CSE 421: Introduction to Algorithms

Induction - Graphs

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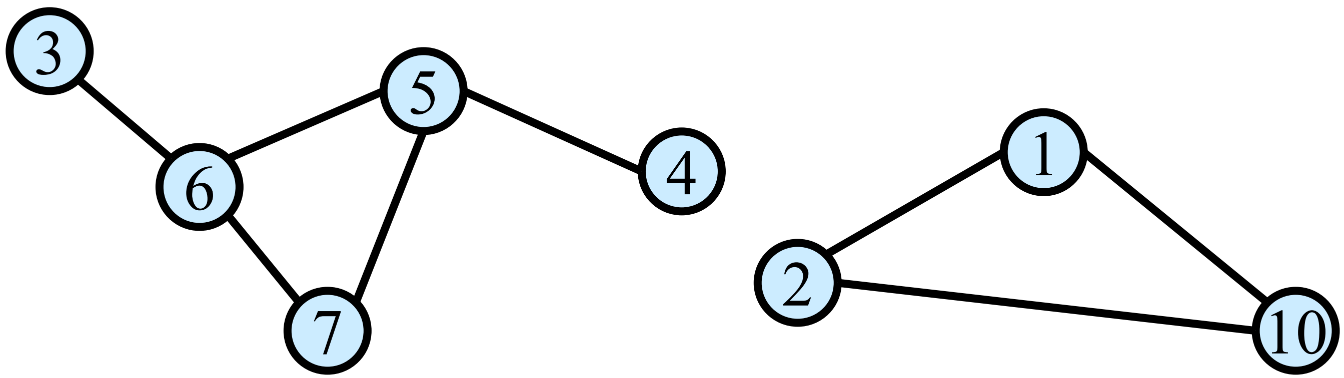
Terminology

- Degree of a vertex: # edges that touch that vertex

deg(6)=3

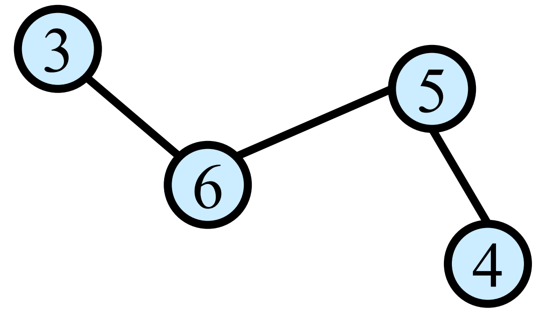


- Connected: Graph is connected if there is a path between every two vertices

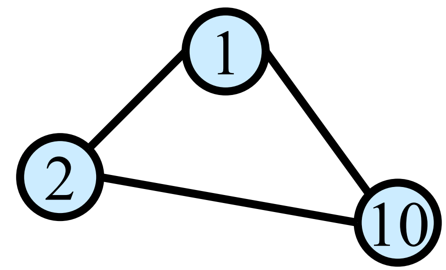
- Connected component: Maximal set of connected vertices

Terminology (cont'd)

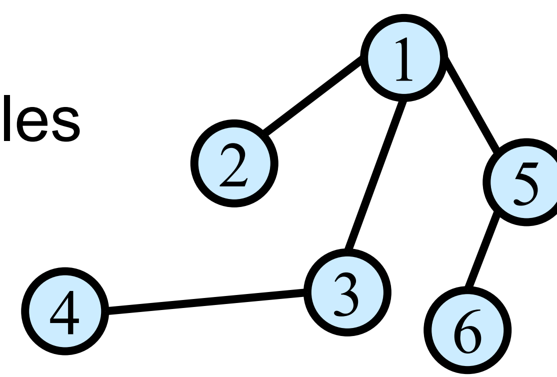
- Path: A sequence of distinct vertices s.t. each vertex is connected to the next vertex with an edge



- Cycle: Path of length > 2 that has the same start and end



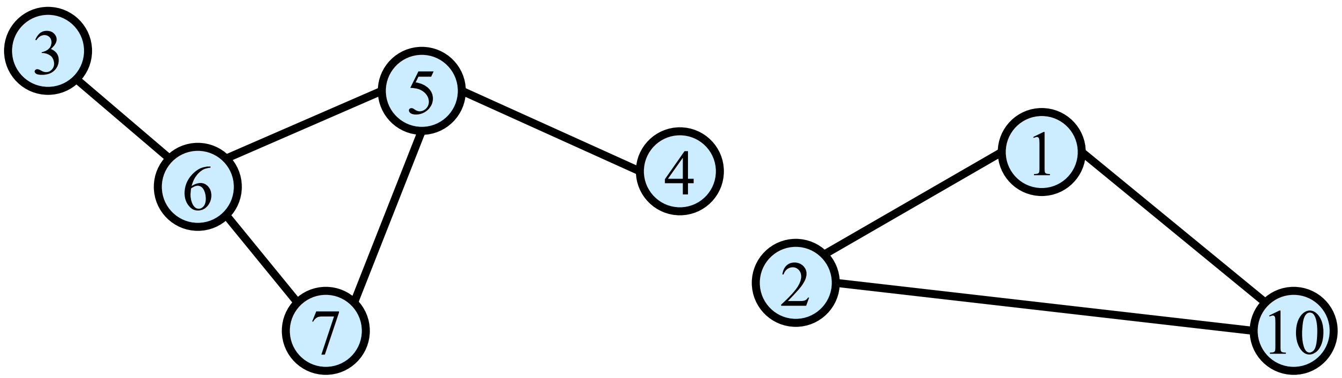
- Tree: A connected graph with no cycles



Degree Sum

Claim: In any undirected graph, the number of edges is equal to $(1/2)\sum_{\text{vertex } v} \deg(v)$

Pf: $\sum_{\text{vertex } v} \deg(v)$ counts every edge of the graph exactly twice; once from each end of the edge.



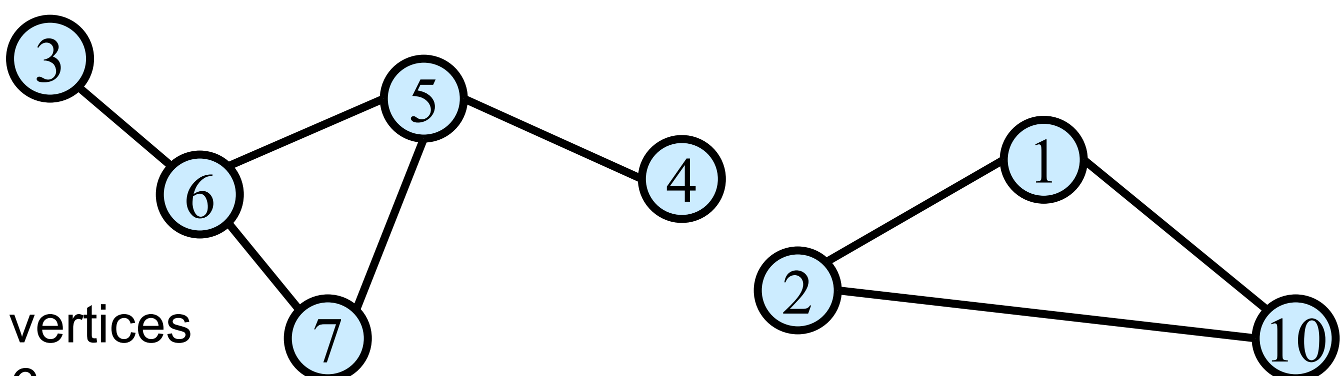
|E|=8

$$\sum_{\text{vertex } v} \deg(v) = 2 + 2 + 1 + 1 + 3 + 2 + 3 + 2 = 16$$

Odd Degree Vertices

Claim: In any undirected graph, the number of odd degree vertices is even

Pf: In previous claim we showed sum of all vertex degrees is even. So there must be even number of odd degree vertices, because sum of odd number of odd numbers is odd.



4 odd degree vertices
3, 4, 5, 6

Degree 1 vertices

Claim: If G has no cycle, then it has a vertex of degree ≤ 1

(So, every tree has a leaf)

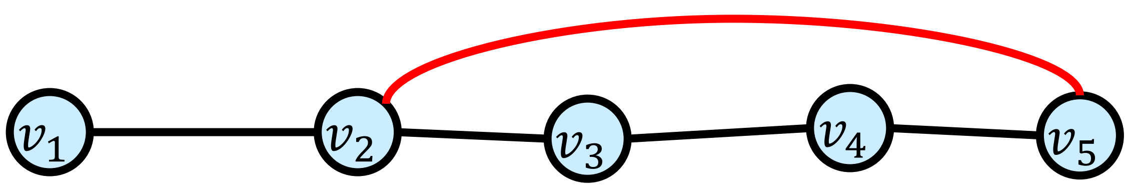
Pf: (By contradiction)

Suppose every vertex has degree ≥ 2.

Start from a vertex v_1 and follow a path, $v_1, \ldots, v_i$ when we are at v_i we choose the next vertex to be different from v_{i-1}. We can do so because $\deg(v_i) \geq 2$.

The first time that we see a repeated vertex ($v_j = v_i$) we get a cycle.

We always get a repeated vertex because G has finitely many vertices



Trees and Induction

Claim: Show that every tree with n vertices has n-1 edges.

Pf: By induction.

Base Case: n=1, the tree has no edge

IH: Suppose every tree with n-1 vertices has n-2 edges

IS: Let T be a tree with n vertices.

So, T has a vertex v of degree 1.

Remove v and the neighboring edge, and let T' be the new graph.

We claim T' is a tree: It has no cycle, and it must be connected.

So, T' has n-2 edges and T has n-1 edges.

#edges

Let $G = (V, E)$ be a graph with $n = |V|$ vertices and $m = |E|$ edges.

Claim: $0 \leq m \leq \binom{n}{2} = \frac{n(n-1)}{2} = O(n^2)$

Pf: Since every edge connects two distinct vertices (i.e., G has no loops)

and no two edges connect the same pair of vertices (i.e., G has no multi-edges)

It has at most $\binom{n}{2}$ edges.

Sparse Graphs

A graph is called sparse if $m \ll n^2$ and it is called dense otherwise.

Sparse graphs are very common in practice

- Friendships in social network
- Planar graphs
- Web braph

Q: Which is a better running time $O(n+m)$ vs $O(n^2)$?

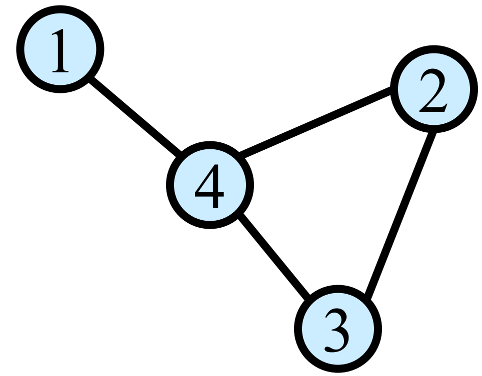
A: $O(n+m) = O(n^2)$, but $O(n+m)$ is usually much better.

Storing Graphs (Internally in ALG)

Vertex set $V = \{v_1, \dots, v_n\}$.

Adjacency Matrix: A

- For all, $i, j, A[i,j] = 1$ iff $(v_i, v_j) \in E$
- Storage: n^2 bits



	1	2	3	4
1	0	0	0	1
2	0	0	1	1
3	0	1	0	1
4	1	1	1	0

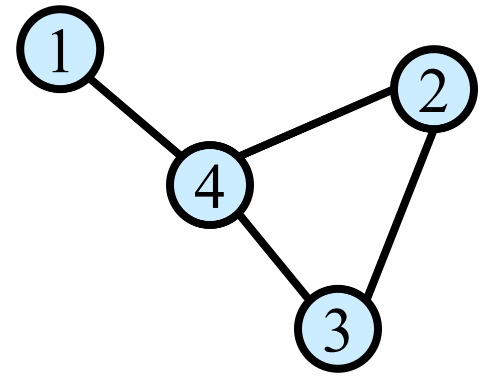
Advantage:

- $O(1)$ test for presence or absence of edges

Disadvantage:

- Inefficient for sparse graphs both in storage and edge-access

Storing Graphs (Internally in ALG)



Adjacency List:

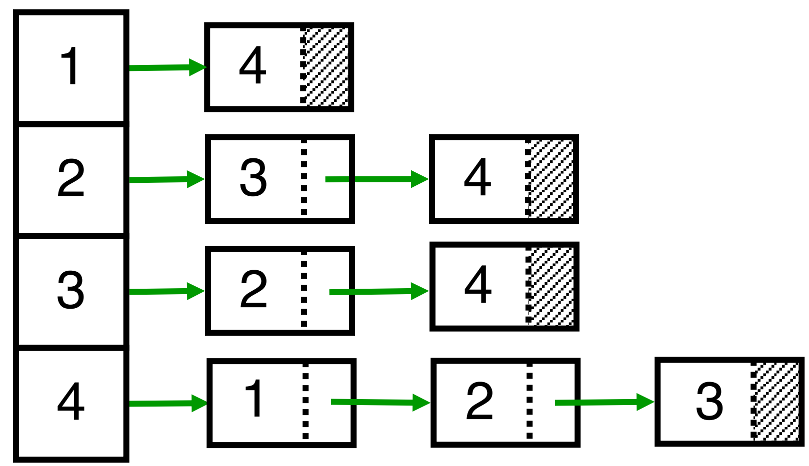
O(n+m) words

Advantage

- Compact for sparse
- Easily see all edges

Disadvantage

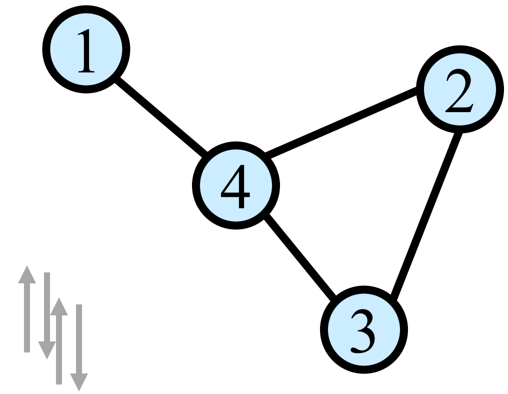
- No O(1) edge test
- More complex data structure



Storing Graphs (Internally in ALG)

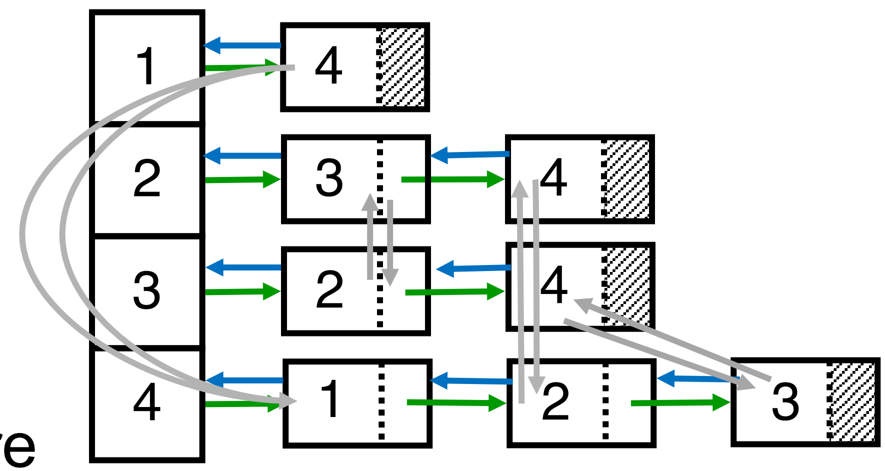
Adjacency List:

O(n+m) words



Advantage

- Compact for sparse
- Easily see all edges



Disadvantage

- No O(1) edge test
- More complex data structure

Graph Traversal

Walk (via edges) from a fixed starting vertex s to all vertices reachable from s.

- Breadth First Search (BFS): Order nodes in successive layers based on distance from s
- Depth First Search (DFS): More natural approach for exploring a maze; many efficient algs build on it.

Applications:

- Finding Connected components of a graph
- Testing Bipartiteness
- Finding Aritculation points

Breadth First Search (BFS)

Completely explore the vertices in order of their distance from s.

Three states of vertices:

- Undiscovered
- Discovered
- Fully-explored

Naturally implemented using a queue

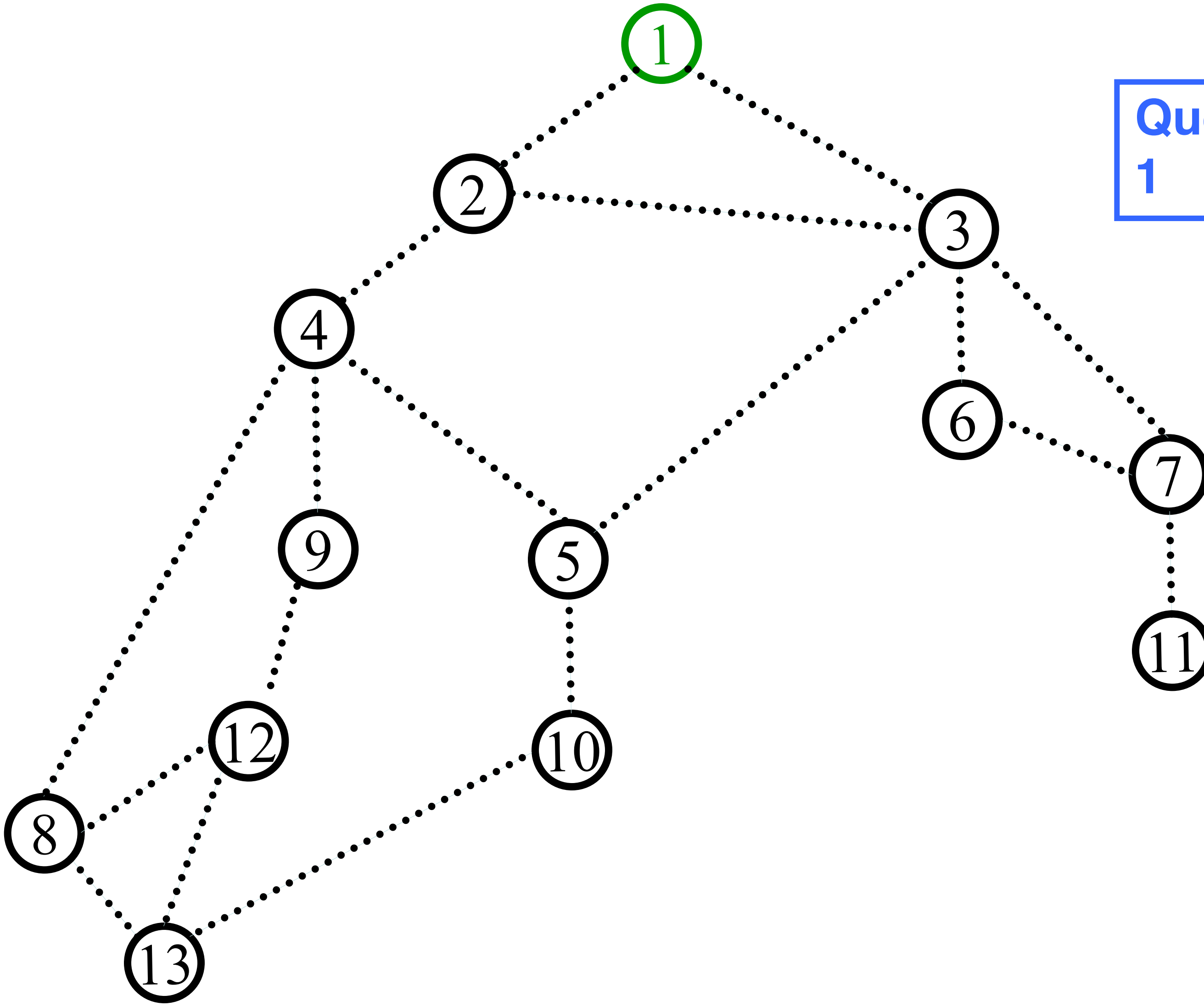
The queue will always have the list of Discovered vertices

BFS implementation

Global initialization: mark all vertices "undiscovered"

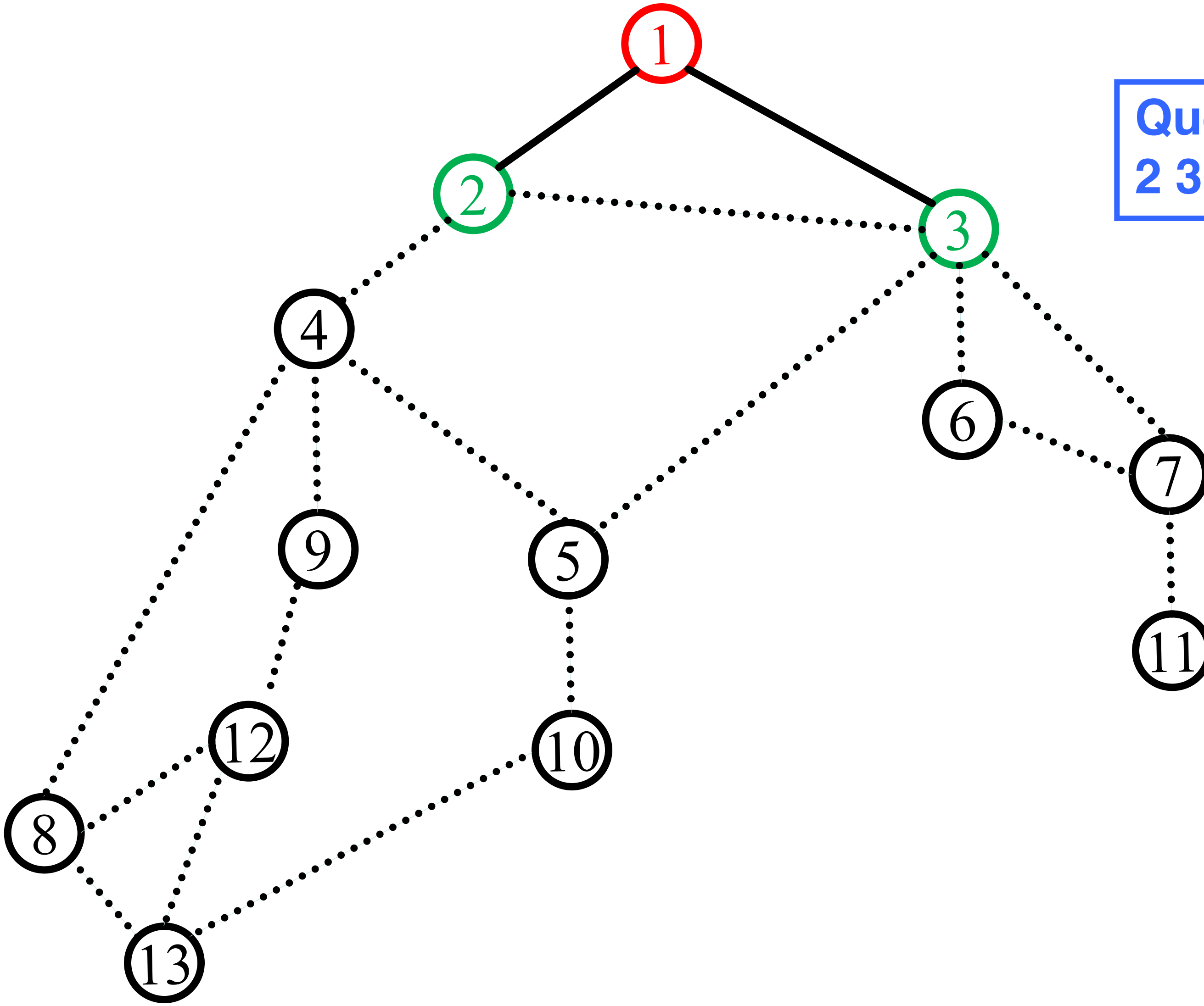
```
BFS(s)
    mark  s "discovered"
    queue = { s }
    while queue not empty
        u = remove_first(queue)
        for each edge {u,x}
            if (x is undiscovered)
                mark x discovered
                append x on queue
        mark u fully-explored
```

BFS(1)



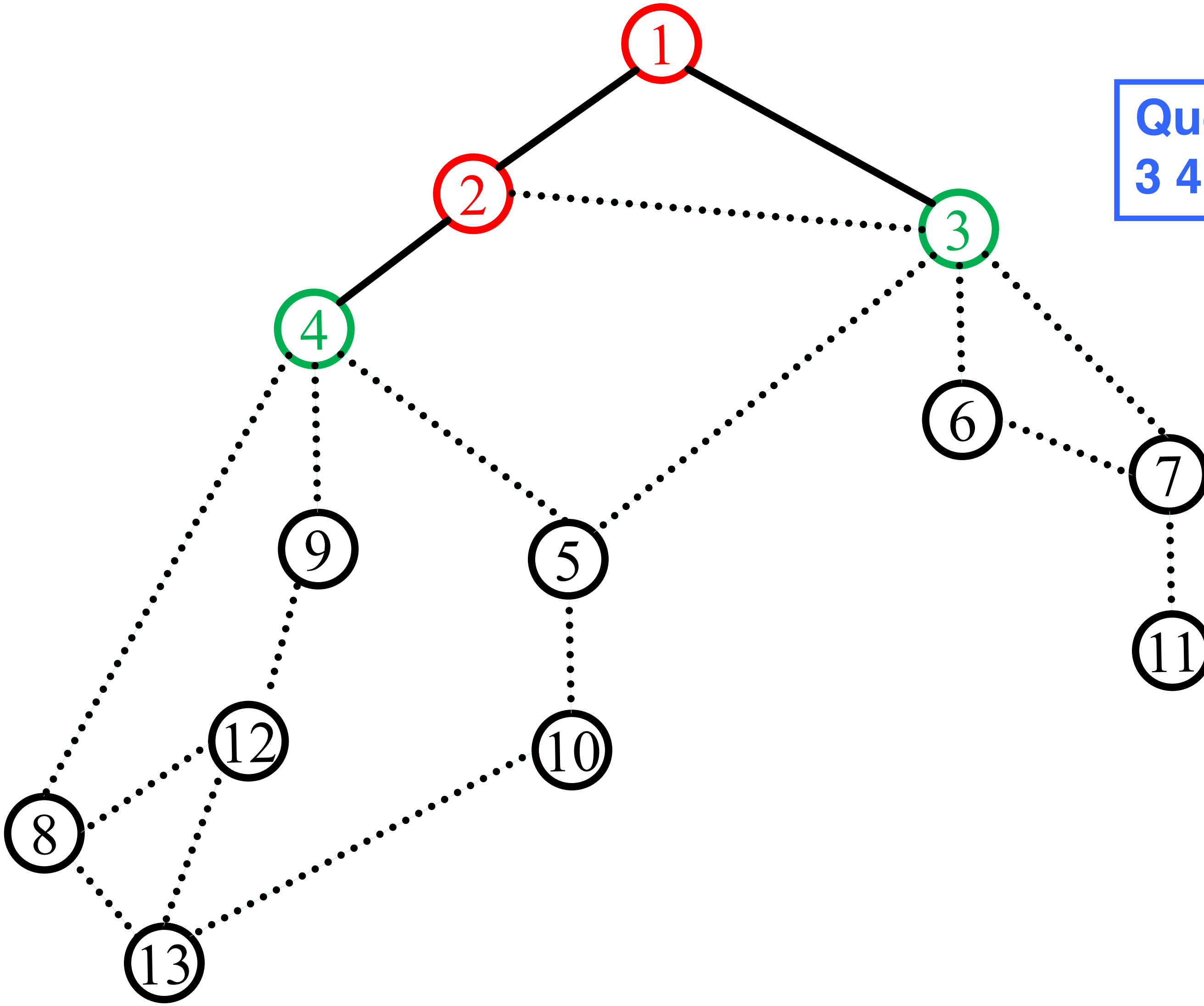
Queue:
1

BFS(1)



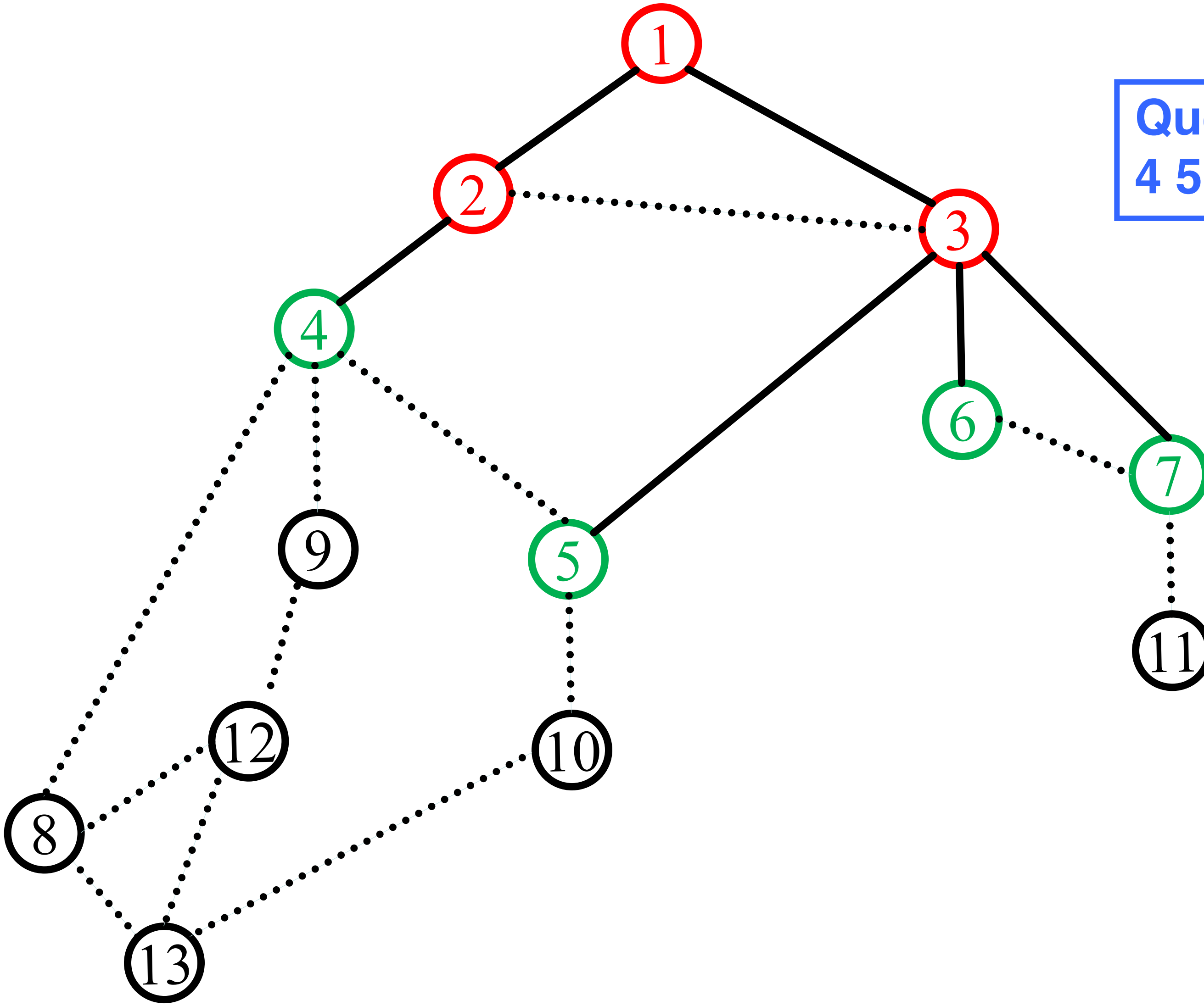
Queue:
2 3

BFS(1)



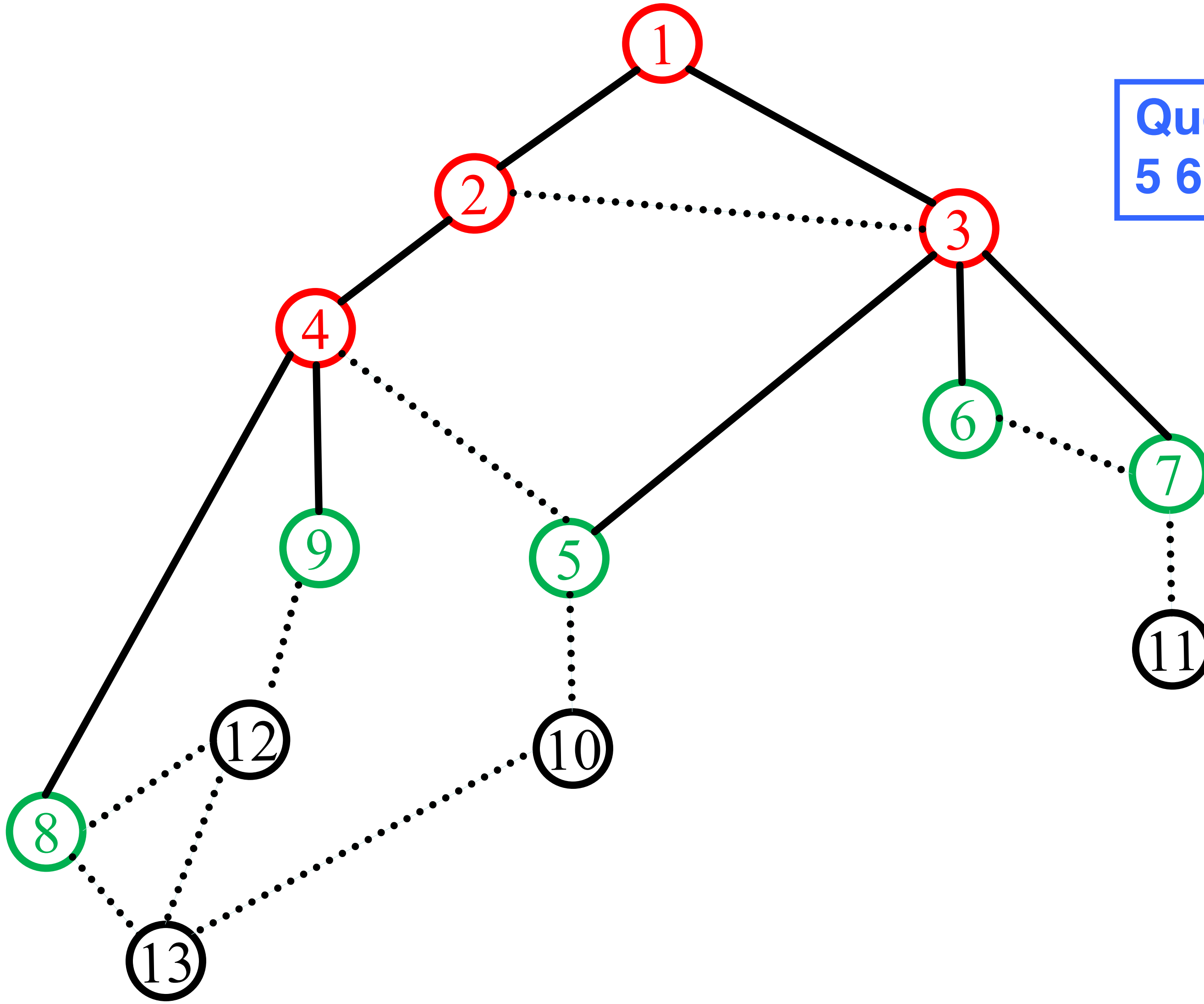
Queue:
3 4

BFS(1)



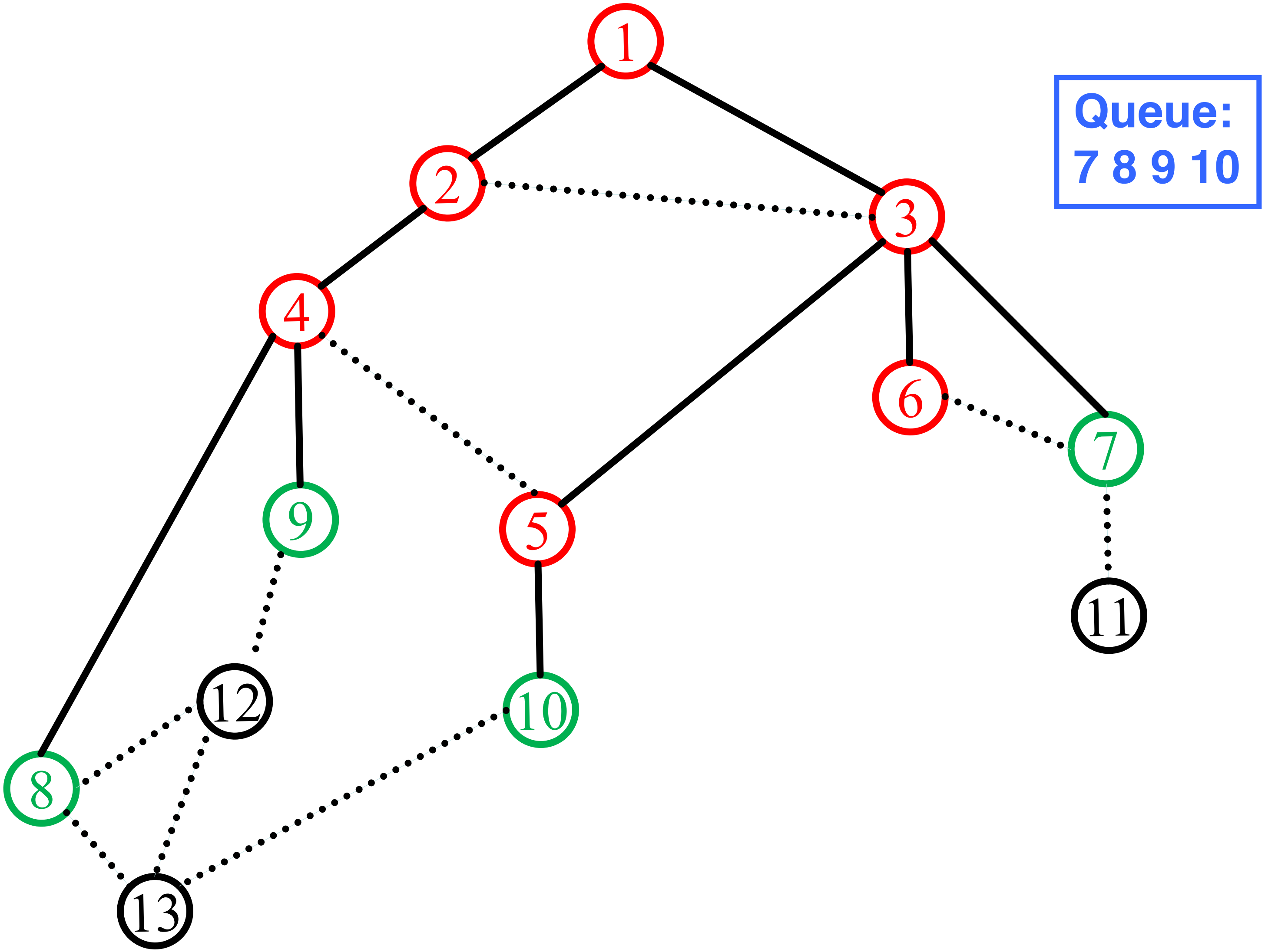
Queue:
4 5 6 7

BFS(1)

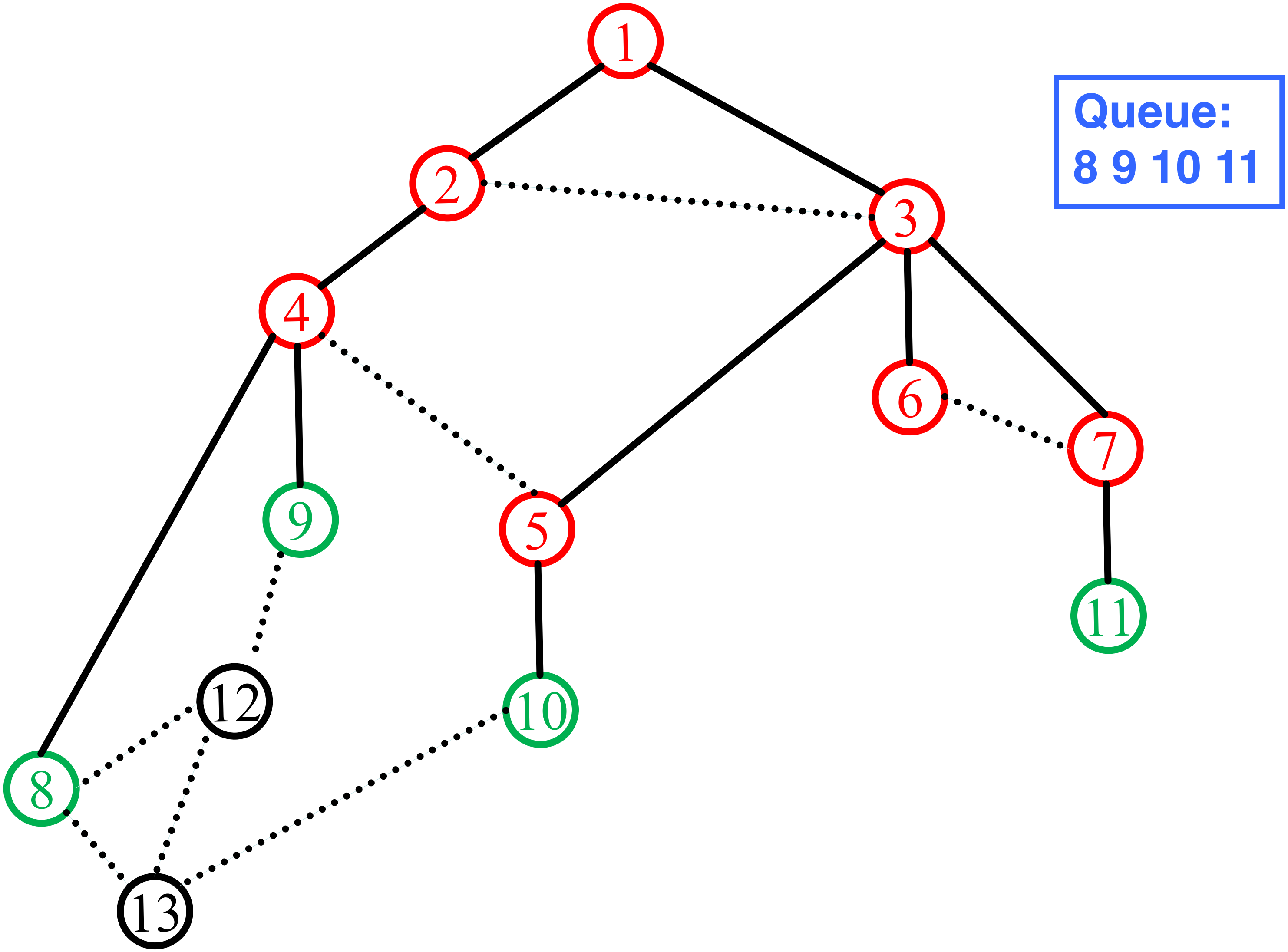


Queue:
5 6 7 8 9

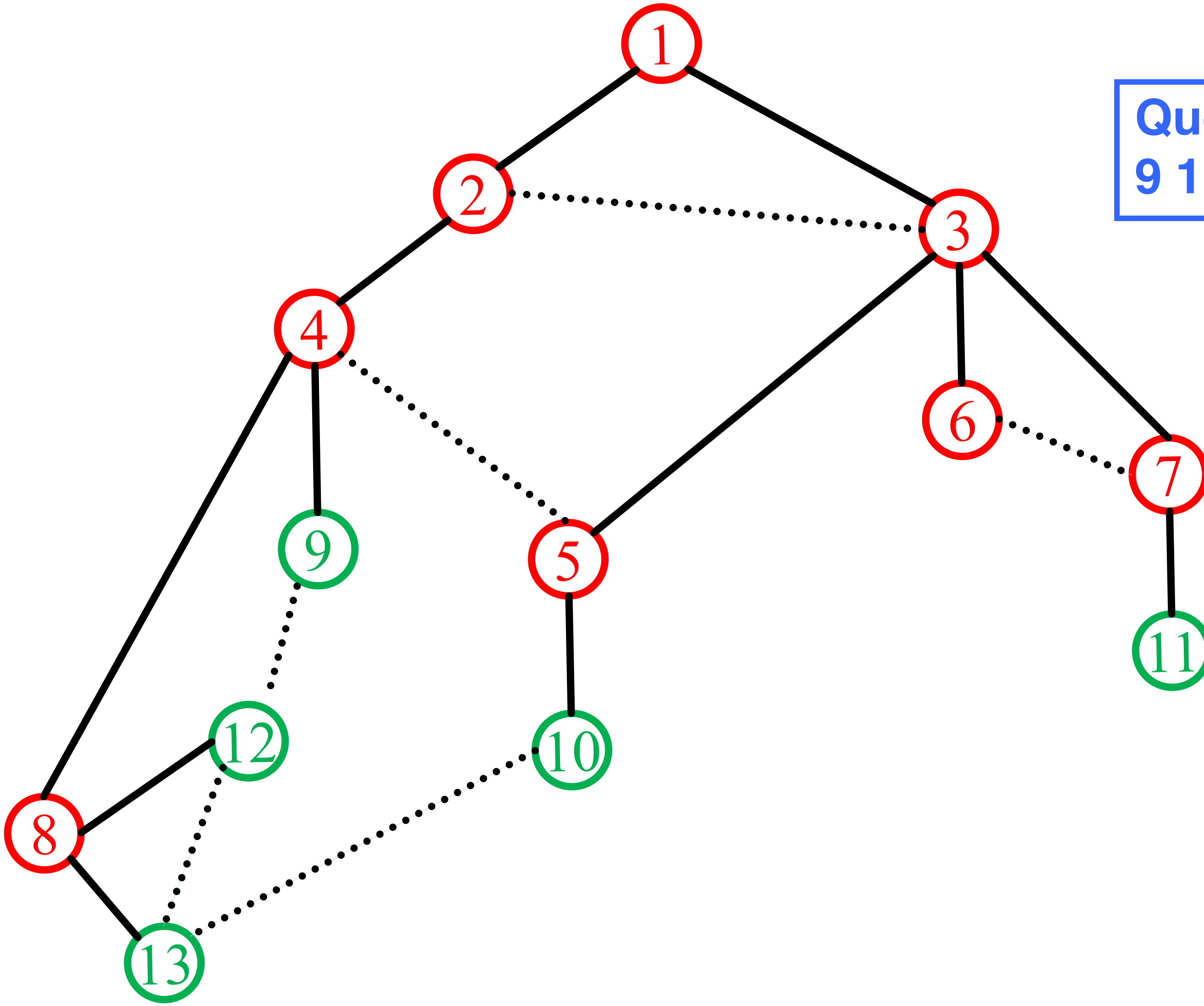
BFS(1)



BFS(1)

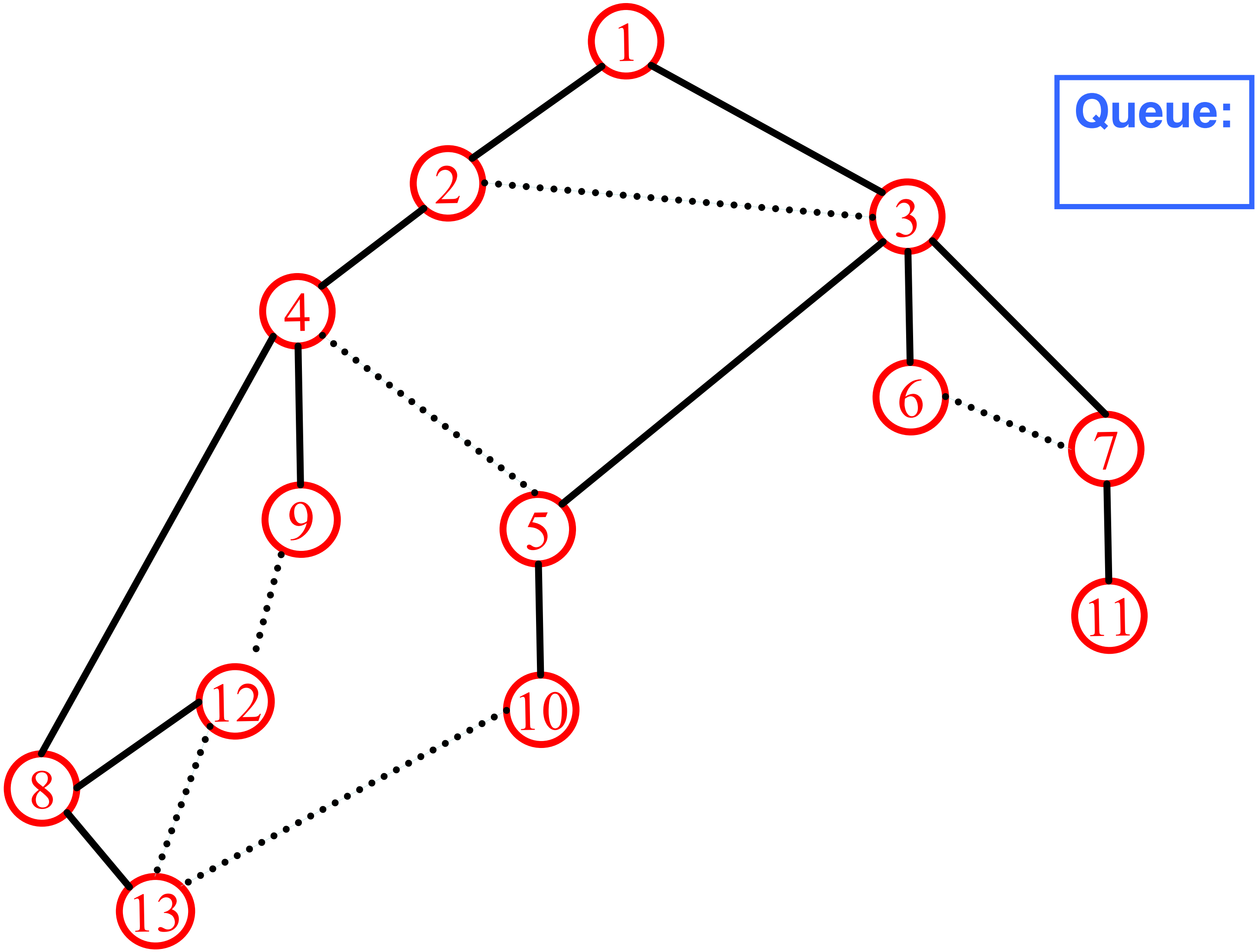


BFS(1)



Queue:
9 10 11 12 13

BFS(1)



BFS Analysis

Global initialization: mark all vertices "undiscovered"

```
BFS(s)
    mark s discovered
    queue = { s }
    while queue not empty
        u = remove_first(queue)
        for each edge {u,x}
            if (x is undiscovered)
                mark x discovered
                append x on queue
        mark u fully-explored
```

O(n) times: Once from every vertex if G is connected

$\mathbf{deg}(\boldsymbol{u}) \leq \boldsymbol{O}(\boldsymbol{n})$ **times**

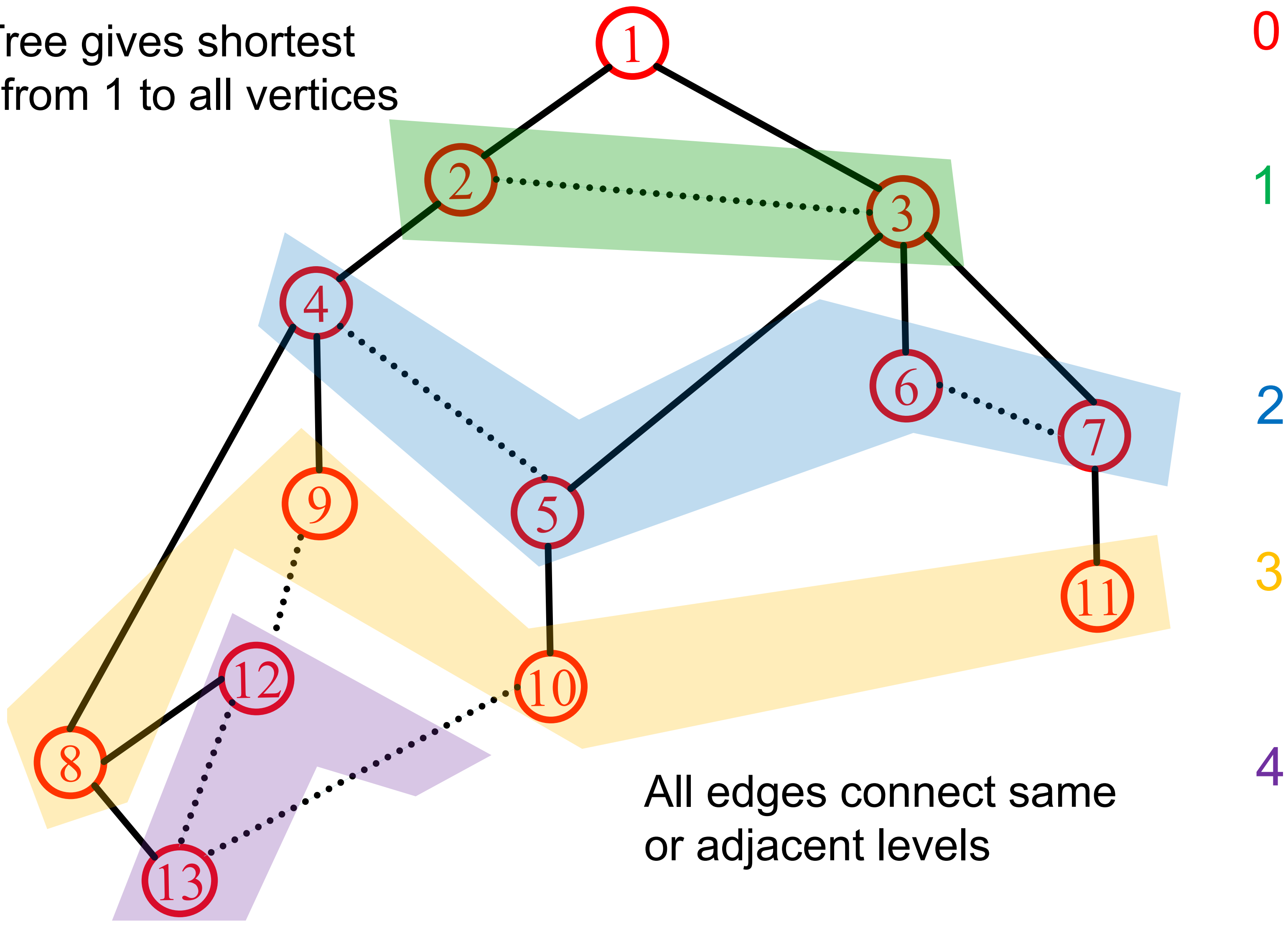
If we use adjacency list: $O(n) + O(\sum_v \deg(v)) = O(m+n)$

Properties of BFS

- BFS(s) visits a vertex v if and only if there is a path from s to v
- Edges into then-undiscovered vertices define a tree – the “Breadth First spanning tree” of G
- Level i in the tree are exactly all vertices v s.t., the shortest path (in G) from the root s to v is of length i
- All nontree edges join vertices on the same or adjacent levels of the tree

BFS Application: Shortest Paths

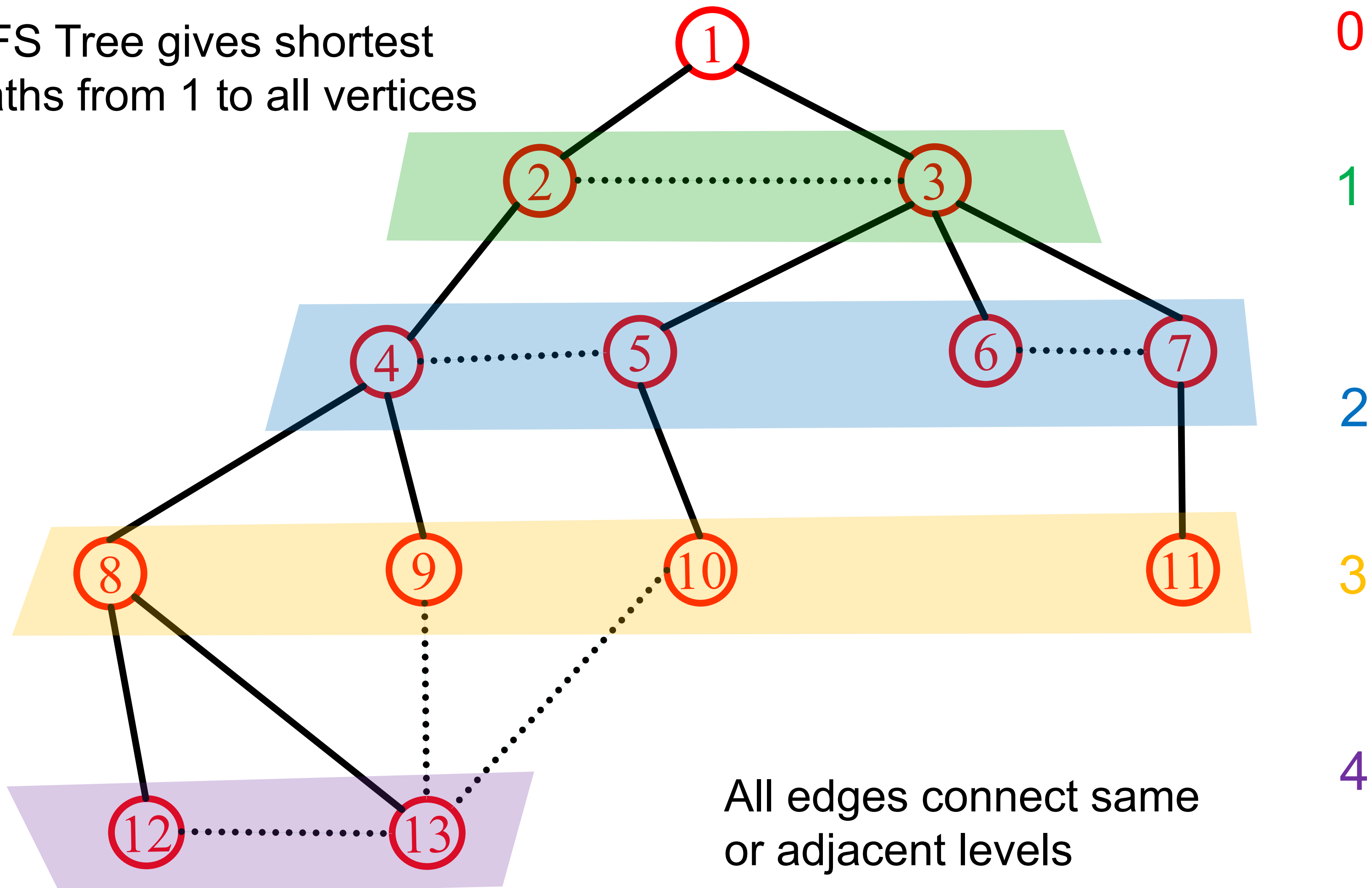
BFS Tree gives shortest paths from 1 to all vertices



All edges connect same or adjacent levels

BFS Application: Shortest Paths

BFS Tree gives shortest paths from 1 to all vertices



All edges connect same or adjacent levels

Properties of BFS

Claim: All nontree edges join vertices on the same or adjacent levels of the tree

Pf: Consider an edge {x,y}

Say x is first discovered and it is added to level i.

We show y will be at level i or $i+1$

This is because when vertices incident to x are considered in the loop, if y is still undiscovered, it will be discovered and added to level $i+1$.