CSE 421

Matching, Connectivity, Image Segmentation

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Perfect Bipartite Matching

Perfect Bipartite Matching

Def. A matching M $\subseteq$ E is perfect if each node appears in exactly one edge in M.

Q. When does a bipartite graph have a perfect matching?

Structure of bipartite graphs with perfect matchings:

- Clearly we must have $|X| = |Y|$.
- What other conditions are necessary?
- What conditions are sufficient?

Marriage Theorem

Thm: [Frobenius 1917, Hall 1935] Let $G = (X \cup Y, E)$ be a bipartite graph with |X| = |Y|.

Then, G has a perfect matching iff $|N(S)| \geq |S|$ for all subsets $S \subseteq X$.

Pf. $\Rightarrow$

This was the previous observation.

If |N(S)| < |S| for some S, then there is no perfect matching.

Marriage Theorem

Pf. $\exists S \subseteq X$ s.t., $|N(S)| < |S|$ $\Leftarrow$ G does not a perfect matching

Formulate as a max-flow and let (A, B) be the min s-t cut

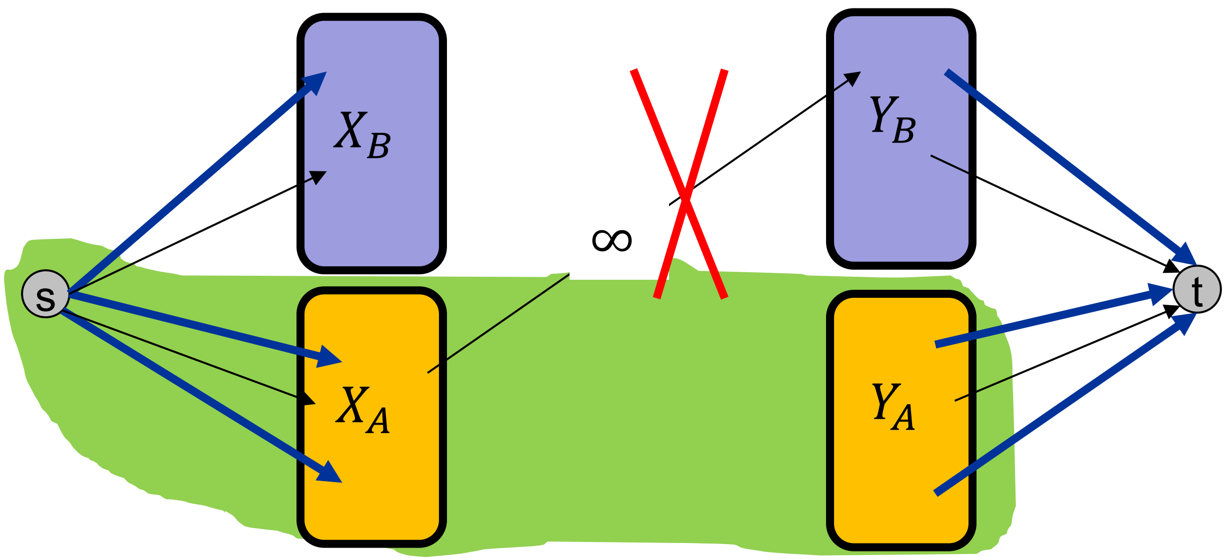
G has no perfect matching => $v(f^*) < |X|$. So, $cap(A, B) < |X|$

Define $X_A = X \cap A, X_B = X \cap B, Y_A = Y \cap A$

Then, $cap(A, B) = |X_B| + |Y_A|$

Since min-cut does not use ∞ edges, $N(X_A) \subseteq Y_A$

$|N(X_A)| \leq |Y_A| = cap(A, B) - |X_B| = cap(A, B) - |X| + |X_A| < |X_A|$



Bipartite Matching Running Time

Which max flow algorithm to use for bipartite matching?

- Generic augmenting path: O(m val(f*)) = O(mn).
- Capacity scaling: $O(m^2 \log C) = O(m^2)$.
- Shortest augmenting path: $O(m\, n^{1/2})$.
- Recent algorithms $O(m^{1+o(1)})$ [Chen-Kyng-Liu-Peng-Gutenberg-Sachdeva'22]

Non-bipartite matching.

- Structure of non-bipartite graphs is more complicated, but well-understood. [Tutte-Berge, Edmonds-Galai]
- Blossom algorithm: $O(n^4)$. [Edmonds 1965]
- Best known: $O(m\, n^{1/2})$. [Micali-Vazirani 1980]

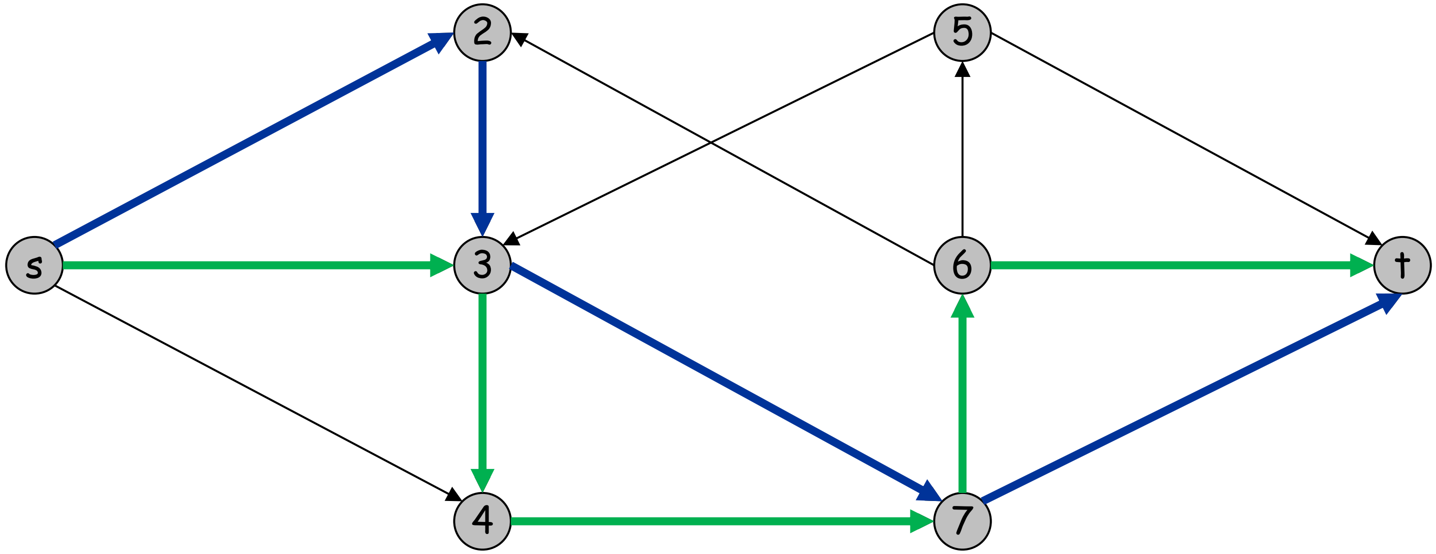
Edge Disjoint Paths

Edge Disjoint Paths Problem

Given a digraph G = (V, E) and two nodes s and t, find the max number of edge-disjoint s-t paths.

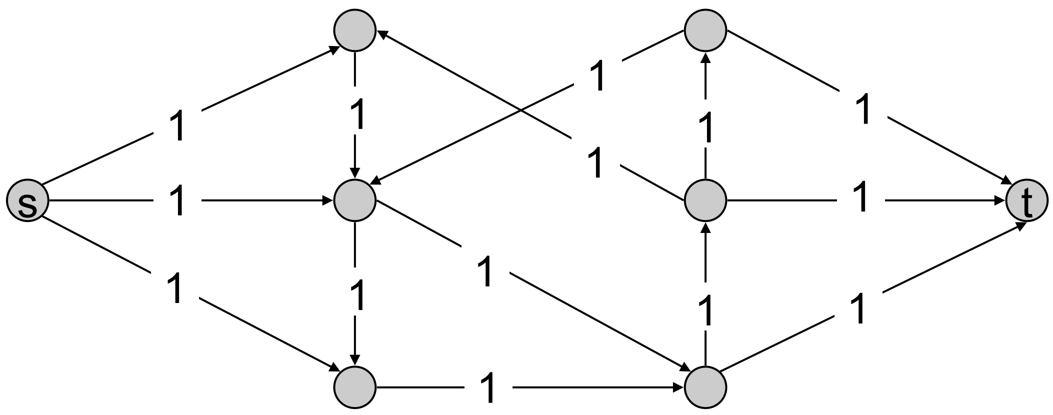
Def. Two paths are edge-disjoint if they have no edge in common.

Ex: communication networks.



Max Flow Formulation

Assign a unit capacitary to every edge. Find Max flow from s to t.



Thm. Max number edge-disjoint s-t paths equals max flow value.

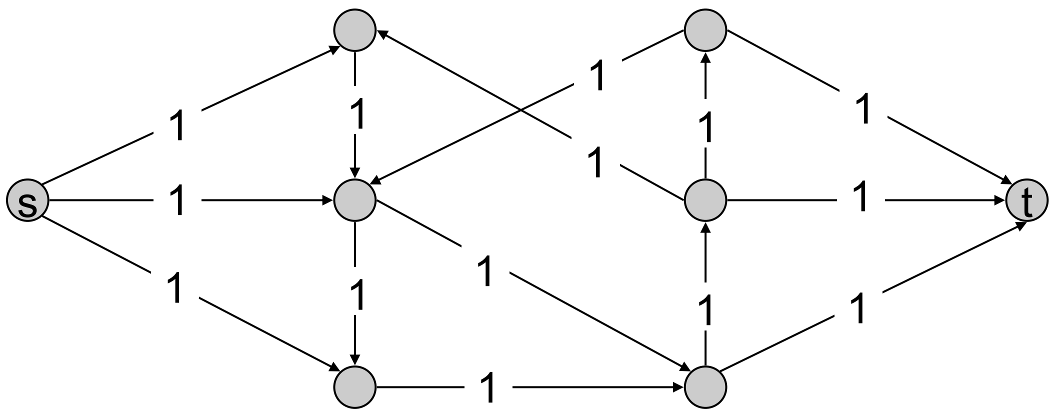
Pf. $\leq$

Suppose there are k edge-disjoint paths $P_1, \dots, P_k$.

Set f(e) = 1 if e participates in some path P_i ; else set f(e) = 0.

Since paths are edge-disjoint, f is a flow of value k. ▪

Max Flow Formulation



Thm. Max number edge-disjoint s-t paths equals max flow value.

Pf. $\geq$ Suppose max flow value is k

Integrality theorem $\Rightarrow$ there exists 0-1 flow f of value k.

Consider edge (s, u) with f(s, u) = 1.

- by conservation, there exists an edge (u, v) with f(u, v) = 1
- continue until reach t, always choosing a new edge

This produces k (not necessarily simple) edge-disjoint paths. ▪

We can return to u so we can have cycles. But we can eliminate cycles if desired

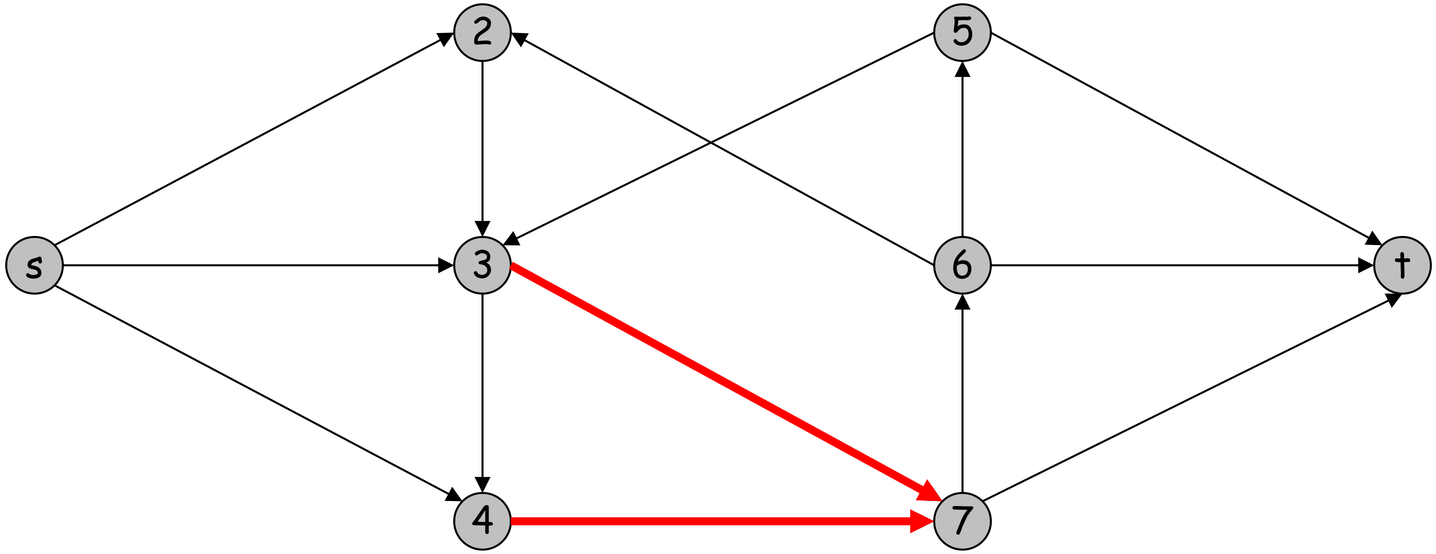
Network Connectivity

Network Connectivity

Given a digraph G = (V, E) and two nodes s and t, find min number of edges whose removal disconnects t from s.

Def. A set of edges $F \subseteq E$ disconnects t from s if all s-t paths uses at least one edge in F.

Ex: In testing network reliability



Network Connectivity using Min Cut

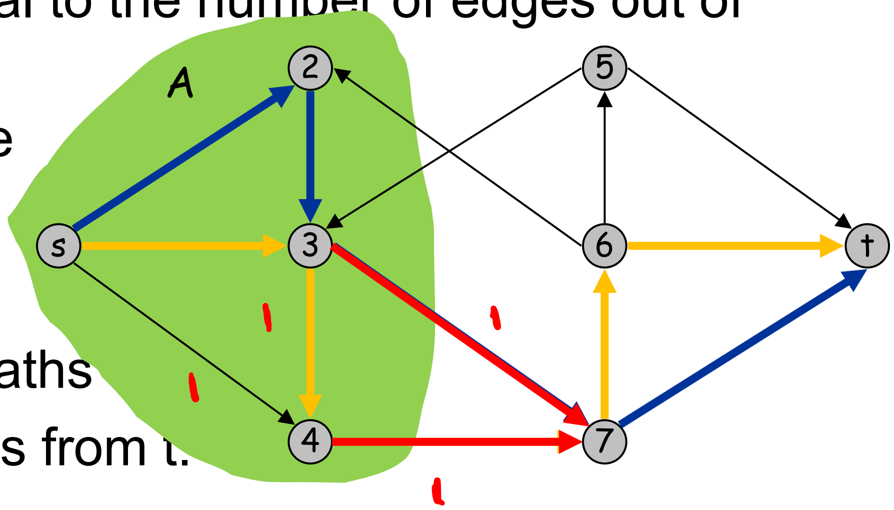
Thm. [Menger 1927] The max number of edge-disjoint s-t paths is equal to the min number of edges whose removal disconnects t from s.

Pf.

i) We showed that max number edge disjoint s-t paths = max flow.

ii) Max-flow Min-cut theorem => min s-t cut = max-flow

iii) For a s-t cut (A,B), cap(A,B) is equal to the number of edges out of A. In other words, every s-t cut (A,B) corresponds to cap(A,B) edges whose removal disconnects s from t.



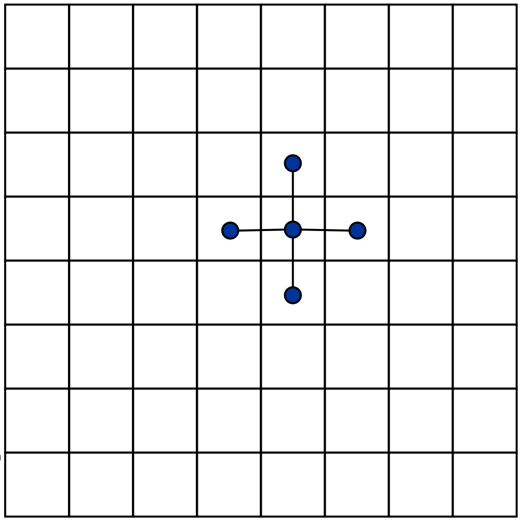
So, max number of edge disjoint s-t paths = min number of edges to disconnect s from t.

Image Segmentation

Given an image we want to separate foreground from background

- Central problem in image processing.
- Divide image into coherent regions.

Foreground / background segmentation



Label each pixel as foreground/background.

- V = set of pixels, E = pairs of neighboring pixels.
- $a_i \geq 0$ is likelihood pixel i in foreground.
- $b_i \geq 0$ is likelihood pixel i in background.
- $p_{i,j} \geq 0$ is separation penalty for labeling one of i and j as foreground, and the other as background.

Goals.

Accuracy: if $a_i > b_i$ in isolation, prefer to label i in foreground.

Smoothness: if many neighbors of i are labeled foreground, we should be inclined to label i as foreground.

Find partition (A, B) that maximizes:

$$\sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ i \in A, j \in B}} p_{i,j}$$

Foreground (→ A)

Background (→ B)

Image Seg: Min Cut Formulation

Difficulties:

- Maximization (as opposed to minimization)
- No source or sink
- Undirected graph

Step 1: Turn into Minimization

Maximizing

$$\sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ i \in A, j \in B}} p_{i,j}$$

Equivalent to minimizing

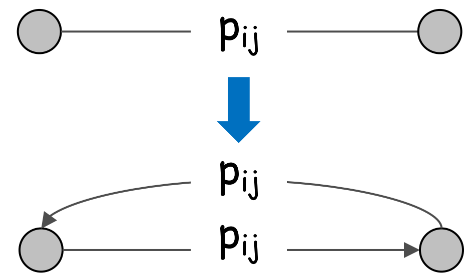
$$+\sum_{i \in V} a_i + \sum_{j \in V} b_j - \sum_{i \in A} a_i - \sum_{j \in B} b_j + \sum_{\substack{(i,j) \in E \\ i \in A, j \in B}} p_{i,j}$$

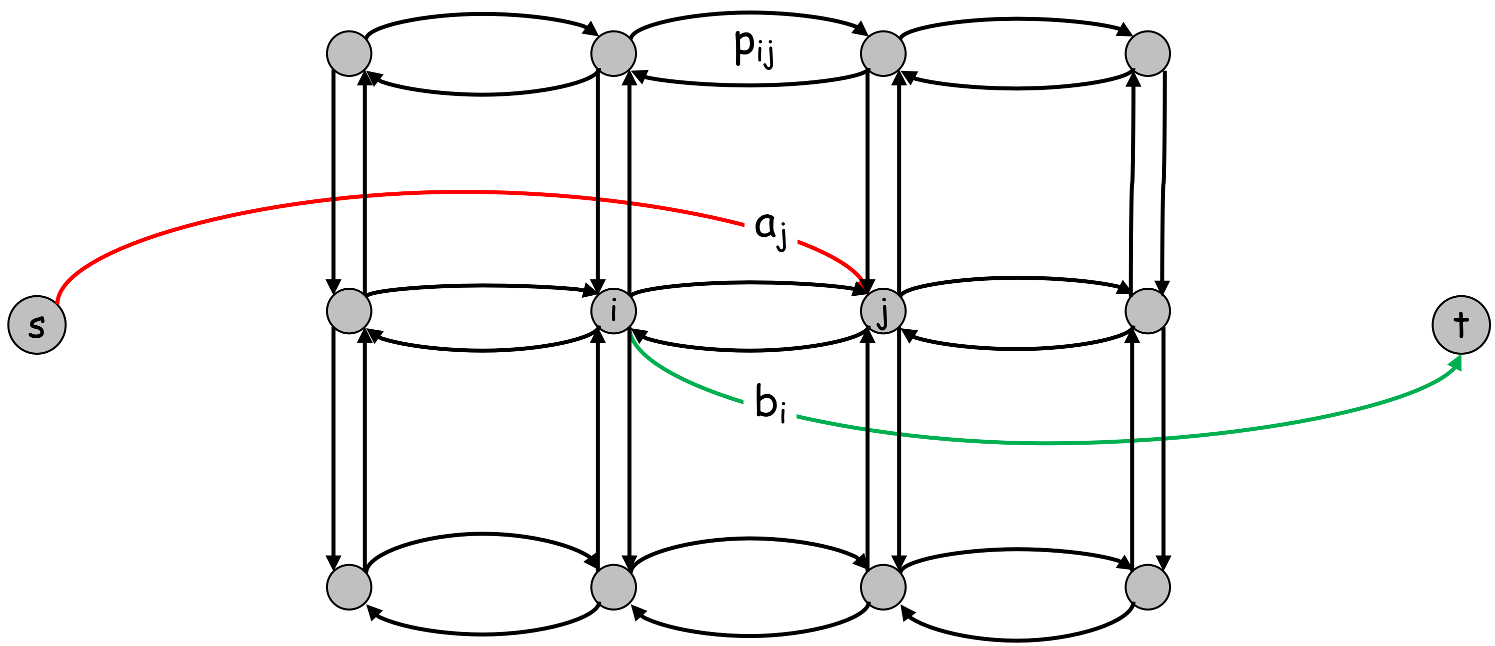
Equivalent to minimizing

$$+\sum_{j \in B} a_j + \sum_{i \in A} b_i + \sum_{\substack{(i,j) \in E \\ i \in A, j \in B}} p_{i,j}$$

Min cut Formulation (cont'd)

G' = (V', E').
Add s to correspond to foreground;
Add t to correspond to background
Use two anti-parallel edges
instead of undirected edge.





G'

Min cut Formulation (cont'd)

Consider min cut (A, B) in G'. (A = foreground.)

$$cap(A,B) = \sum_{j \in B} a_j + \sum_{i \in A} b_i + \sum_{\substack{(i,j) \in E \\ i \in A, j \in B}} p_{i,j}$$

Precisely the quantity we want to minimize.