# CSE 421

## Polynomial Time Reductions

Shayan Oveis Gharan

# Reductions & NP-Completeness

# Computational Complexity

Goal: Classify problems according to the amount of computational resources used by the best algorithms that solve them

Here we focus on time complexity

Recall: worst-case running time of an algorithm

- **max** # steps algorithm takes on any input of size **n**

# Computational Complexity and Reduction

In most cases, we cannot characterize the true hardness of a computational problem

So?

We only reduce the number of problems

Want to be able to make statements of the form

- "If we could solve problem B in polynomial time then we can solve problem A in polynomial time"
- "Problem B is at least as hard as problem A"

# Polynomial Time Reduction

Def $A \leq_P B$: if there is an algorithm for problem A using a 'black box' (subroutine) that solve problem B s.t.,

- Algorithm uses only a polynomial number of steps
- Makes only a polynomial number of calls to a subroutine for **B**

So,

B is Polynomial time solvable → A is Polynomial time solvable

Conversely,

No efficient Algorithm for A → No efficient Algorithm for B

In words, B is as hard as A (it can be even harder)

# $\leq_p^1$ Reductions

In this lecture we see a restricted form of polynomial-time reduction often called Karp or many-to-one reduction

$A \leq_p^1 B$: if and only if there is an algorithm for A given a black box solving B that on input x

- Runs for polynomial time computing an input f(x) of B
- Makes one call to the black box for B for input f(x)
- Returns the answer that the black box gave

We say that the function f(.) is the reduction

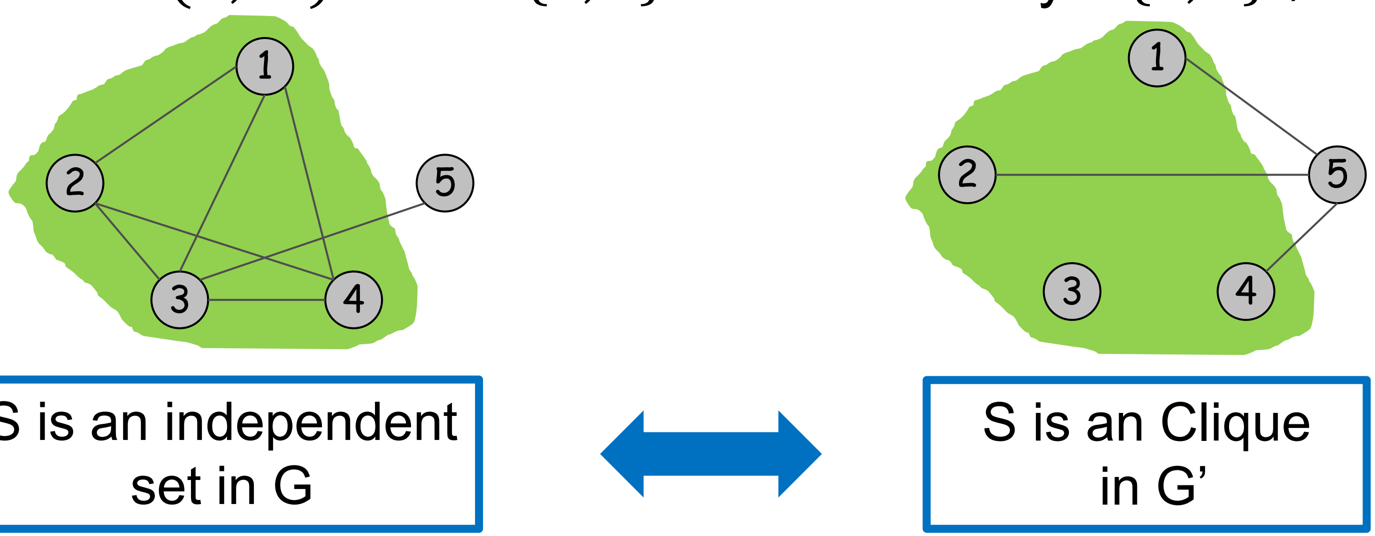
# Example 1: Indep Set $\leq_p$ Clique

Indep Set: Given G=(V,E) and an integer k, is there $S \subseteq V$ s.t. $|S| \geq k$ an no two vertices in S are joined by an edge?

Clique: Given a graph G=(V,E) and an integer k, is there $S \subseteq V$, |U| ≥ k s.t., every pair of vertices in S is joined by an edge?

Claim: Indep Set $\leq_p$ Clique

Pf: Given $G = (V, E)$ and instance of indep Set. Construct a new graph $G' = (V, E')$ where $\{u, v\} \in E'$ if and only if $\{u, v\} \notin E$.



# Example 2: Vertex Cover $\leq_p$ Indep Set

Vertex Cover: Given a graph G=(V,E) and an integer k, is there a vertex cover of size at most k?

Claim: For any graph $G = (V, E)$, S is an independent set iff $V - S$ is a vertex cover

Pf:

=> Let S be a independent set of G

Then, $S$ has at most one endpoint of every edge of G

So, $V - S$ has at least one endpoint of every edge of G

So, $V - S$ is a vertex cover.

<= Suppose $V - S$ is a vertex cover

Then, there is no edge between vertices of S (otherwise, $V - S$ is not a vertex cover)

So, $S$ is an independent set.

# Example 3: Vertex Cover $\leq_p$ Set Cover

Set Cover: Given a set U, collection of subsets $S_1, \dots, S_m$ of U and an integer k, is there a collection of k sets that contain all elements of U?

Claim: Vertex Cover $\leq_p$ Set Cover

Pf:

Given $(G = (V, E), k)$ of vertex cover we construct a set cover input $f(G, k)$

- $U = E$
- For each $v \in V$ we create a set $S_v$ of all edges connected to $v$

This clearly is a polynomial-time reduction

So, we need to prove it gives the right answer

# Example 3: Vertex Cover $\leq_p$ Set Cover

Claim: Vertex Cover $\leq_p$ Set Cover
Pf: Given $(G = (V, E), k)$ of vertex cover we construct a set cover input $f(G, k)$

- $U = E$
- For each $v \in V$ we create a set $S_v$ of all edges connected to $v$

Vertex-Cover (G,k) is yes => Set-Cover f(G,k) is yes
If a set $W \subseteq V$ covers all edges,, just choose $S_v$ for all $v \in W$, it covers all $U$.

Set-Cover f(G,k) is yes => Vertex-Cover (G,k) is yes
If $(S_{v_1}, \ldots, S_{v_k})$ covers all $U$, the set $\{v_1, \ldots, v_k\}$ covers all edges of G.

# Decision Problems

A decision problem is a computational problem where the answer is just yes/no

Here, we study computational complexity of decision Problems.

Why?

- much simpler to deal with
- Decision version is not harder than Search version, so it is easier to lower bound Decision version
- Less important, usually, you can use decider multiple times to find an answer .

# Polynomial Time

Define P (polynomial-time) to be the set of all decision problems solvable by algorithms whose worst-case running time is bounded by some polynomial in the input size.

Do we well understand P?

- We can prove that a problem is in P by exhibiting a polynomial time algorithm
- It is in most cases very hard to prove a problem is not in P.

# Beyond P?

We have seen many problems that seem hard

- Independent Set
- 3-coloring
- Min Vertex Cover
- 3-SAT

The independent set S

The 3-coloring

The vertex cover S

The T/F assignment

Given a 3-CNF $(x_1 \vee \overline{x_2} \vee x_9) \wedge (\overline{x_2} \vee x_3 \vee x_7) \wedge \cdots$ is there a satisfying assignment?

Common Property: If the answer is yes, there is a "short" proof (a.k.a., certificate), that allows you to verify (in polynomial-time) that the answer is yes.

- The proof may be hard to find

# NP

Certifier: algorithm C(x, t) is a certifier for problem A if for every string x, the answer is “yes” iff there exists a string t such that C(x, t) = `yes`.

Intuition: Certifier doesn't determine whether answer is “yes” on its own; rather, it checks a proposed proof that answer is “yes”.

NP: Decision problems for which there exists a poly-time certifier.

Remark. NP stands for nondeterministic polynomial-time.

# Example: 3SAT is in NP

Given a 3-CNF formula, is there a satisfying assignment?

Certificate: An assignment of truth values to the n boolean variables.

Verifier: Check that each clause has at least one true literal.

Ex:$(x_1 \vee \overline{x_3} \vee x_4) \wedge (x_2 \vee \overline{x_4} \vee x_3) \wedge (x_2 \vee \overline{x_1} \vee x_3)$

Certificate: $x_1 = T, x_2 = F, x_3 = T, x_4 = F$

Conclusion: 3-SAT is in NP

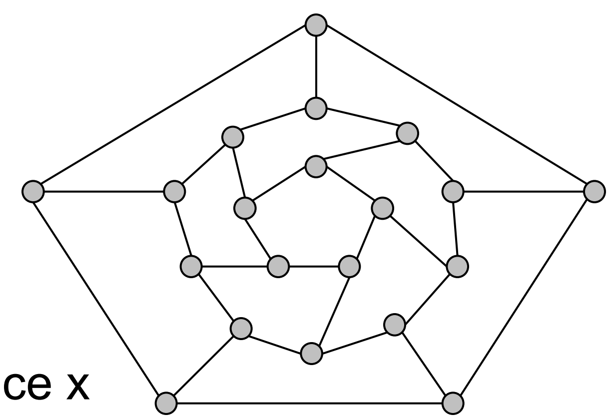
# Example: Hamil-Cycle is in NP

HAM-CYCLE. Given an undirected graph G = (V, E), does there exist a simple cycle C that visits every node?

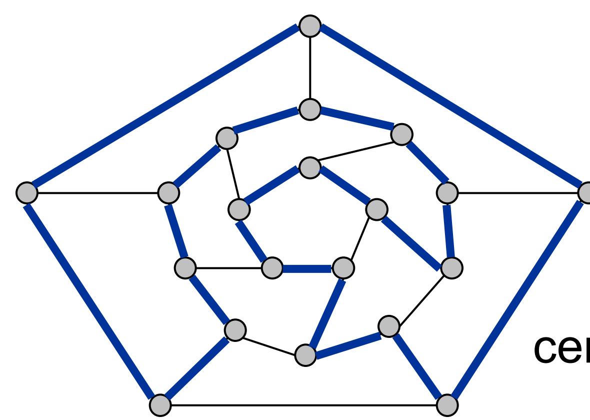
Certificate. A permutation of the n nodes.

Certifier. Check that the permutation contains each node in V exactly once, and that there is an edge between each pair of adjacent nodes in the permutation.

Conclusion. HAM-CYCLE is in NP.



Instance x



certificate t

# Example: Min s,t-cut in in NP

MIN-CUT. Given a flow network, and a number k, does there exist a min-cut of capacity at most k?

Certificate. A min-cut T.

Certifier.Check that the capacity of the min-cut is at most T.

Conclusion. MIN-CUT is in NP.

# P, NP, EXP

P. Decision problems for which there is a poly-time algorithm.

EXP. Decision problems for which there is an exponential-time algorithm.

NP. Decision problems for which there is a poly-time certifier.

Claim. $P \subseteq NP$.

Pf. Consider any problem X in P.

- By definition, there exists a poly-time algorithm A(x) that solves X.
- Certificate: t = empty string, certifier C(x, t) = A(x). ▪

Claim. $NP \subseteq EXP$.

Pf. Consider any problem X in NP.

- By definition, there exists a poly-time certifier C(x, t) for X.
- To solve input x, run C(x, t) on all strings t with $|t| \leq p(|x|)$
- Return `yes`, if C(x, t) returns `yes` for any of these.

# The main question: P vs NP

Does P = NP? [Cook 1971, Edmonds, Levin, Yablonski, Gödel]

Is the decision problem as easy as the certification problem?

Clay $1 million prize.


If yes: Efficient algorithms for 3-COLOR, TSP, FACTOR, SAT, …

If no: No efficient algorithms possible for 3-COLOR, TSP, SAT, …

# What do we know about NP?

- Nobody knows if all problems in NP can be done in polynomial time, i.e. does P=NP?
  - one of the most important open questions in all of science.
  - Huge practical implications specially if answer is yes

- To show Hamil-cycle $\notin P$ we have to prove that there is no poly-time algorithm for it even using all mathematical theorem that will be discovered in future!

# NP Completeness

Complexity Theorists Approach: We don't know how to prove any problem in NP is hard. So, let's find hardest problems in NP.

NP-hard: A problem B is NP-hard iff for any problem $A \in NP$, we have $A \leq_p B$

NP-Completeness: A problem B is NP-complete iff B is NP-hard and $B \in NP$.

Motivations:

- If $P \neq NP$, then every NP-Complete problems is not in P. So, we shouldn't try to design Polytime algorithms
- To show $P = NP$, it is enough to design a polynomial time algorithm for just one NP-complete problem.

# Cook-Levin Theorem

Theorem (Cook 71, Levin 73): 3-SAT is NP-complete, i.e., for all problems $A \in NP$, $A \leq_p$ 3-SAT.

- So, 3-SAT is the hardest problem in NP.

What does this say about other problems of interest? Like Independent set, Vertex Cover, …

Fact: If $A \leq_p B$ and $B \leq_p C$ then, $A \leq_p C$
Pf idea: Just compose the reductions from A to B and B to C

So, if we prove 3-SAT $\leq_p$ Independent set, then Independent Set, Clique, Vertex cover, Set cover are all NP-complete

3-SAT $\leq_p$ Independent Set $\leq_p$ Vertex Cover $\leq_p$ Set Cover