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CSE 421  
Midterm Exam  
February 15, 2002

**Instructions:**

- You have 50 minutes to complete the exam.
- Please do not turn the page until you are instructed to do so.
- Good luck!

1. (14 points, 2 each) Indicate for each of the following if it is **true or false** by circling the appropriate answer. (Remember, a statement is true only if it is true in all instances.)
  - True or False: There is an algorithm to determine whether a given undirected graph on  $n$  nodes is a tree that runs in  $O(n)$  time.
  - True or False: Suppose that  $G$  is a connected, undirected graph. If removing edge  $e$  from  $G$  disconnects the graph, then  $e$  is a tree edge in the depth-first search of  $G$ .
  - True or False: Let  $G = (V, E)$  be a directed graph, where  $(u, v) \in E$ . If in a depth-first search on  $G$  the edge  $(u, v)$  is a cross edge, then  $u$  is visited before  $v$ .
  - True or False: Suppose that  $G$  is a directed, acyclic graph and has a topological ordering with  $v_1$  the first node in the ordering and  $v_n$  the last node in the ordering. Then there is a path in  $G$  from  $v_1$  to  $v_n$ .
  - True or False: It is possible for a directed graph  $G$  with exactly two topological orderings to have exactly one node of in-degree 0.
  
2. (10 points, 2 each) For the following problems, let  $G$  be an undirected, weighted graph that is connected. Please indicate whether each statement is **true or false** by circling the appropriate answer.
  - True or False: Suppose that  $e$  is a minimum weight edge  $G$ , and all the edge weights are distinct. Then  $e$  is always contained in the minimum spanning tree of  $G$ .
  - True or False: The path between a pair of vertices in a minimum spanning tree for  $G$  must be a shortest (*i.e.* least-cost) path between the two vertices in  $G$ .
  - True or False: Suppose that  $G$  has a unique minimum spanning tree. Then all edges have distinct weights.
  - True or False: Say edge  $e$  is the cheapest edge in some cycle of  $G$  and all edge weights are distinct. Then  $e$  is in the minimum spanning tree of  $G$ .
  - True or False: Suppose we run Dijkstra's algorithm on  $G$  starting at some vertex,  $s$ . Recall that at each step of the algorithm, we add another vertex to the set  $S$ . Suppose we add vertex  $u$  to  $S$  before we add the vertex  $v$  to  $S$ . Then the distance from  $s$  to  $u$  in  $G$  is no greater than the distance from  $s$  to  $v$ .

3. (10 points) Consider the generalization of the stable matching problem, in which certain man-woman pairs are explicitly *forbidden*. Specifically, we have a set  $M$  of  $n$  men, a set  $W$  of  $n$  women, and a set  $F \subseteq M \times W$  of pairs who are simply *not allowed* to get married. Each man  $m$  ranks all the women  $w$  for which  $(m, w) \notin F$ , and each woman  $w$  ranks all the men  $m'$  for which  $(m', w) \notin F$ . Note that under these more general definitions, a stable matching need not be a perfect matching.

Consider the following algorithm for finding a stable matching that consists only of unforbidden pairs.

```

Initially all  $m \in M$  and  $w \in W$  are free (i.e.,  $S = \emptyset$ )
While there is a man  $m$  who is free and hasn't proposed to
  every woman  $w$  for which  $(m, w) \notin F$ 
  Choose such a man  $m$ 
  Let  $w$  be the highest-ranked woman in  $m$ 's preference list
    to which  $m$  has not yet proposed
  If  $w$  is free then
     $(m, w)$  become engaged
  Else  $w$  is currently engaged to  $m'$ 
    If  $w$  prefers  $m'$  to  $m$  then
       $m$  remains free
    Else  $w$  prefers  $m$  to  $m'$ 
       $(m, w)$  become engaged
       $m'$  becomes free
    Endif
  Endif
Endwhile
Return the set  $S$  of engaged pairs

```

(2 points each) For each of the following statements about this algorithm, indicate whether or not the statement is **true or false**, by circling the appropriate answer.

- True or False: Any woman  $w$  remains engaged from the point at which she receives her first proposal, and the sequence of partners to which she is engaged get better and better (in terms of her preference list).
- True or False: If a man  $m$  is free at the end of the algorithm, then he must have proposed to every non-forbidden woman.
- True or False: If a woman  $w$  is free at the end of the algorithm, then it must be that no man ever proposed to  $w$ .
- True or False: At the end of the algorithm, there can be a man  $m$  and a woman  $w$ , such that  $(m, w) \notin F$ , but neither of which is part of any pair in the matching  $S$ .
- True or False: At the end of the algorithm, there can be a pair  $(m, w) \in S$  and a man  $m'$  that is free,  $(m', w) \notin F$ , but such that  $w$  prefers  $m'$  to  $m$ .

4. (6 points) **Fill in the blanks:**

- (3 points) Give an example of preference lists for two men and two women such that the male-optimal stable matching (where each man is paired with his best valid partner) and the female-optimal stable matching (where each woman is paired with her best valid partner) are different.

man 1's preference list: .....

man 2's preference list: .....

woman 1's preference list: .....

woman 2's preference list: .....

- (3 points) Suppose that  $e$  is the maximum weight edge in an undirected, weighted graph  $G$ , where the weights on all edges are distinct. Give a necessary and sufficient condition for  $e$  to be in the minimum spanning tree of  $G$ .

- Consider a divide and conquer algorithm where the running time  $T(n)$  satisfies

$$T(n) = 5T(n/2) + n$$

for  $n \geq 1$  and  $T(1) = 0$ . The running time of the algorithm is  $\Theta(\dots\dots\dots)$ .

5. (7 points) Consider the following problem. You are given two sequences over some alphabet: sequence  $S$  consists of  $s_1, \dots, s_n$  and sequence  $S'$  consists of  $s'_1, \dots, s'_m$ . Your goal is to determine whether  $S'$  is a subsequence of  $S$  (i.e.,  $S'$  can be obtained from  $S$  by deleting  $n - m$  of the letters in  $S$ .) For example,  $S' = abbc$  is a subsequence of  $S = adbbc$ , but  $S' = abcc$  is not.

Now consider the following greedy algorithm for this problem. Find the first letter in  $S$  that is the same as  $s'_1$ , match these two letters, then find the first letter after this that is the same as  $s'_2$ , and so on. We will use  $k_1, k_2, \dots$  to denote the positions of the match have we found so far,  $i$  to denote the current position in  $S$ , and  $j$  the current position in  $S'$ . We obtain the following pseudocode for this algorithm:

```

Initially  $i = j = 1$ 
While  $i \leq n$  and  $j \leq m$ 
    If  $s_i$  is the same as  $s'_j$ , then
        let  $k_j = i$  (so that  $s_{k_j} = s'_j$ )
        let  $i = i + 1$  and  $j = j + 1$ 
    otherwise let  $i = i + 1$ 
EndWhile
If  $j = m + 1$  return the subsequence found:

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positions  $k_1, \dots, k_m$ .  
Else return that " $S'$  is not a subsequence of  $S$ "

For each of the following, fill in the blanks.

- (2 points) The running time of this algorithm is .....
- (5 points) Suppose that there is a match, i.e.  $S'$  is the same as the subsequence at positions  $l_1, \dots, l_m$  of  $S$ . (Note that there could be other subsequences of  $S$  that also give  $S'$ .) We'd like to prove that in this case, the greedy algorithm is guaranteed to find a match. Fill in the statement that you would prove by induction in order to show this. (You do not need to supply a proof of the statement.)

$\forall j, 1 \leq j \leq m, \dots$

6. Given an array of  $n$  real numbers, consider the problem of finding the maximum sum in any contiguous subvector of the input. For example, in the array

$$\{31, -41, 59, 26, -53, 58, 97, -93, -23, 84\}$$

the maximum is achieved by summing the third through seventh elements, where  $59 + 26 + (-53) + 58 + 97 = 187$ . When all numbers are positive, the entire array is the answer, while when all numbers are negative, the empty array maximizes the total at 0.

Give an  $O(n \log n)$  divide and conquer algorithm for solving this problem.