rediscovered mergesort!
- Even unbalanced partitioning is good, but less good
  - Bubble-sort improved with a 0.1/0.9 split:

    $$(.1n)^2 + (.9n)^2 + n = .82n^2 + n$$

    The 18% savings compounds significantly if you carry recursion to more levels, actually giving $O(n \log n)$, but with a bigger constant.
- This is why Quicksort with random splitter is good – badly unbalanced splits are rare, and not instantly fatal.

Finding the Root of a Function

Finding the Root of a Function

Given a continuous function f and two points a < b such that

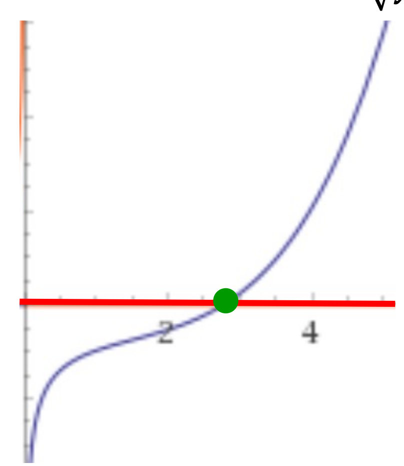
$$f(a) \leq 0$$
$$f(b) \geq 0$$

Find an approximate root of f (a point c where there is r s.t., $|r - c| \leq \epsilon$ and $f(r) = 0$).

$$\text{f}(x) = \sin(x) - \frac{100}{\sqrt{x}} + x^4$$

Note f has a root in $[a, b]$ by

intermediate value theorem



a

b

Note that roots of f may be irrational,

So, we want to approximate

the root with an arbitrary precision!

A Naiive Approch

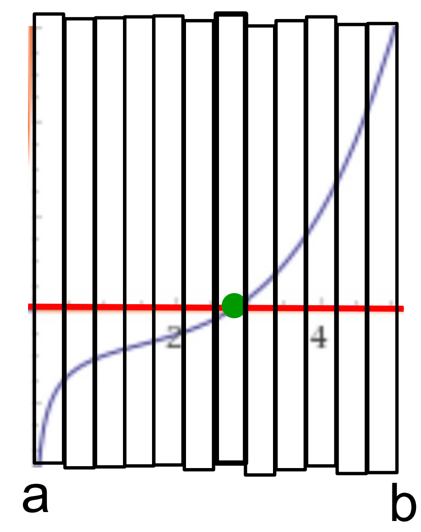
Suppose we want ϵ approximation to a root.

Divide [a,b] into $n = \frac{b-a}{\epsilon}$ intervals. For each interval check

$$f(x) \leq 0, f(x+\epsilon) \geq 0$$

This runs in time $O(n) = O(\frac{b-a}{\epsilon})$

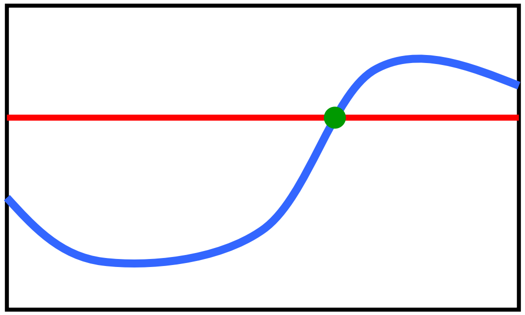
Can we do faster?

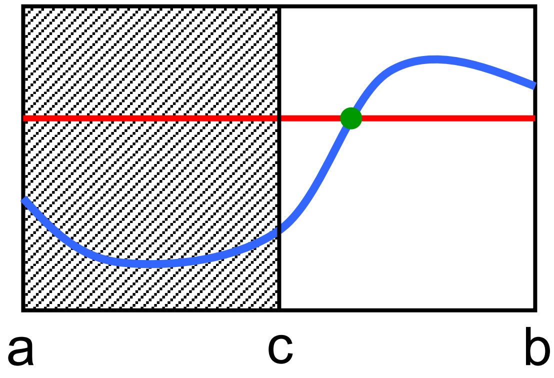


D&C Approach (Based on Binary Search)

Bisection(a,b, ε)

    if $(b - a) < \epsilon$ then
        return (a)
    else
        $m \leftarrow (a + b)/2$
        if $f(m) \leq 0$ then
            return(Bisection(c, b, ε))
        else
            return(Bisection(a, c, ε))





Time Analysis

Let $n = \frac{a-b}{\epsilon}$

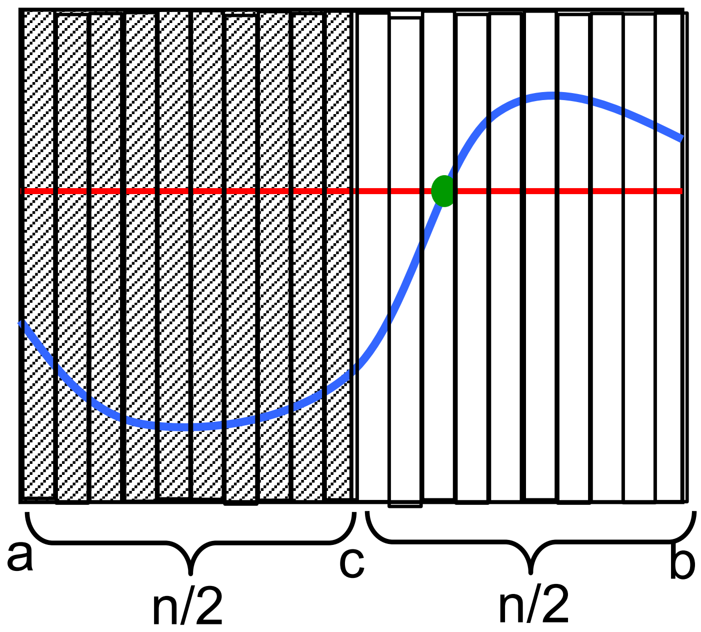
And $c = (a+b)/2$

Always half of the intervals lie to the left and half lie to the right of c

So,

$$T(n) = T\left(\frac{n}{2}\right) + O(1)$$

i.e., $T(n) = O(\log n) = O(\log \frac{a-b}{\epsilon})$



Correctness Proof

P(k) = "For any a, b such that $k\epsilon \leq |a-b| \leq (k+1)\epsilon$ if $f(a)f(b) \leq 0$, then we find an ϵ approx to a root using $\log k$ queries to f"

Base Case: P(1): Output $a + \epsilon$

IH: Assume P(k).

IS: Show P(2k). Consider an arbitrary a, b s.t.,

$$2k\epsilon \leq |a-b| < (2k+1)\epsilon$$

If $f(a+k\epsilon) = 0$ output $a + k\epsilon$.

If $f(\mathrm{a})f(a+k\epsilon) < 0$, solve for interval $a, a+k\epsilon$ using log(k) queries to f.

Otherwise, we must have $f(b)f(a+k\epsilon) < 0$ since $f(a)f(b) < 0$ and $f(a)f(a+k\epsilon) \geq 0$. Solve for interval $a+k\epsilon, b$.

Overall we use at most $\log(k) + 1 = \log(2k)$ queries to *f*.