CSE 421: Introduction to Algorithms

DFS - DAGs

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HW1 Grade

Q: I received low grade in HW1 what should I do?

- Understand what was your mistake. Did you understand the problem statement correctly?
- Show up to office hours and ask for hints or to explain your solution
- Review materials of 311 on proofs/induction
- Do exercises from the book/Problem Solving Sessions

Q: My HW1 grade is low, will I be able to receive 4.0?

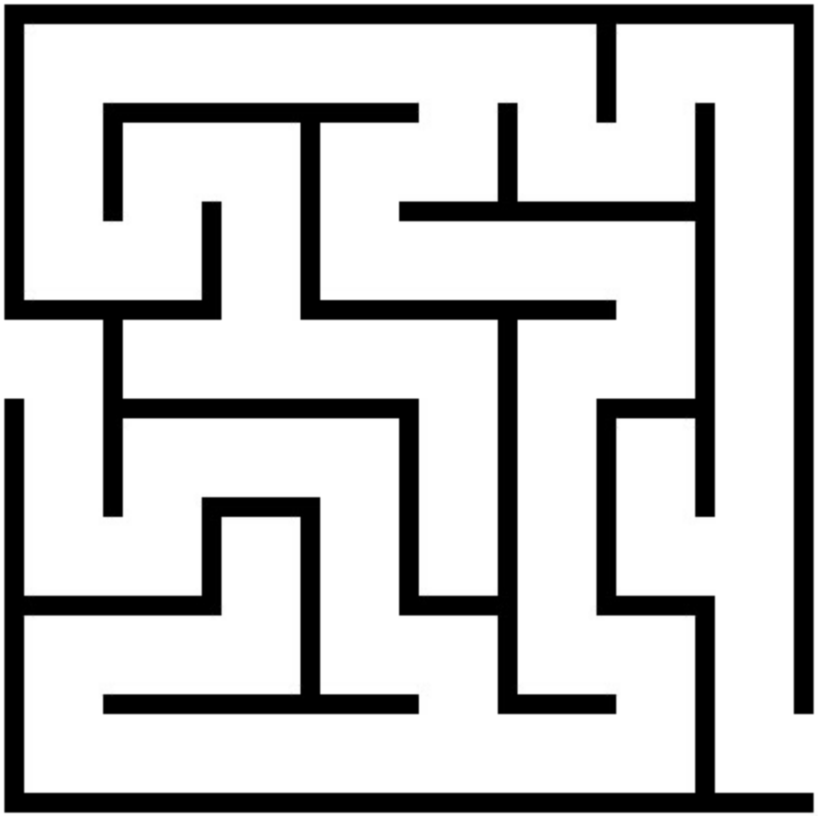
- Yes! I look at your progress. Many students are behind at beginning but by practice they catch up and receive 4.0

Q: I have filled out a regrade request, but was not convinced, what should I do?

- Show up to my office hour and discuss your solution

Depth First Search

Follow the first path you find as far as you can go; back up to last unexplored edge when you reach a dead end, then go as far you can



Naturally implemented using recursive calls or a stack

DFS(s) – Recursive version

Global Initialization: mark all vertices undiscovered

DFS(v)
    Mark v discovered

    for each edge {v,x}
        if (x is undiscovered)
            Mark x discovered
            DFS(x)

    Mark v full-discovered

DFS(A)

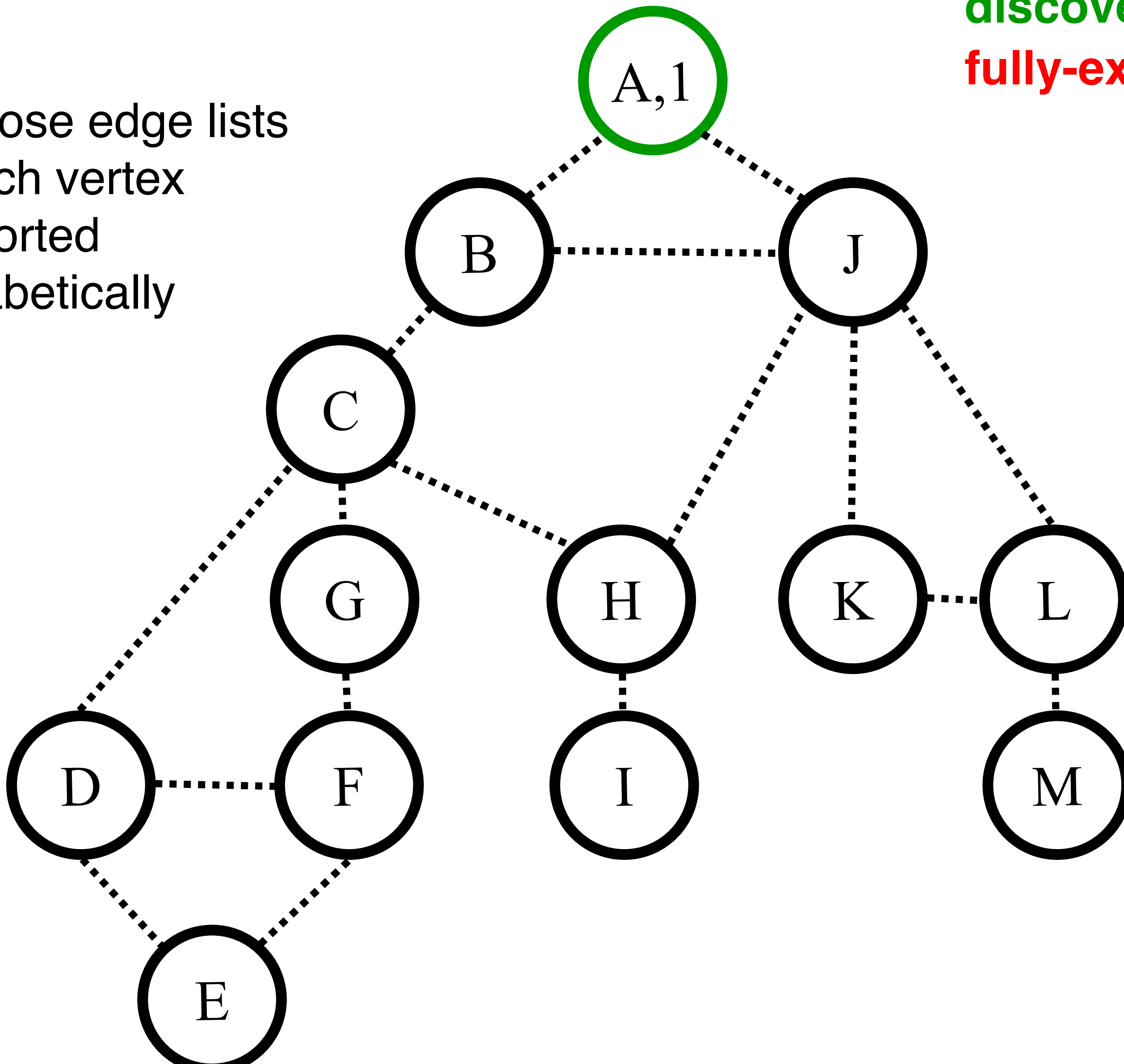
Color code:

undiscovered

discovered

fully-explored

Suppose edge lists at each vertex are sorted alphabetically

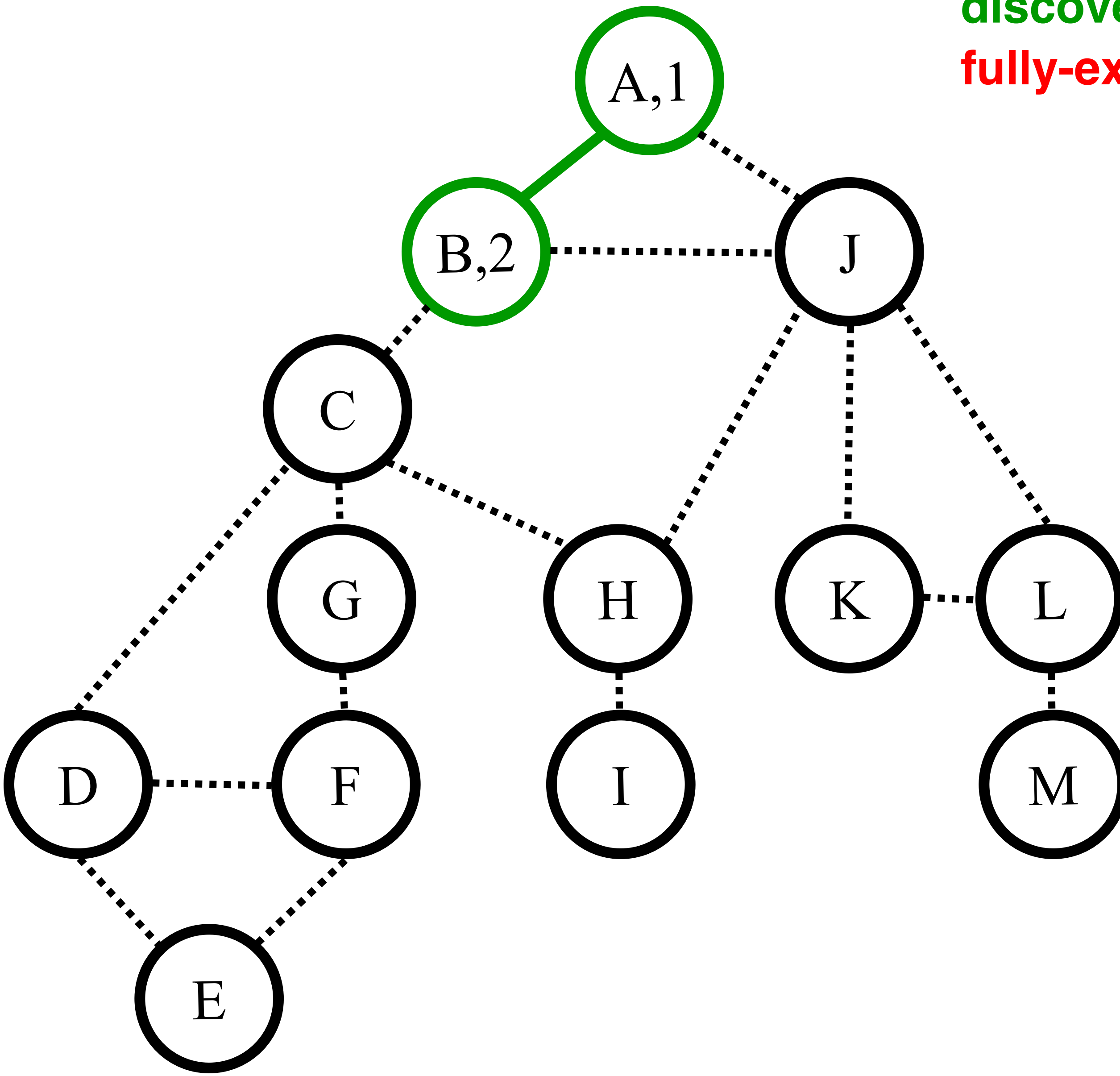


Call Stack (Edge list):

A (B,J)

st[] = {1}

DFS(A)



Color code:

undiscovered

discovered

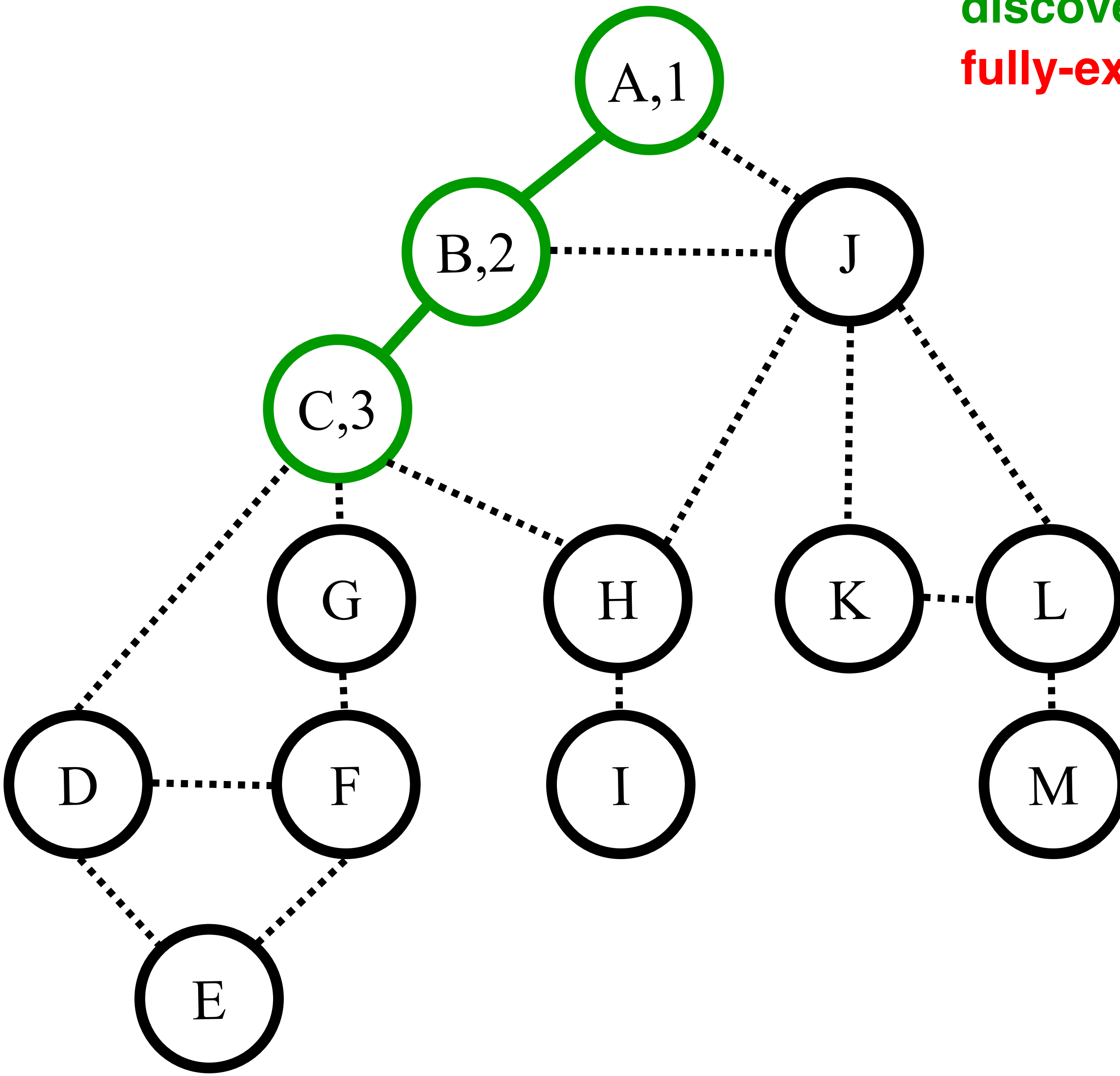
fully-explored

Call Stack:
(Edge list)

A (~~B~~,J)
B (A,C,J)

st[] =
{1,2}

DFS(A)



Color code:

undiscovered

discovered

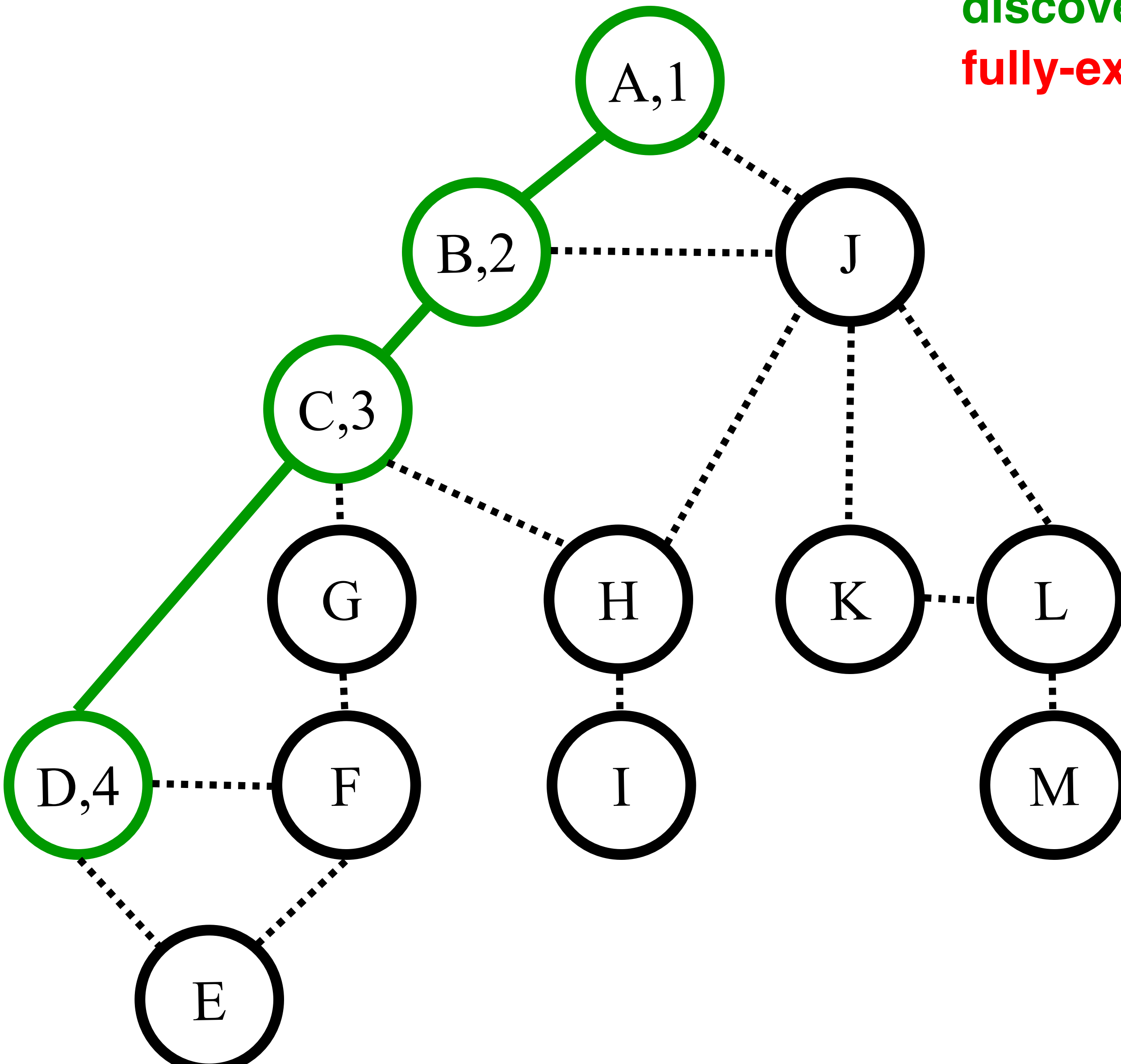
fully-explored

Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (B,D,G,H)

st[] =
{1,2,3}

DFS(A)



Color code:

undiscovered

discovered

fully-explored

Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (C,E,F)

st[] = {1,2,3,4}

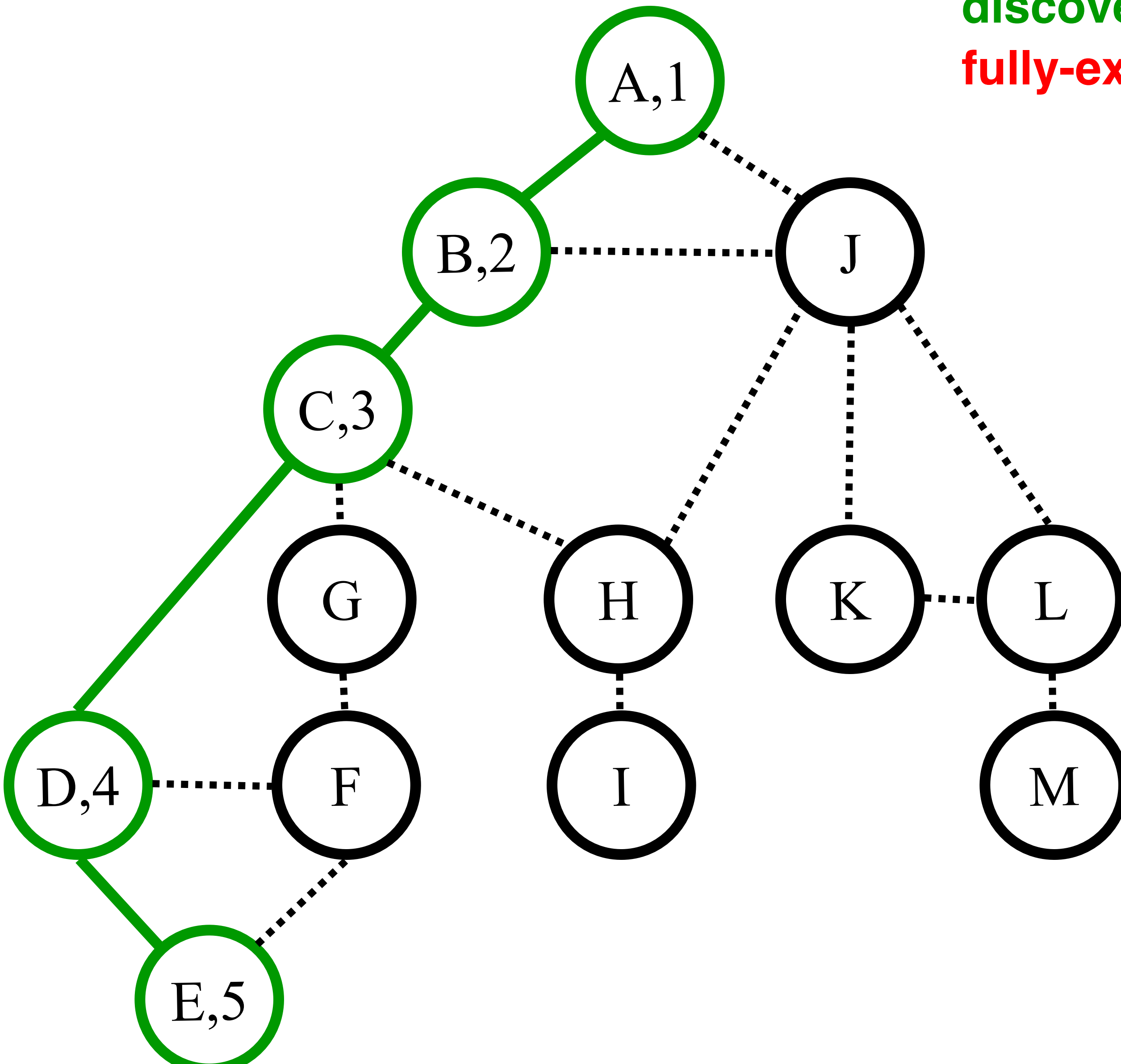
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (D,F)

st[] =
{1,2,3,4,5}

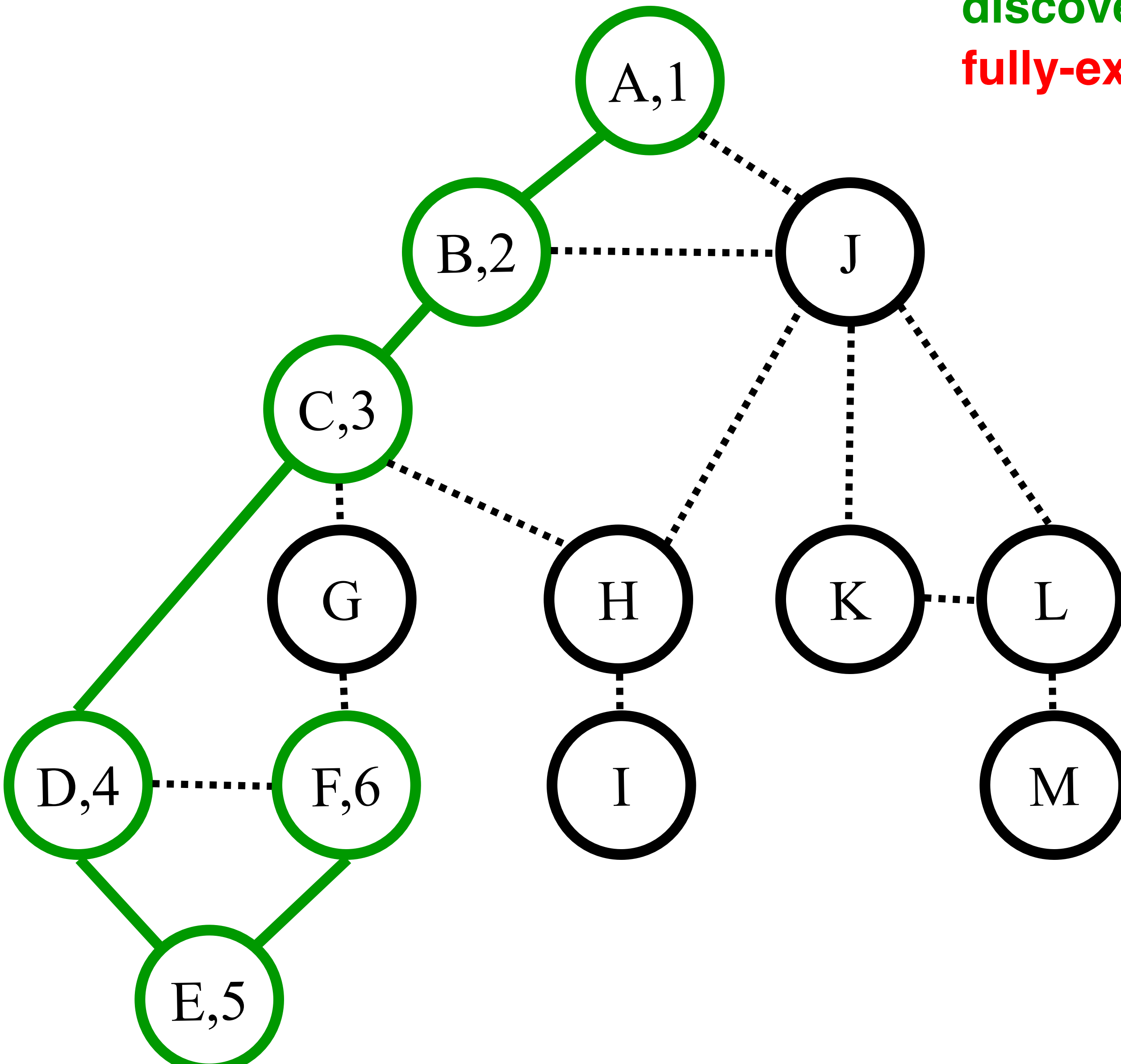
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (D,E,G)

st[] = {1,2,3,4,5, 6}

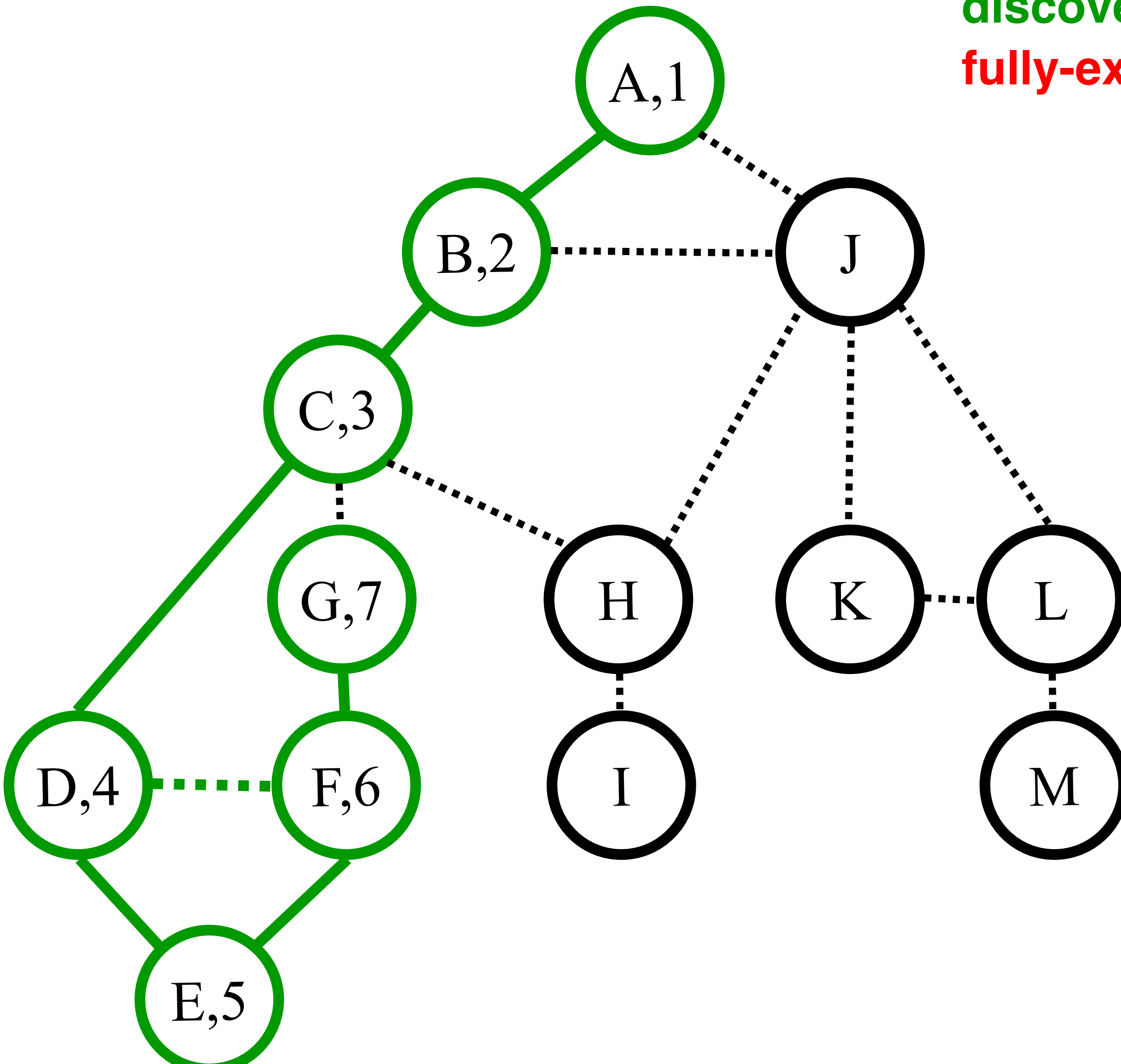
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)
G (C,F)

st[] =
{1,2,3,4,5,
6,7}

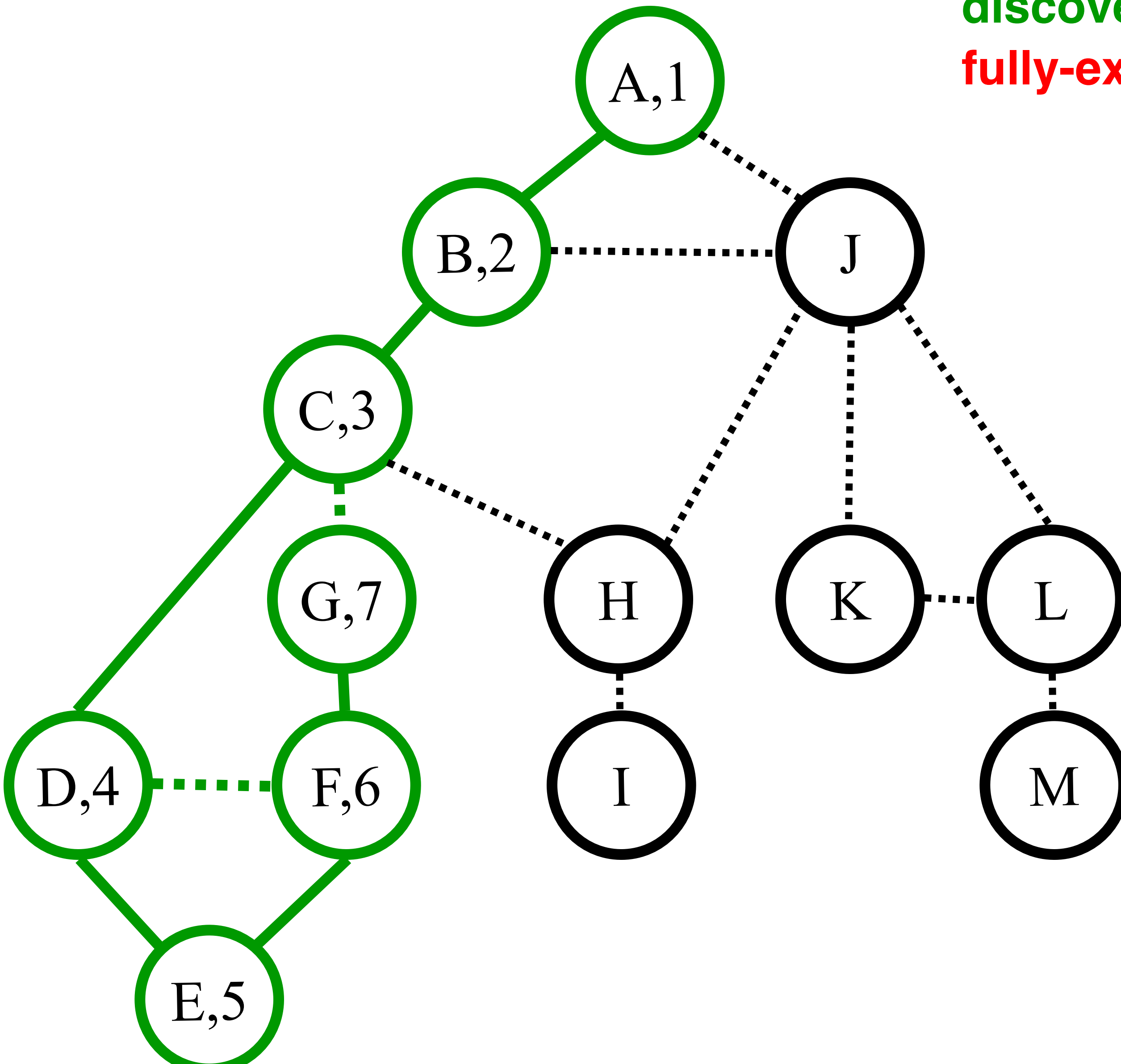
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)
G (~~C~~,~~F~~)

st[] =
{1,2,3,4,5,
6,7}

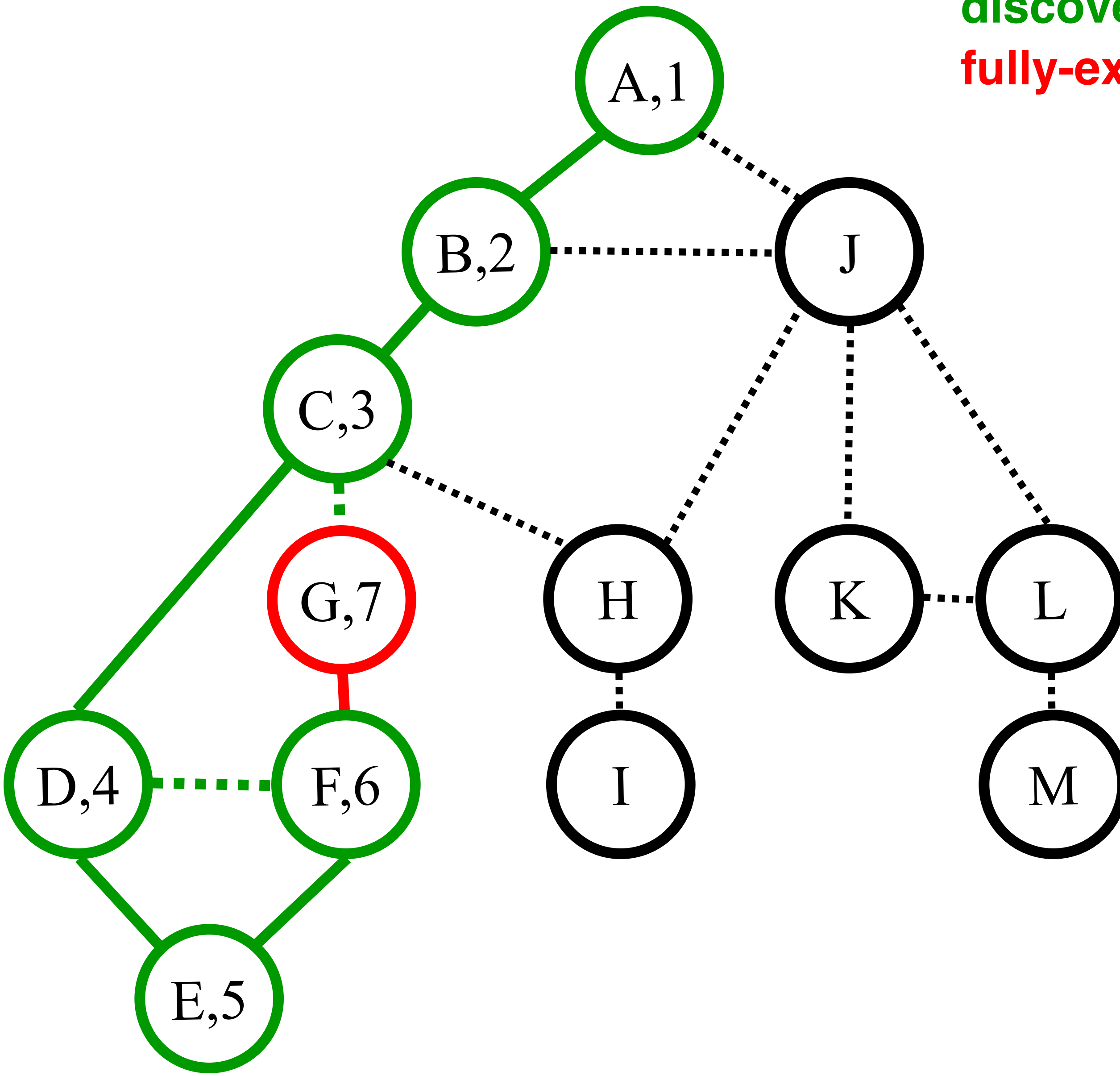
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)

st[] =
{1,2,3,4,5,
6}

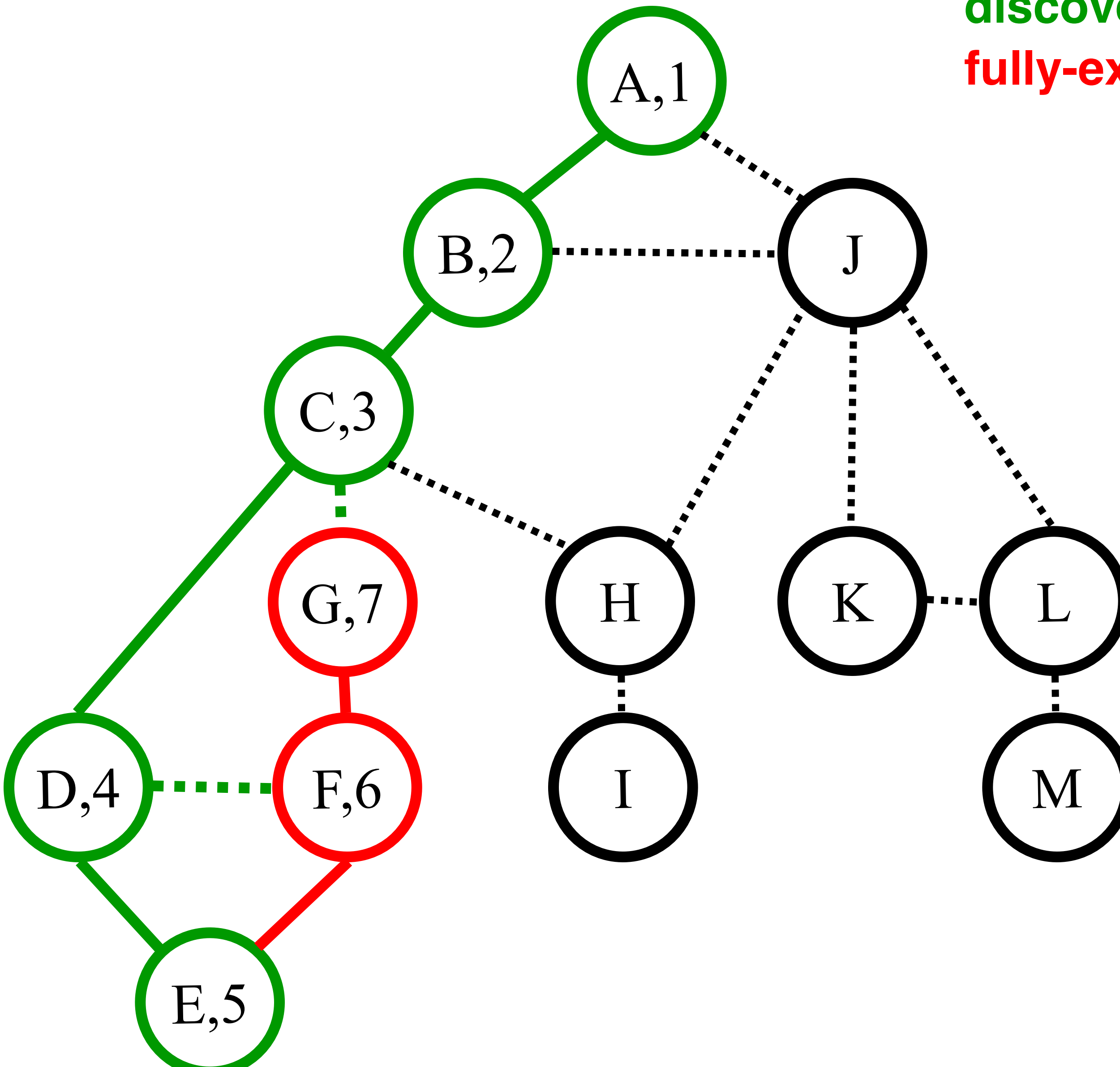
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)

st[] =
{1,2,3,4,5}

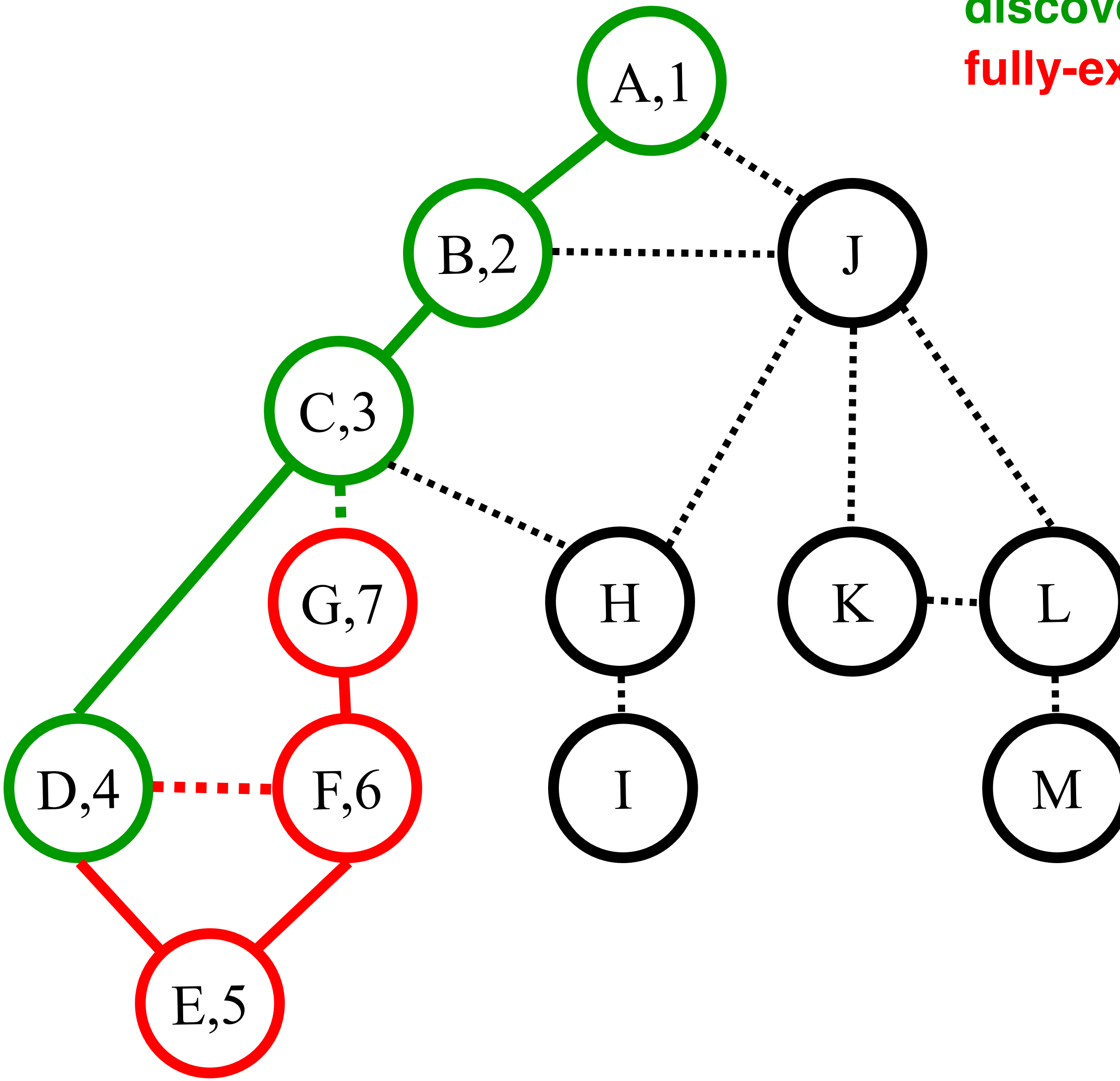
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,~~F~~)

st[] =
{1,2,3,4}

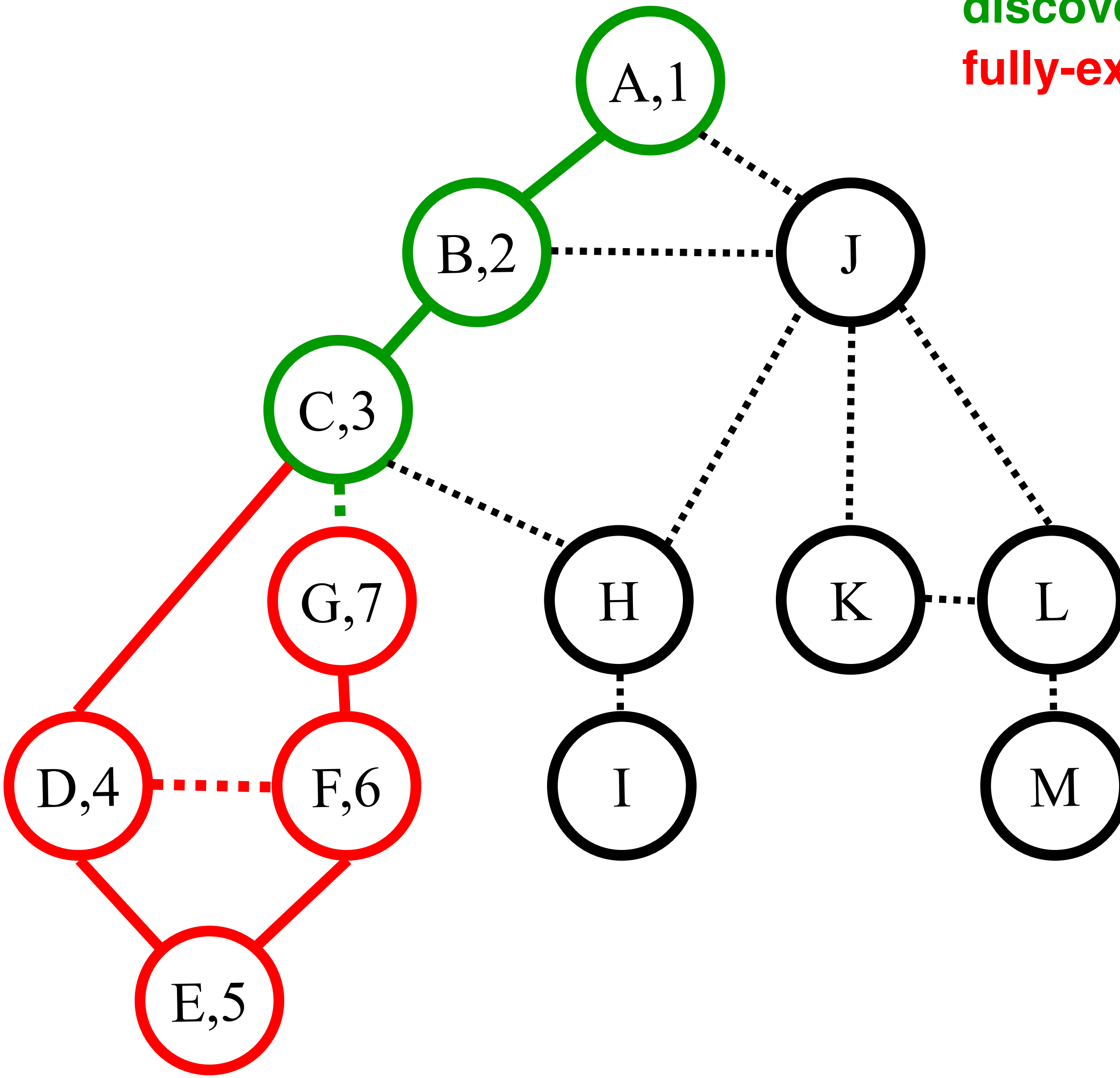
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)

st[] = {1,2,3}

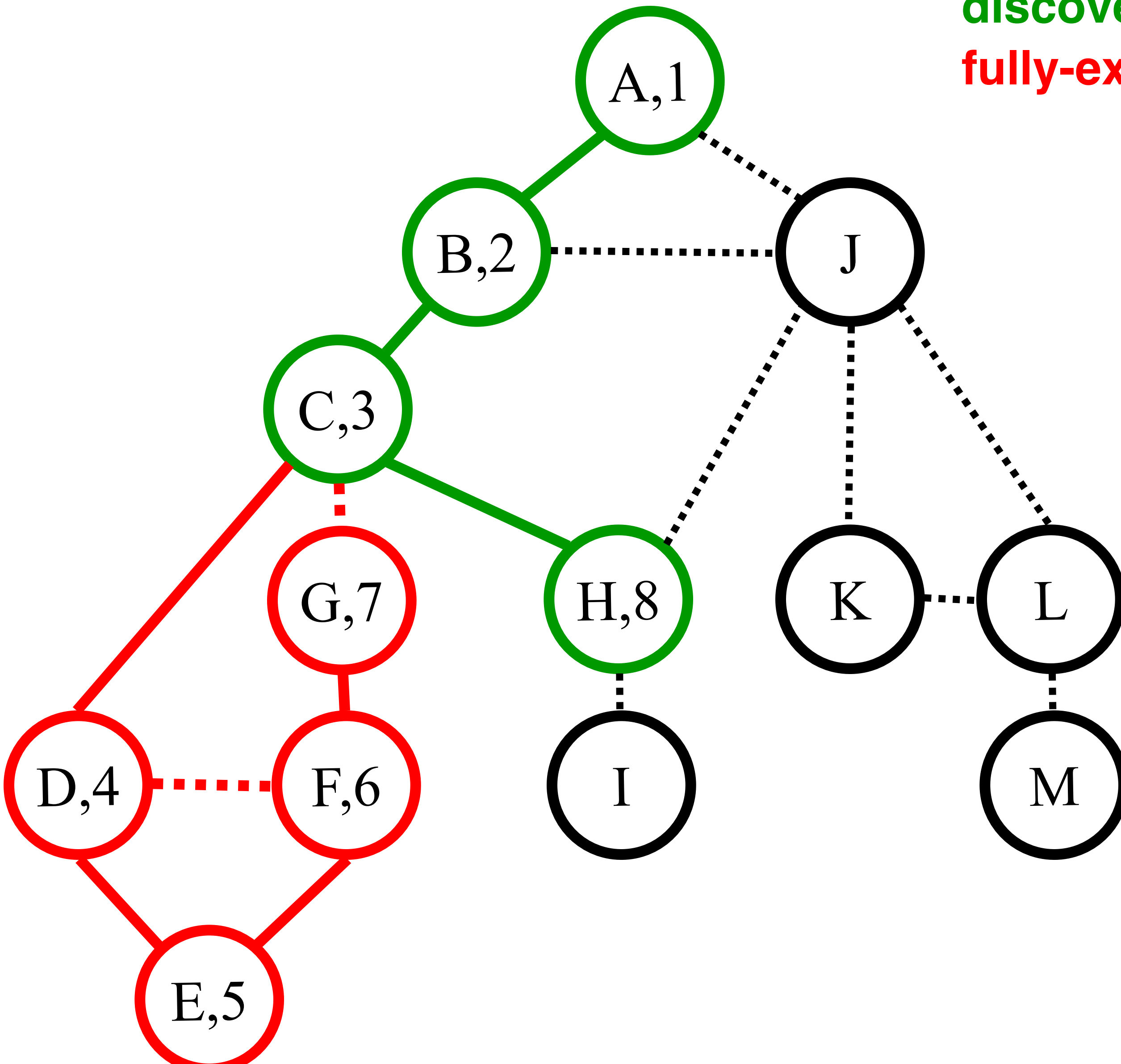
DFS(A)

Color code:

undiscovered

discovered

fullyexplored



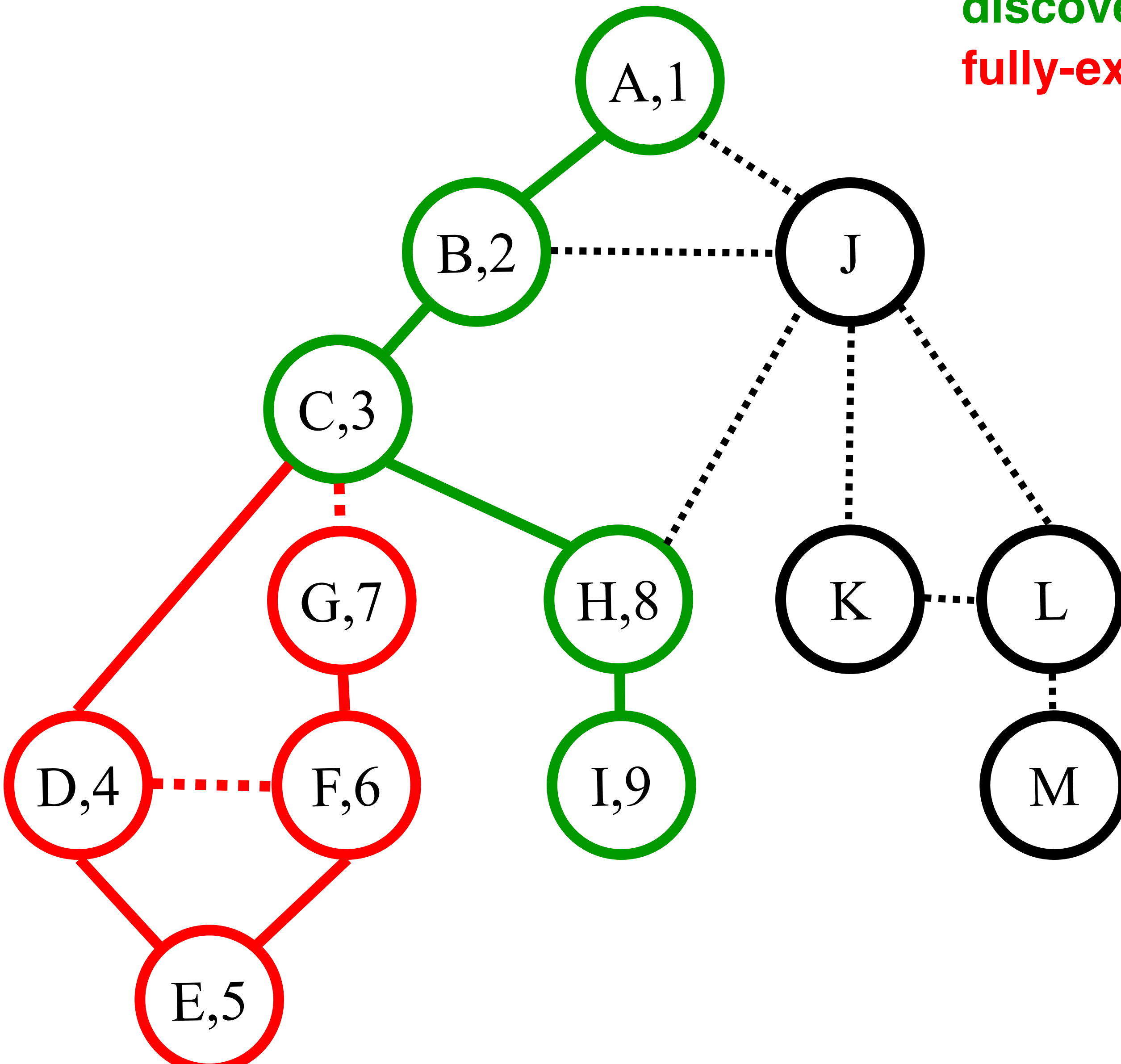
Call Stack: (Edge list)
A (~~B~~,J) B (~~A~~,~~C~~,J) C (~~B~~,~~D~~,~~G~~,~~H~~) H (C,I,J)

st[] = {1,2,3,8}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,J)
I (H)

st[] =
{1,2,3,8,9}

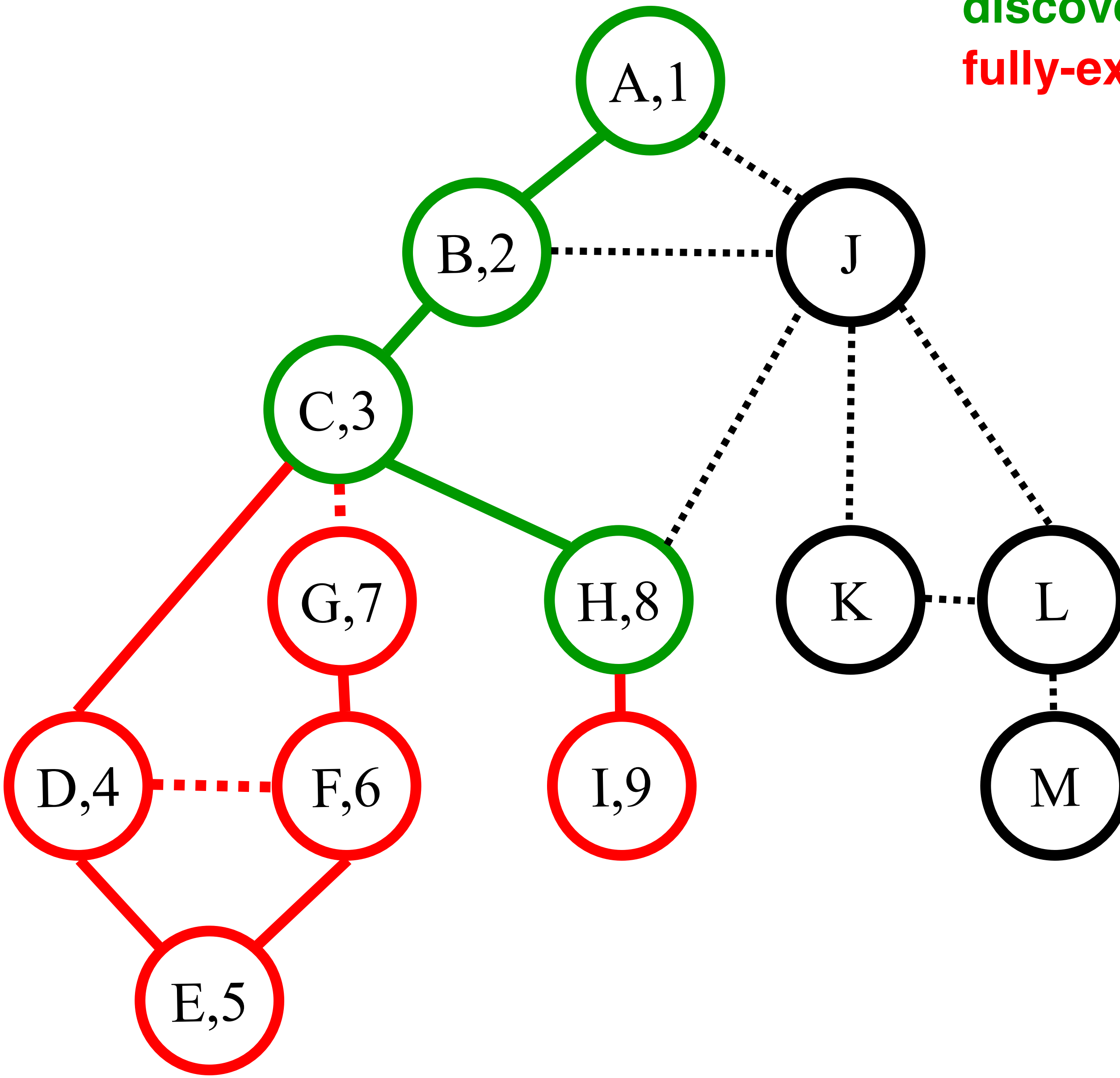
DFS(A)

Color code:

undiscovered

discovered

fully-explored

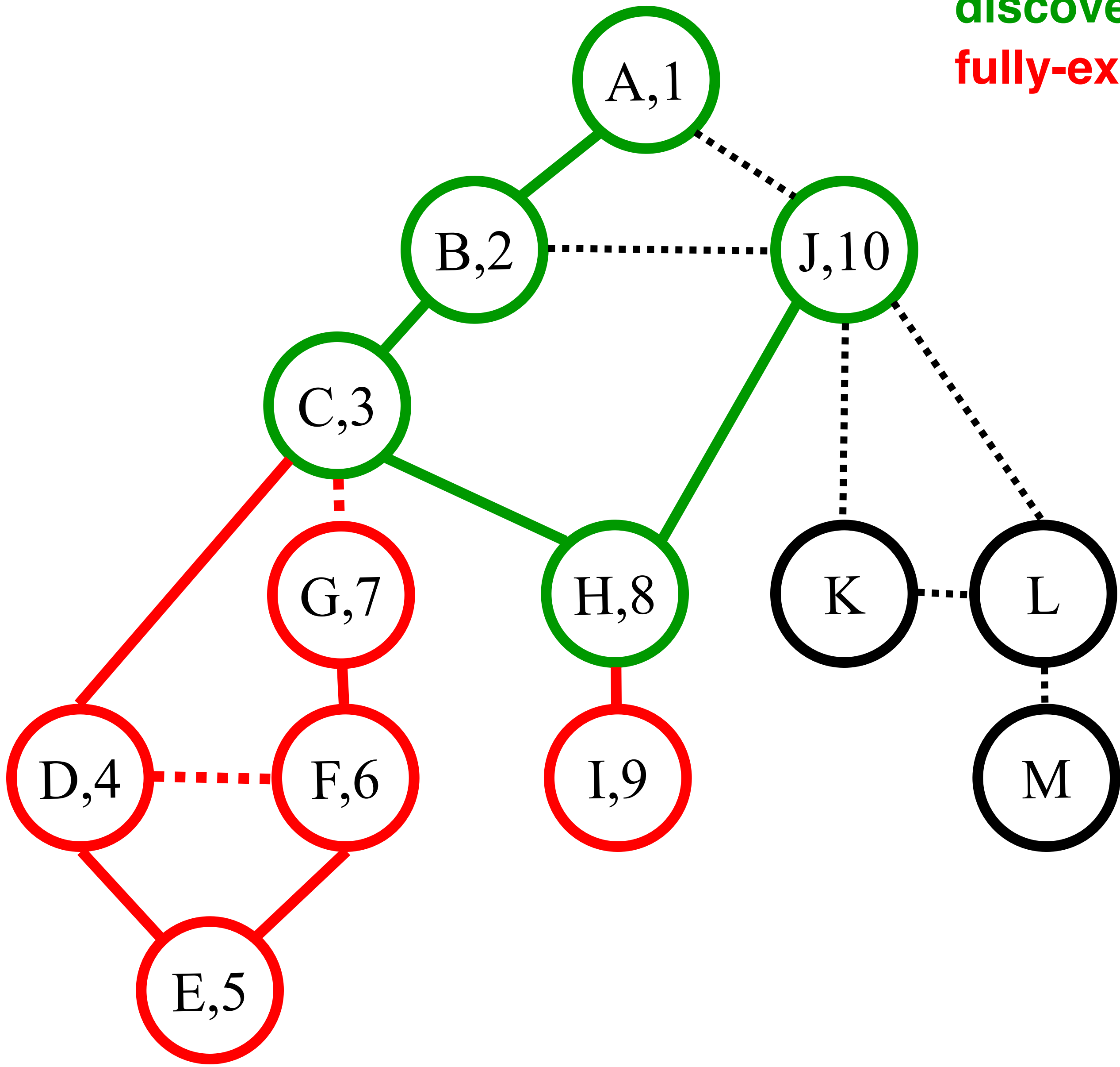


Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,J)

st[] =
{1,2,3,8}

DFS(A)



Color code:

undiscovered

discovered

fully-explored

Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (A,B,H,K,L)

st[] =
{1,2,3,8,
10}

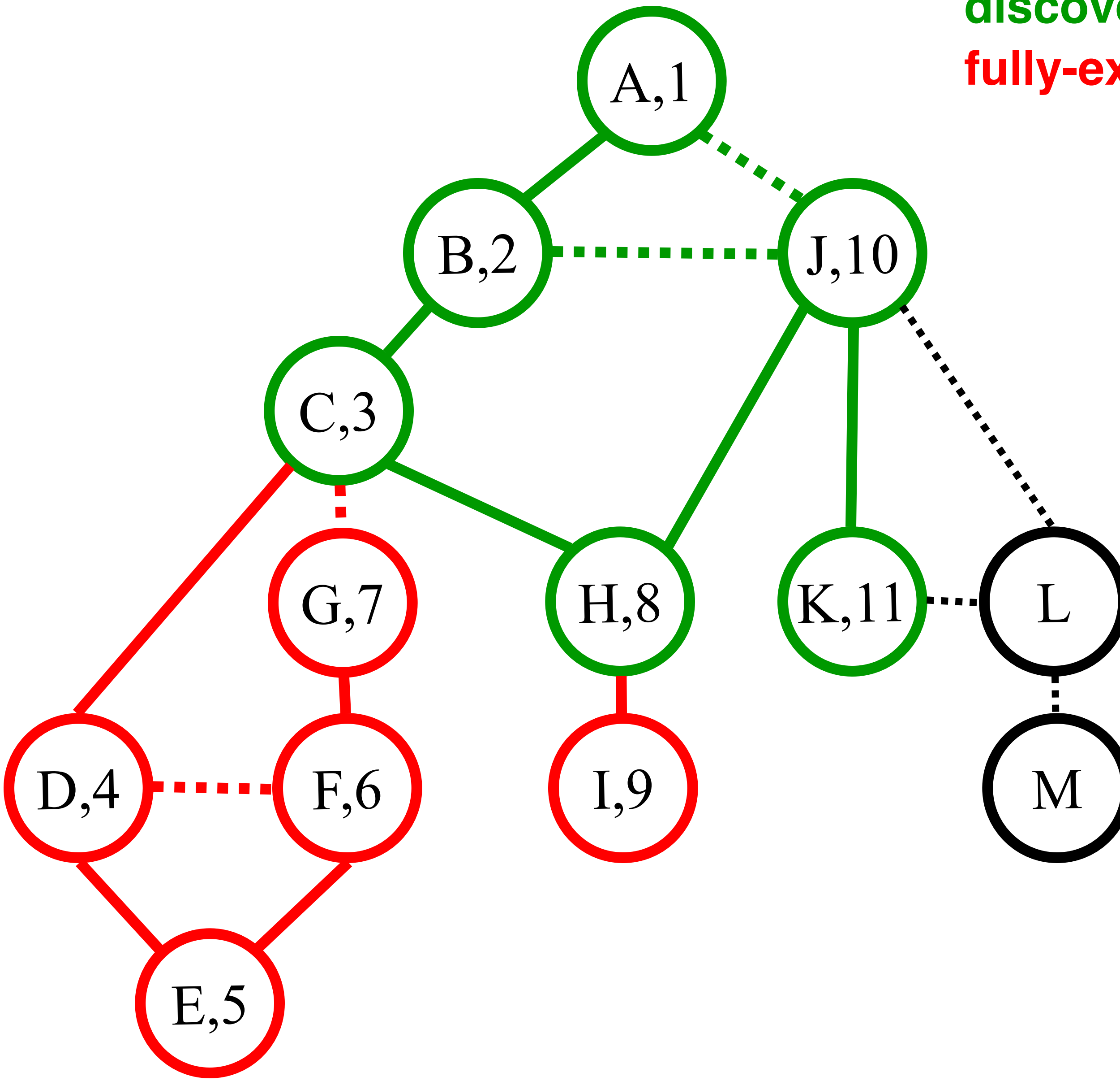
DFS(A)

Color code:

undiscovered

discovered

fully-explored

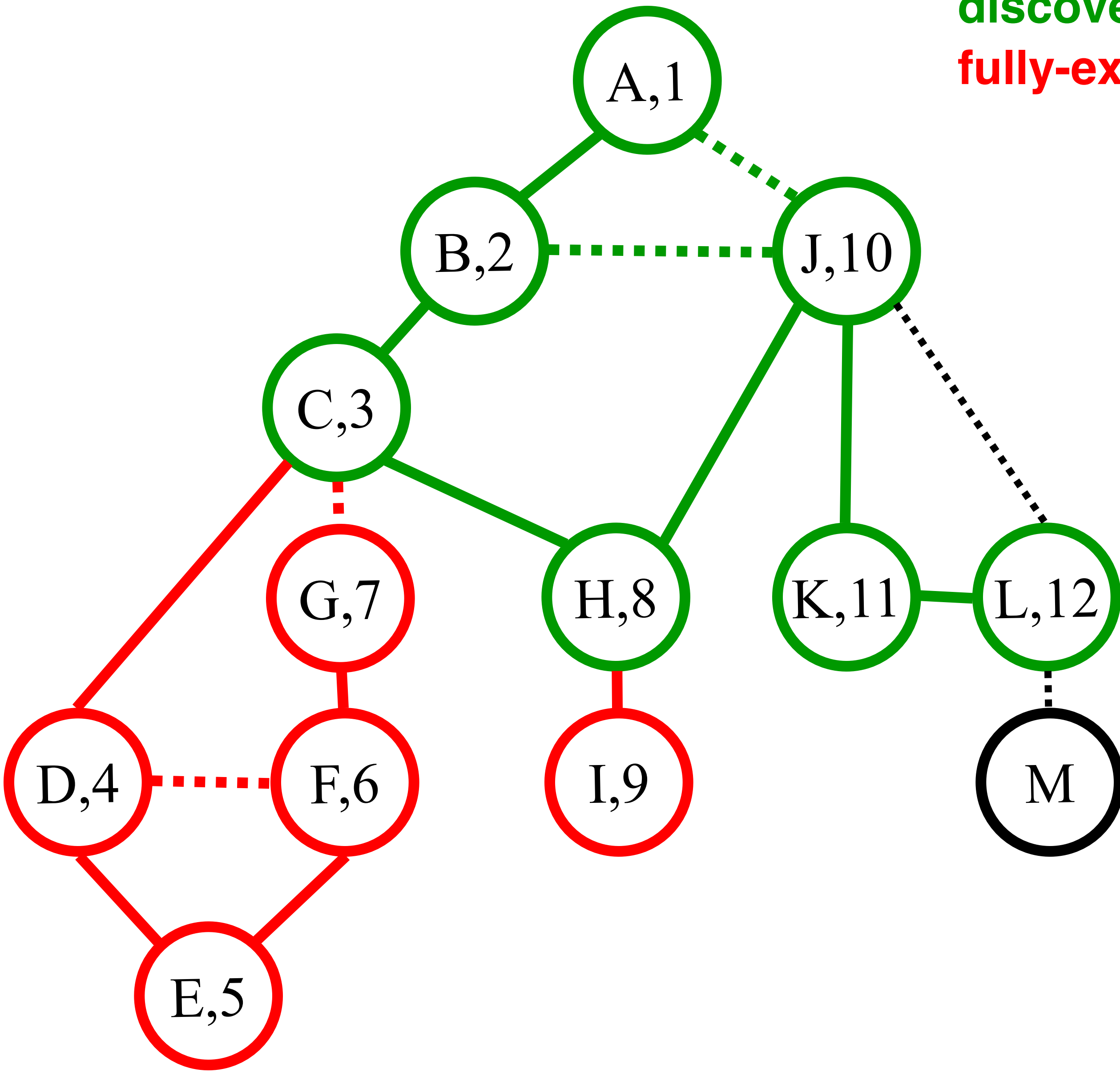


Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (J,L)

st[] = {1,2,3,8,10,11}

DFS(A)



Color code:

undiscovered

discovered

fully-explored

Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (~~J~~,~~L~~)
L (J,K,M)

st[] =
{1,2,3,8,10
,11,12}

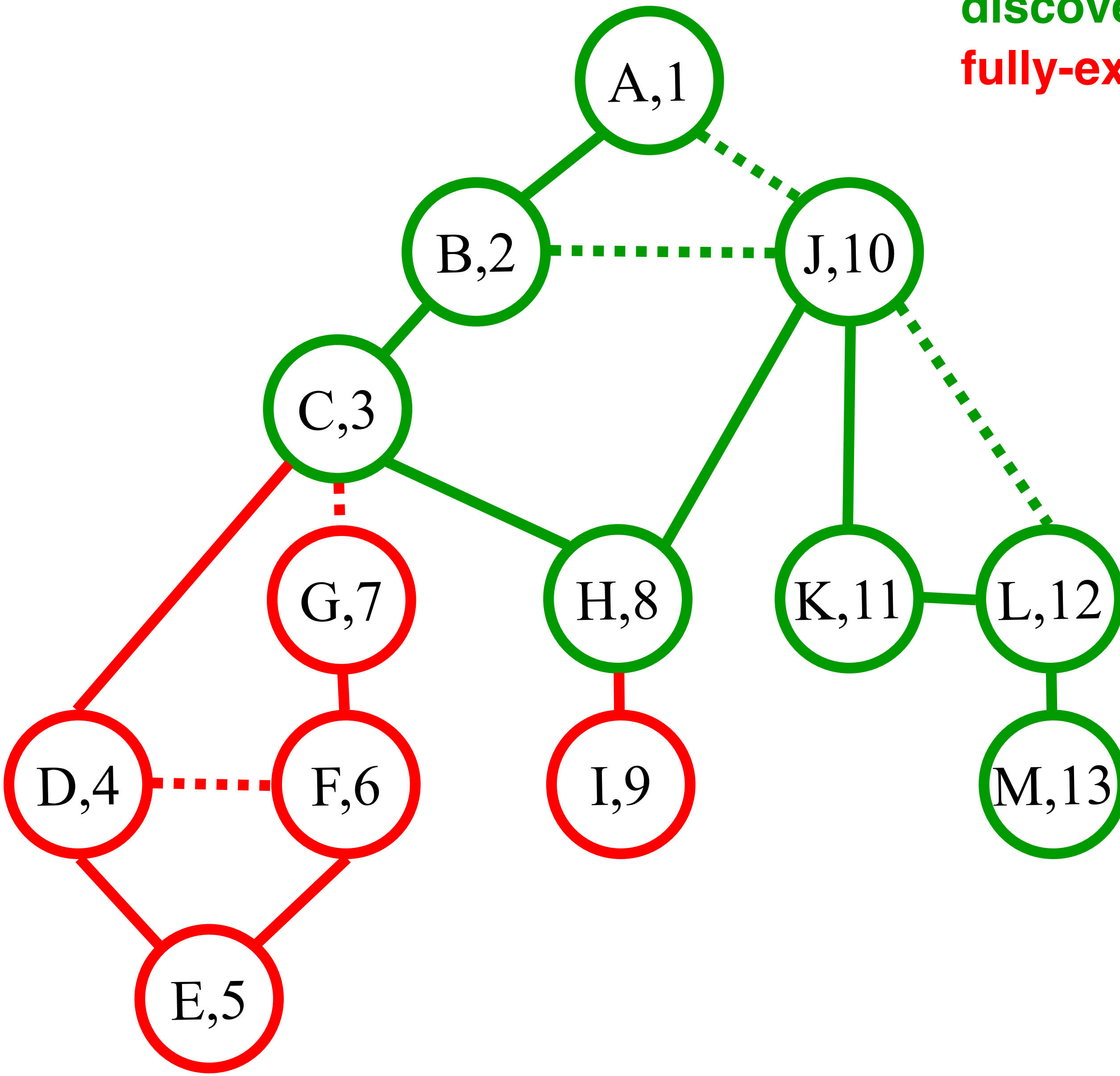
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (~~J~~,~~L~~)
L (~~J~~,~~K~~,~~M~~)
M (L)

st[] =
{1,2,3,8,10
,11,12,13}

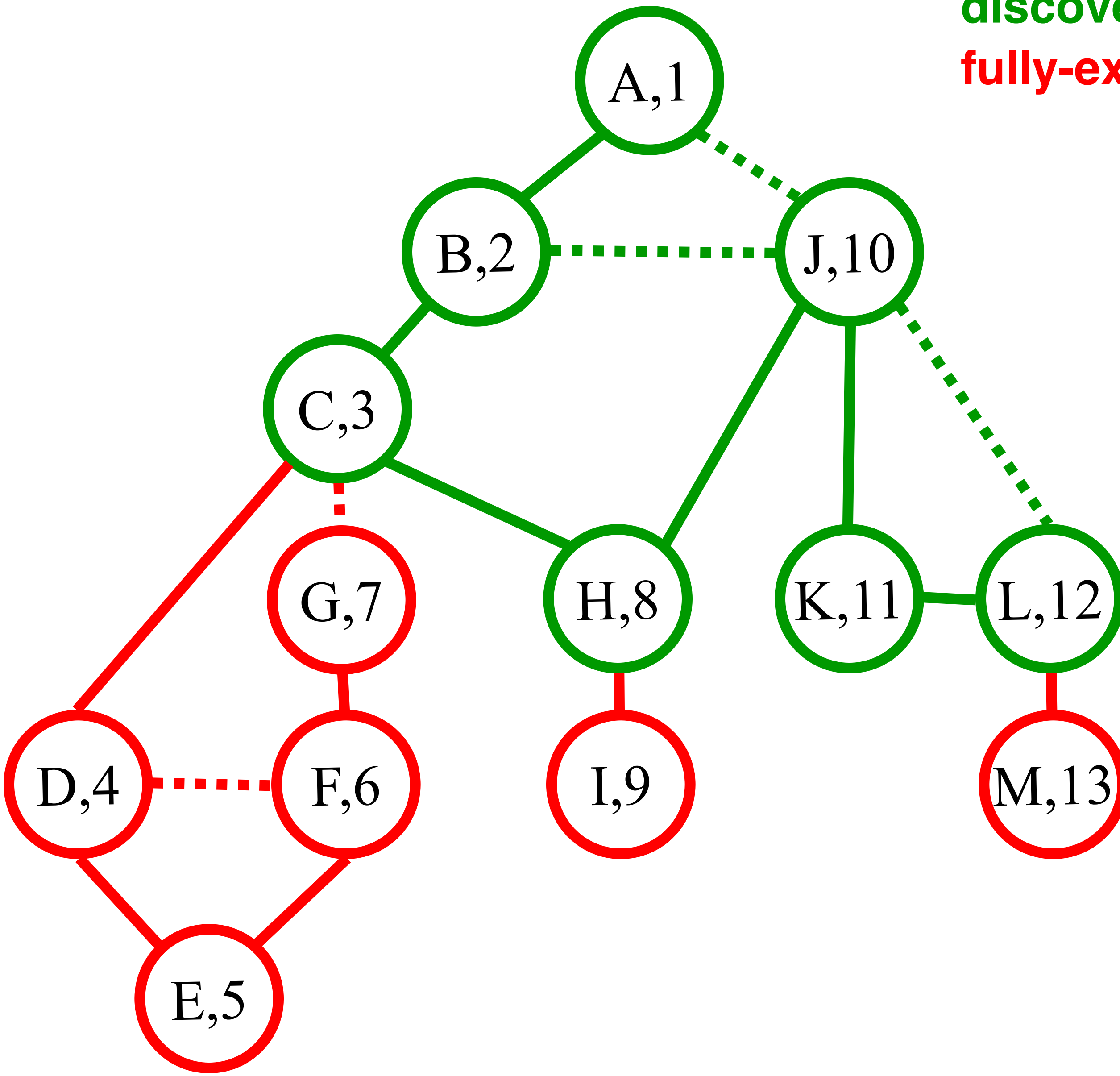
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

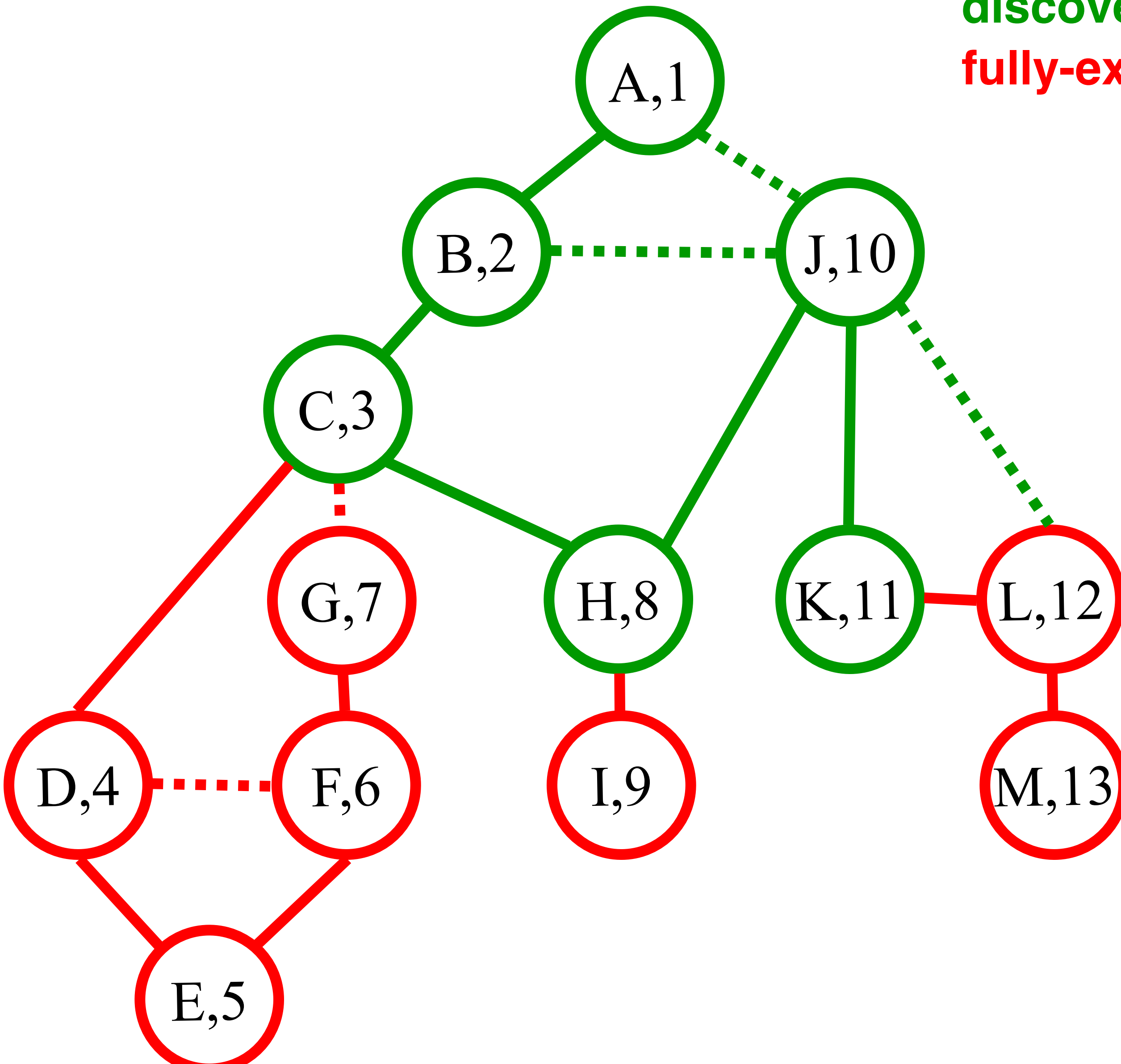
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (~~J~~,~~L~~)
L (~~J~~,~~K~~,~~M~~)

st[] =
{1,2,3,8,10
,11,12}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

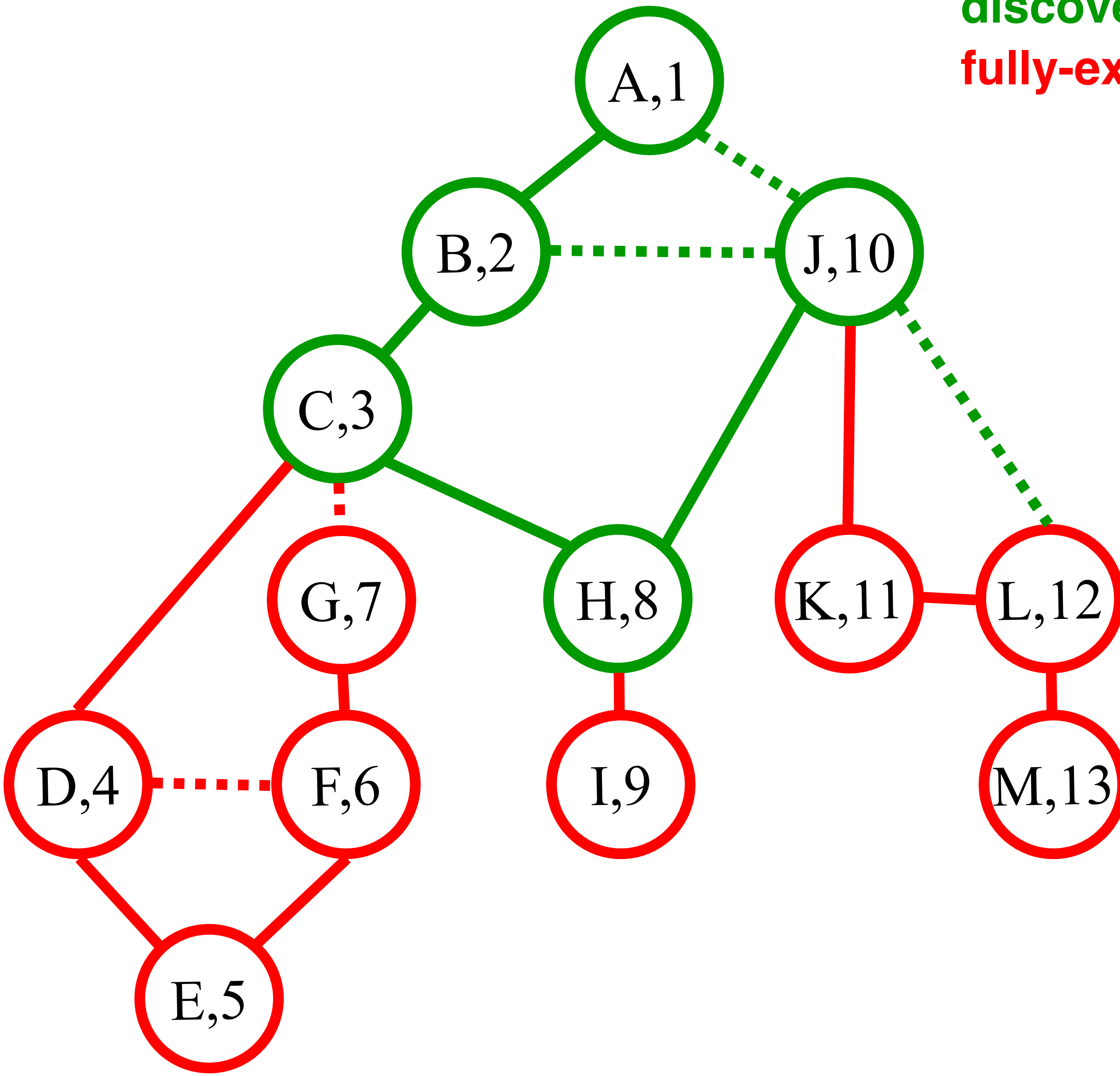
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (~~J~~,~~L~~)

st[] =
{1,2,3,8,10
,11}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)

st[] =
{1,2,3,8,
10}

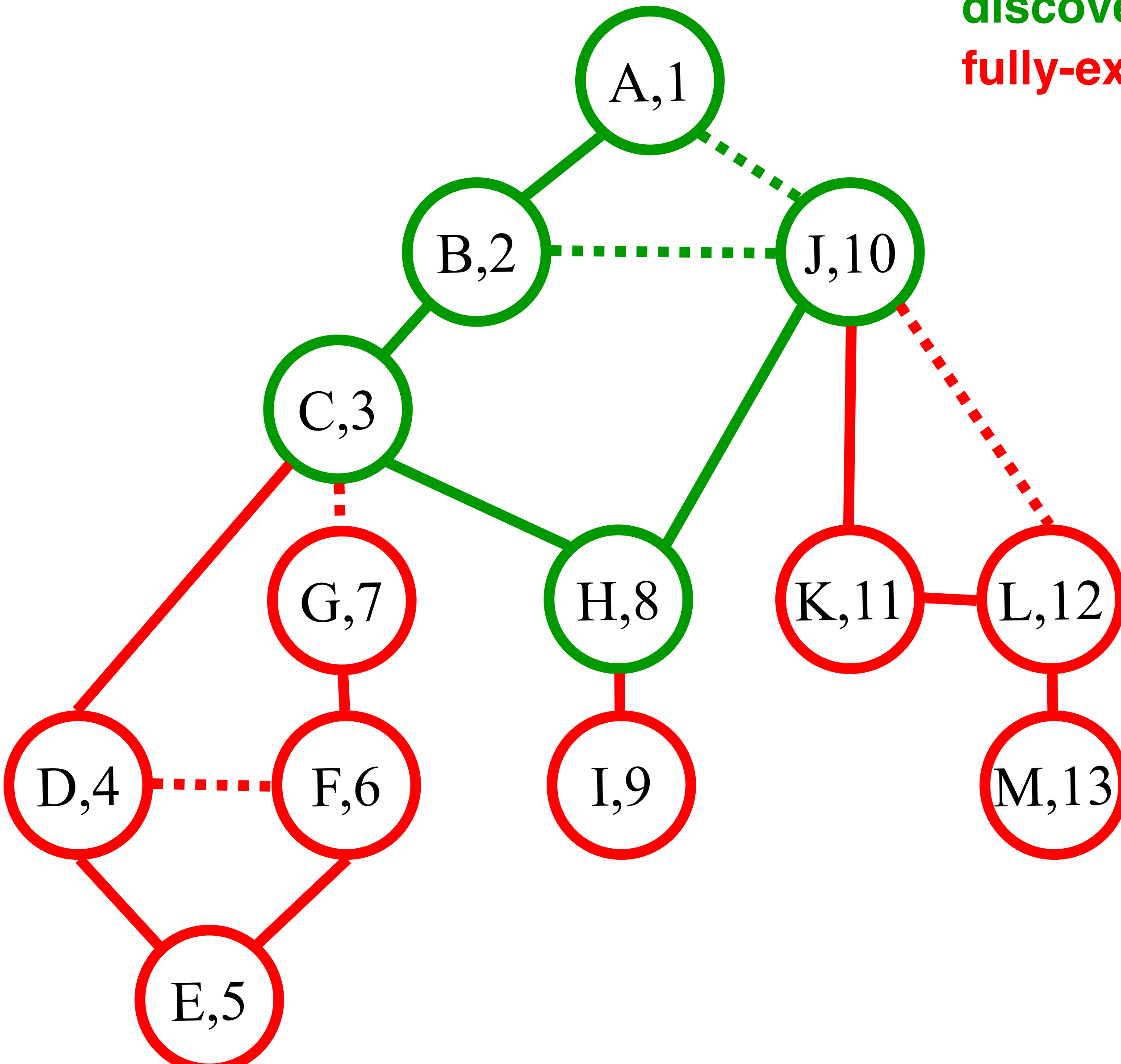
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

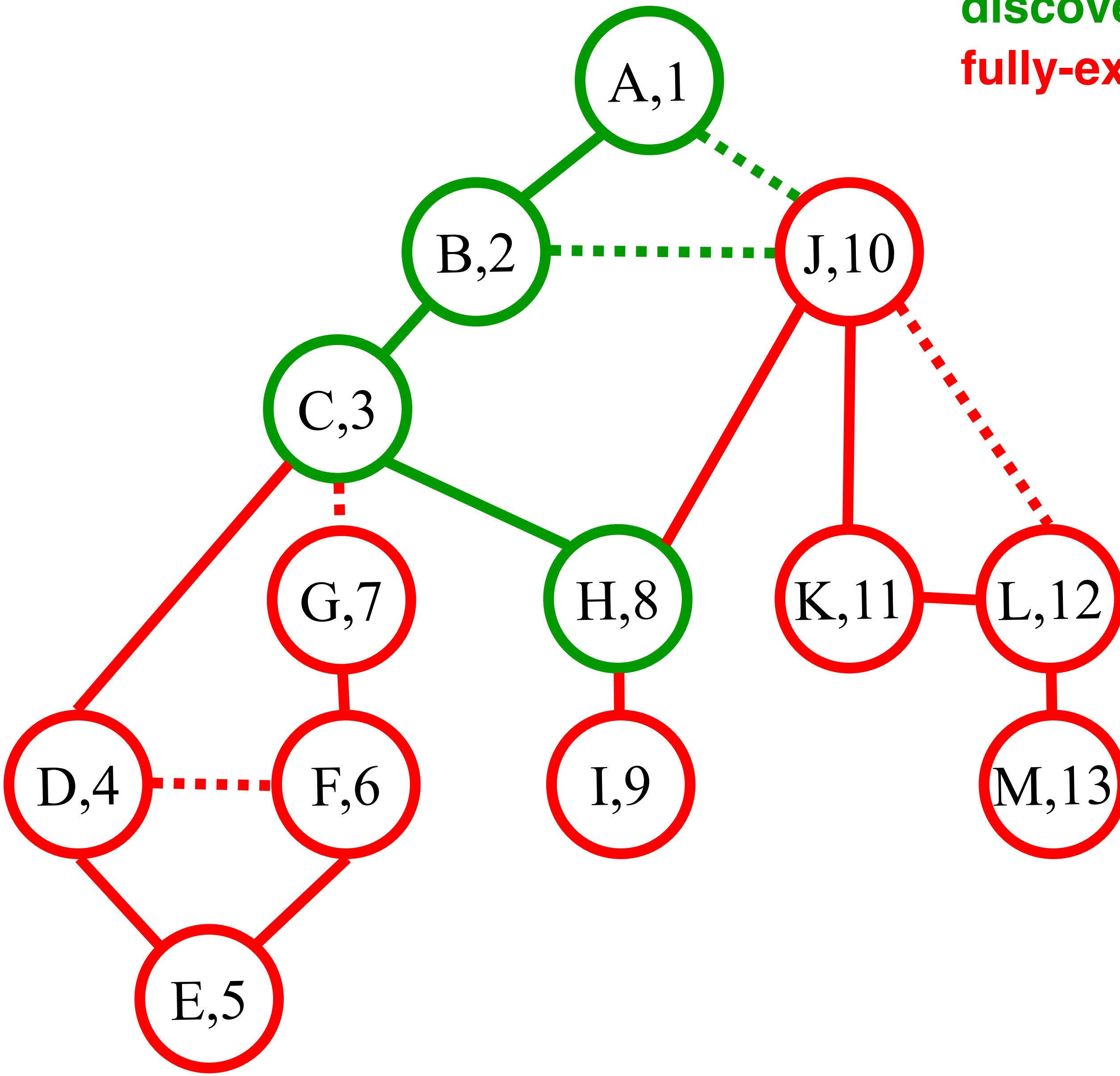
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,~~L~~)

st[] =
{1,2,3,8,
10}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)

st[] =
{1,2,3,8}

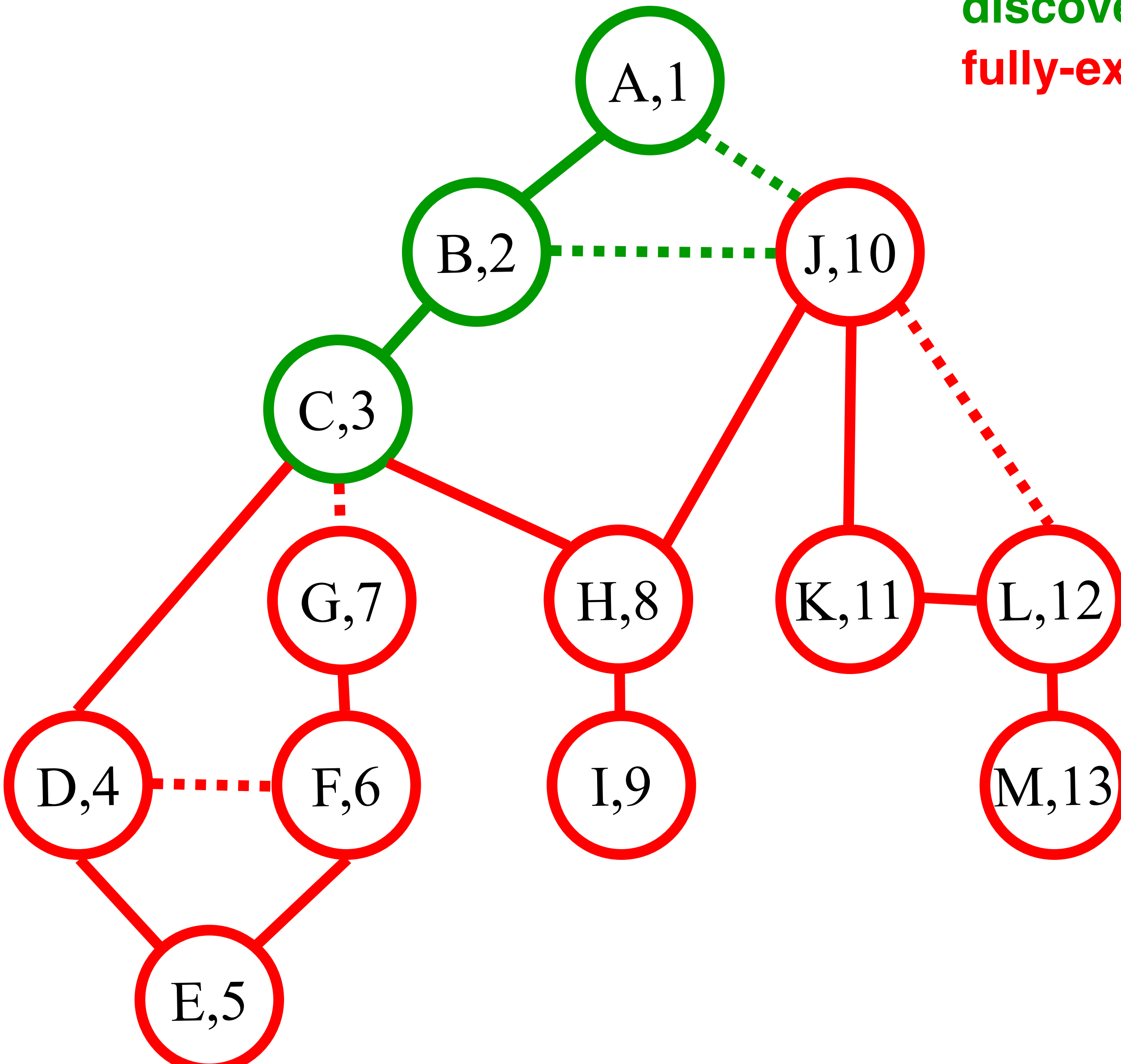
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

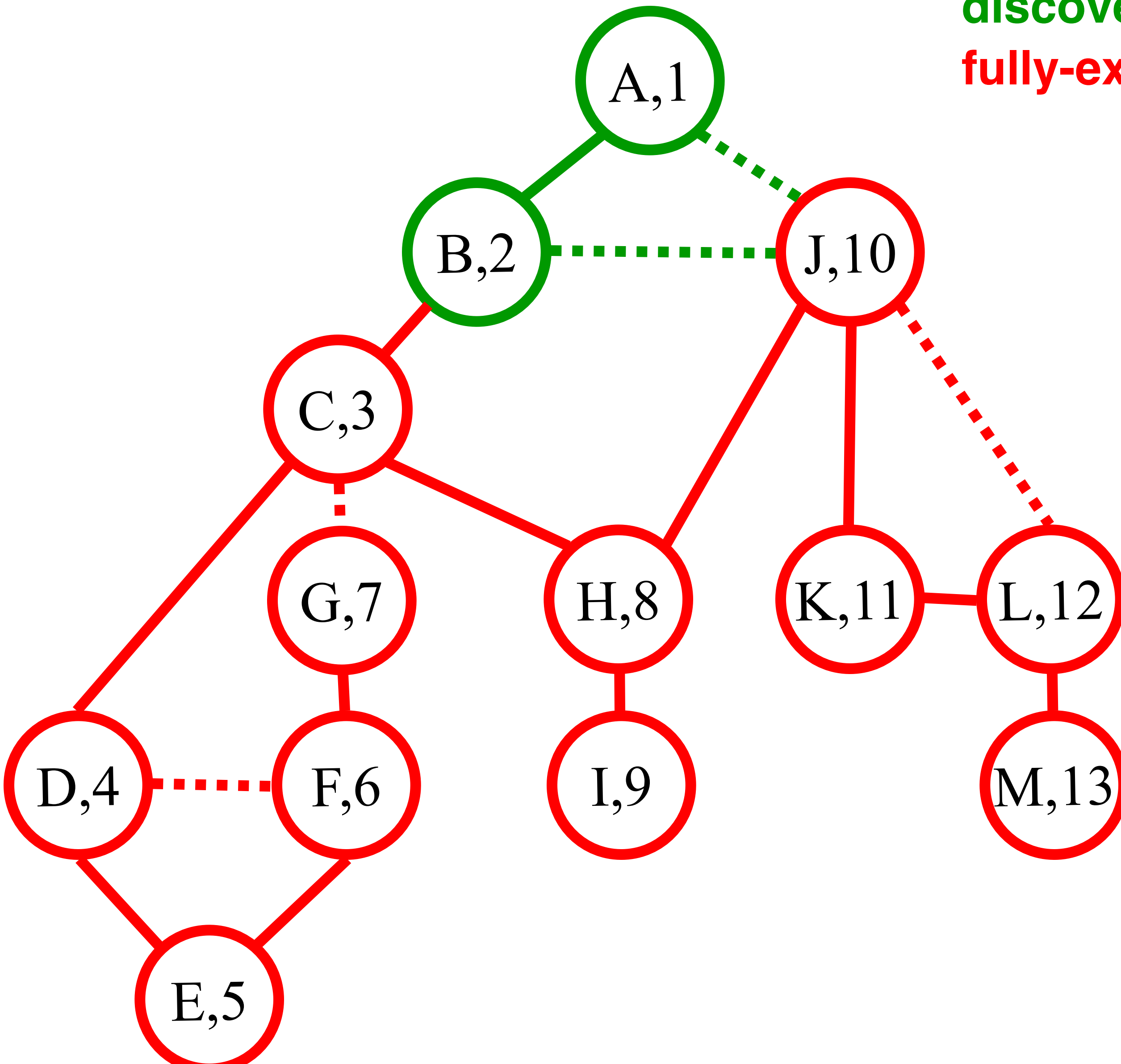
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)

st[] =
{1,2,3}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)

st[] =
{1,2}

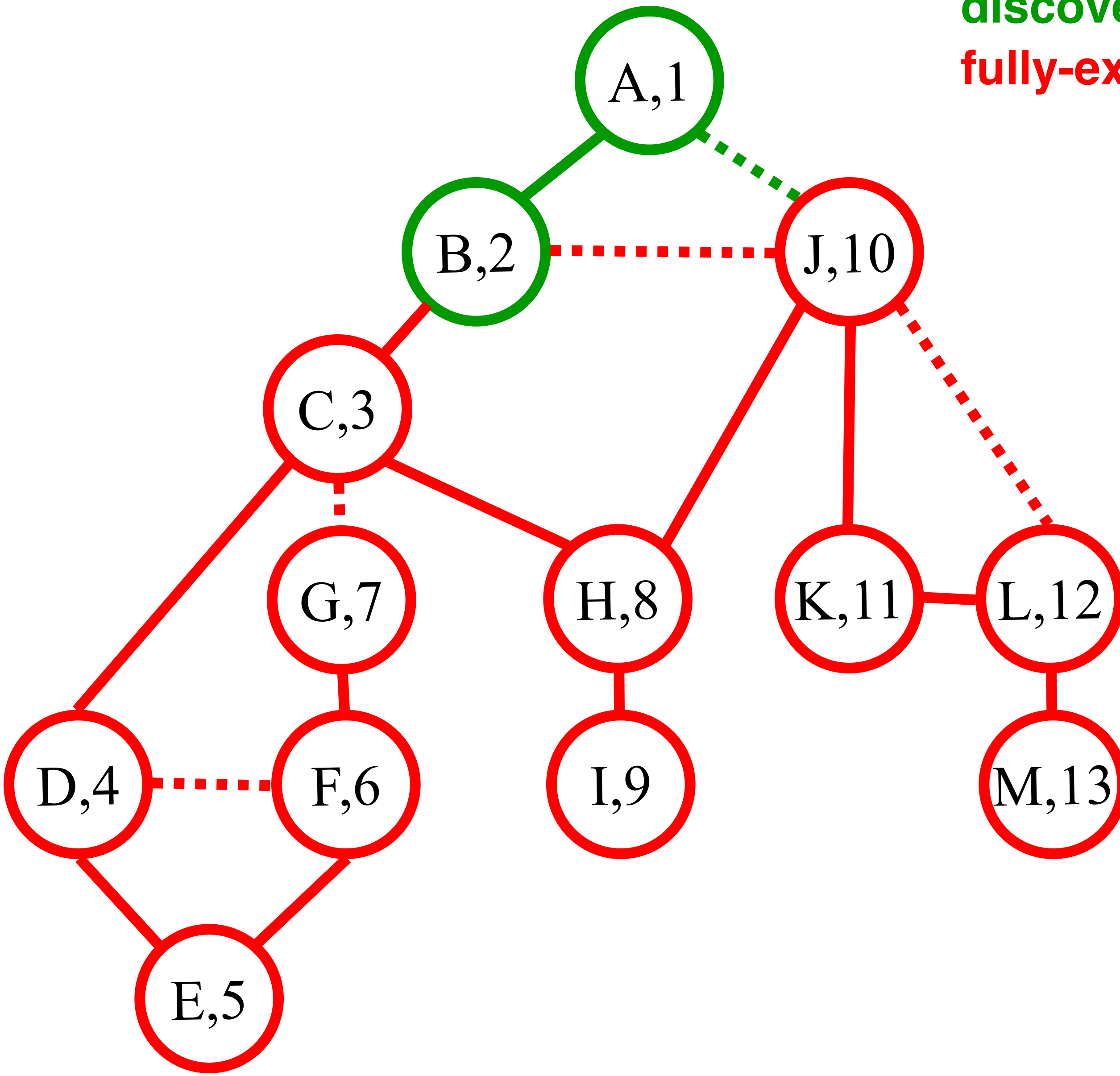
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,~~J~~)

st[] =
{1,2}

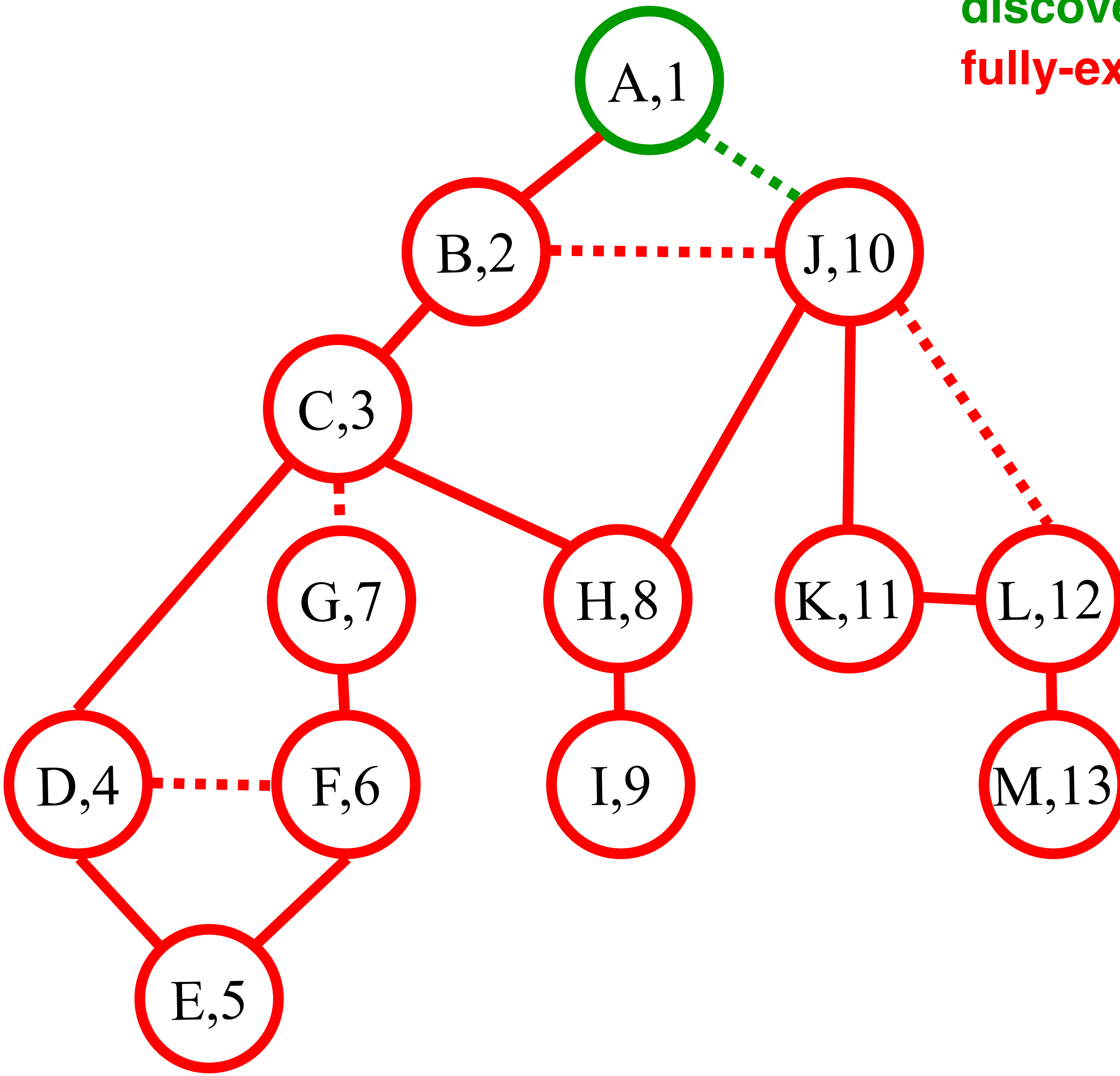
DFS(A)

Color code:

undiscovered

discovered

fully-explored

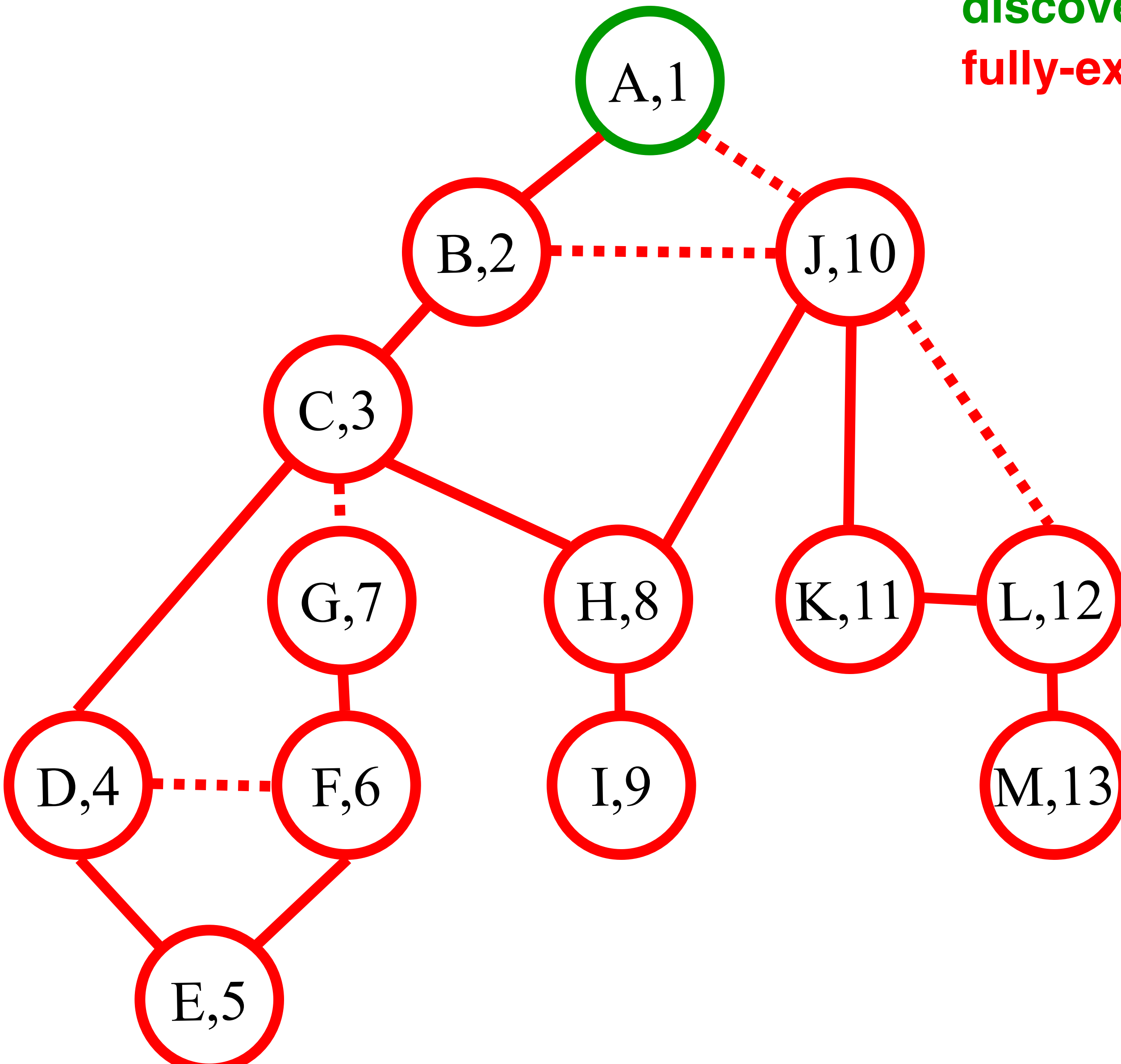


Call Stack:
(Edge list)

A (~~B~~,J)

st[] =
{1}

DFS(A)



Color code:

undiscovered

discovered

fully-explored

Call Stack:
(Edge list)

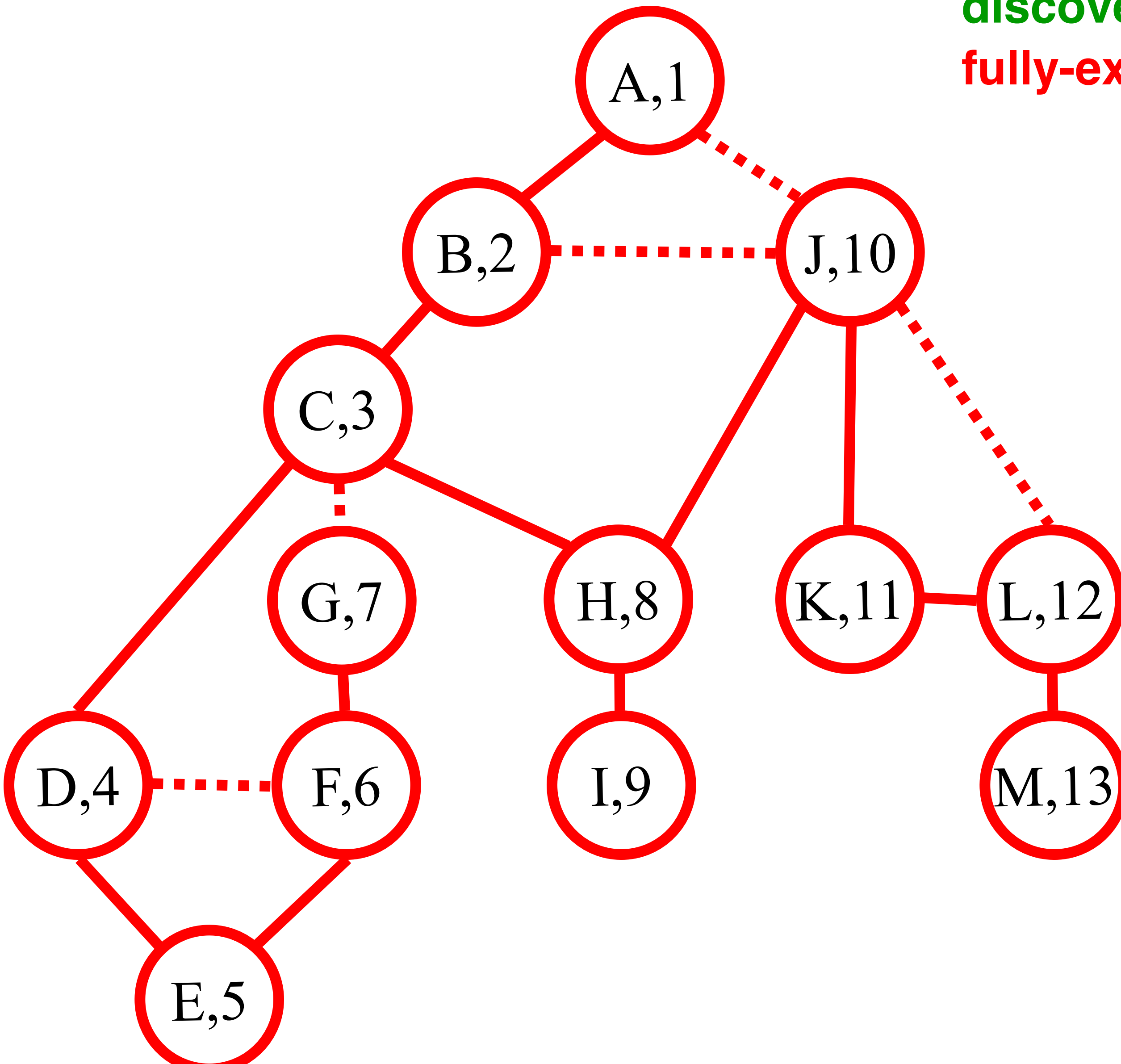
A ($\cancel{B}$,$\cancel{J}$)

st[] =
{1}

DFS(A)

Color code:

undiscovered
discovered
fully-explored

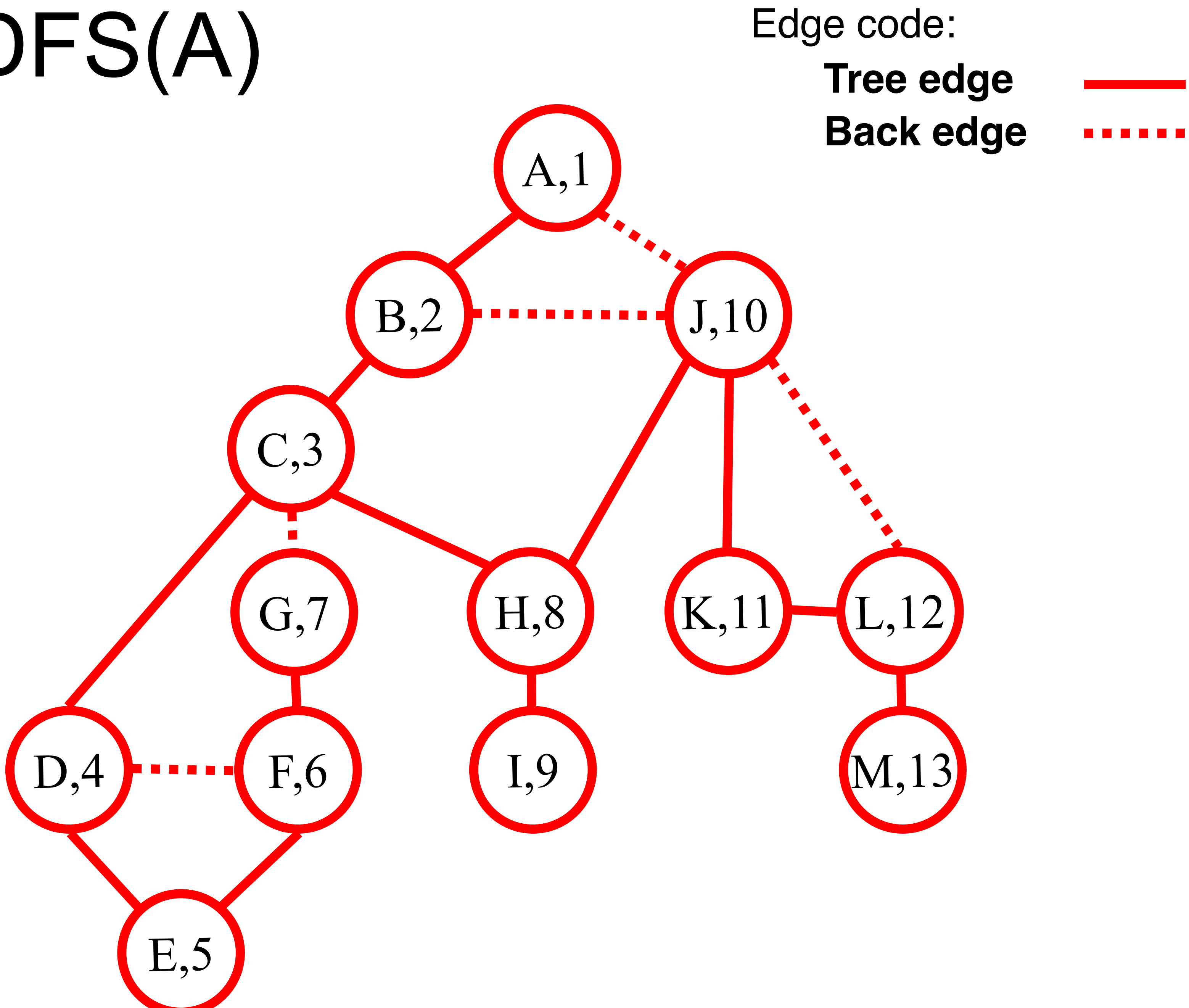


Call Stack:
(Edge list)

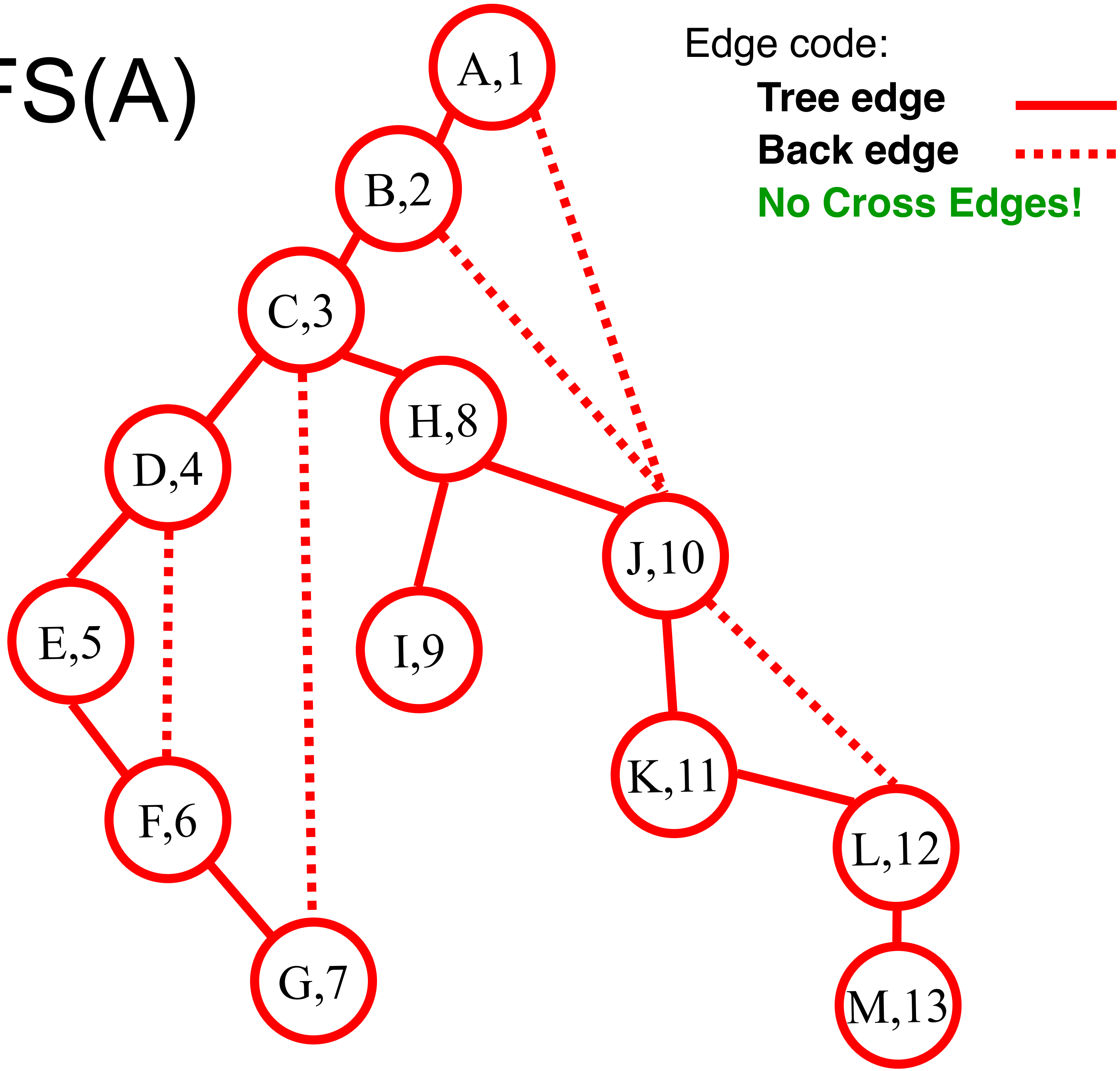
TA-DA!!

st[] = {}

DFS(A)


Edge code:
Tree edge
Back edge
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
D,4
F,6
I,9
M,13
E,5

DFS(A)


A,1
B,2
C,3
D,4
E,5
F,6
G,7
H,8
I,9
J,10
K,11
L,12
M,13
Edge code:
Tree edge
Back edge
No Cross Edges!

Properties of (undirected) DFS

Like BFS(s):

- DFS(s) visits x iff there is a path in G from s to x

  So, we can use DFS to find connected components
- Edges into then-undiscovered vertices define a ***tree*** – the "depth first spanning tree" of G

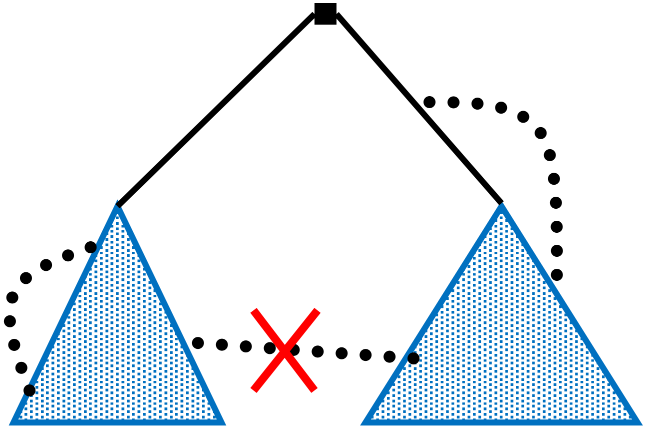
Unlike the BFS tree:

- The DFS spanning tree isn't minimum depth
- Its levels don't reflect min distance from the root
- Non-tree edges never join vertices on the same or adjacent levels

Non-Tree Edges in DFS

All non-tree edges join a vertex and one of its descendants/ancestors in the DFS tree

BFS tree ≠ DFS tree, but, as with BFS, DFS has found a tree in the graph s.t. non-tree edges are "simple" – only descendant/ancestor



Non-Tree Edges in DFS

Obs: During DFS(x) every vertex marked visited is a descendant of x in the DFS tree

Lemma: For every edge $\{x, y\}$, if $\{x, y\}$ is not in DFS tree, then one of x or y is an ancestor of the other in the tree.

Proof:

One of x or y is discovered first, suppose WLOG that x is discovered first and therefore DFS(x) was called before DFS(y)

Since $\{x, y\}$ is not in DFS tree, y was fully-explored when the edge {x,y} was examined during DFS(x)

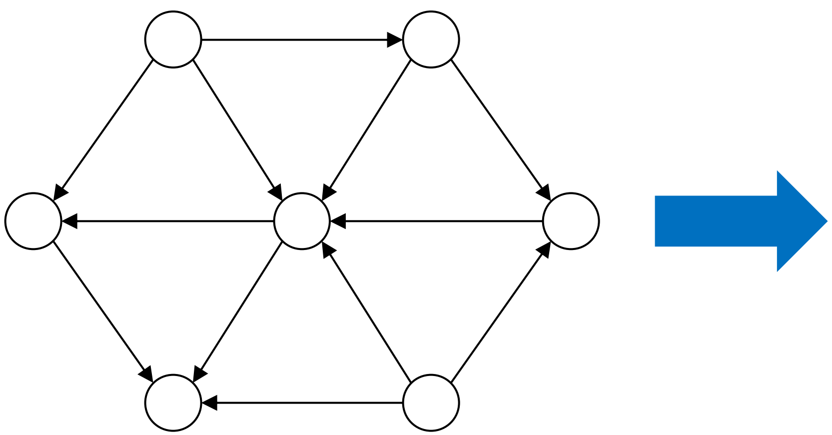
- Therefore y was discovered during the call to DFS(x) so y is a descendant of x by observation.

DAGs and Topological Ordering

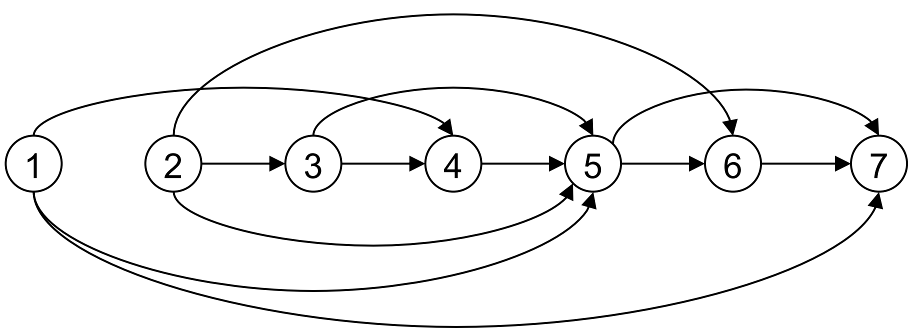
Directed Acyclic Graphs (DAG)

A DAG is a directed acyclic graph, i.e., one that contains no directed cycles.

Def: A topological order of a directed graph $G = (V, E)$ is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge (v_i, v_j) we have $i < j$.



a DAG



a topological ordering of that DAG– all edges left-to-right

DAGs: A Sufficient Condition

Lemma: If G has a topological order, then G is a DAG.

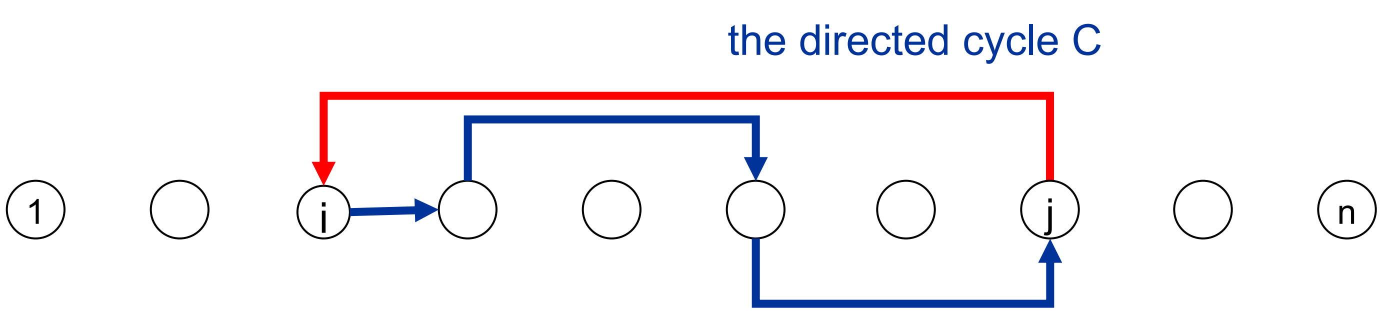
Pf. (by contradiction)

Suppose that G has a topological order $1{,}2, \ldots, n$ and that G also has a directed cycle C.

Let i be the lowest-indexed node in C, and let j be the node just before i; thus (j, i) is an (directed) edge.

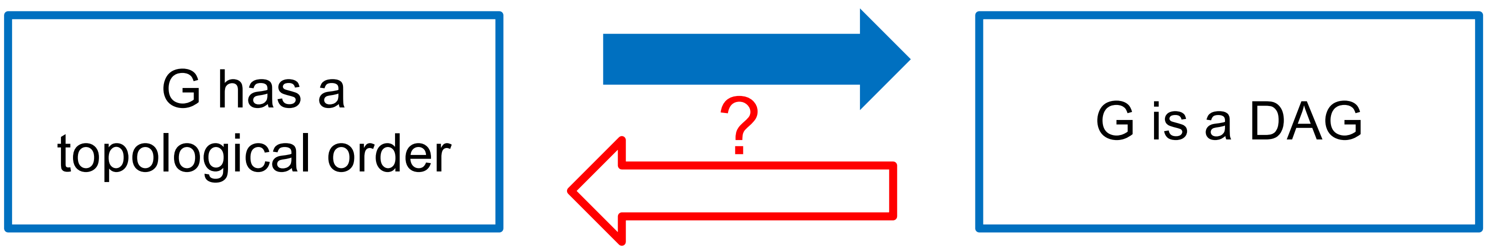
By our choice of i, we have $i < j$.

On the other hand, since (j, i) is an edge and $1, \ldots, n$ is a topological order, we must have $j < i$, a contradiction



the supposed topological order: 1,2,…,n

DAGs: A Sufficient Condition



Every DAG has a source node

Lemma: If G is a DAG, then G has a node with no incoming edges (i.e., a source).

Pf. (by contradiction)

Suppose that G is a DAG and and it has no source

Pick any node v, and begin following edges backward from v. Since v has at least one incoming edge (u, v) we can walk backward to u.

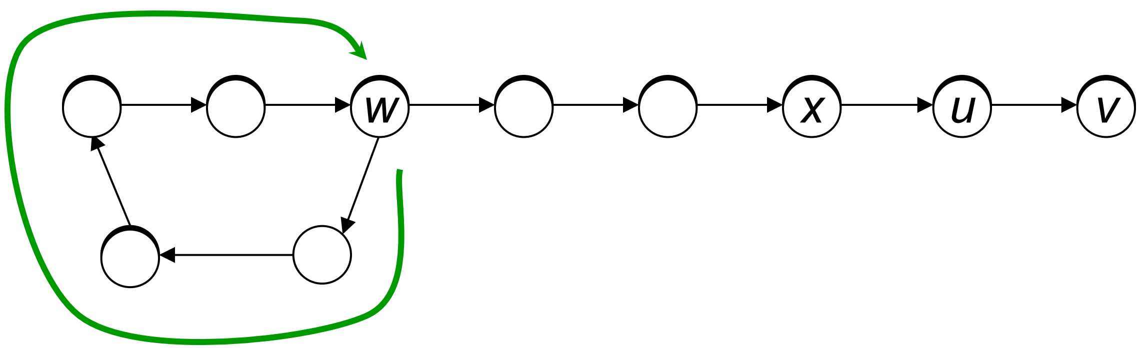
Then, since u has at least one incoming edge (x, u), we can walk backward to x.

Repeat until we visit a node, say w, twice.

Is this similar to a previous proof?

Let C be the sequence of nodes encountered between successive visits to w. C is a cycle.

C



DAG => Topological Order

Lemma: If G is a DAG, then G has a topological order

Pf. (by induction on n)
Base case: true if n = 1.

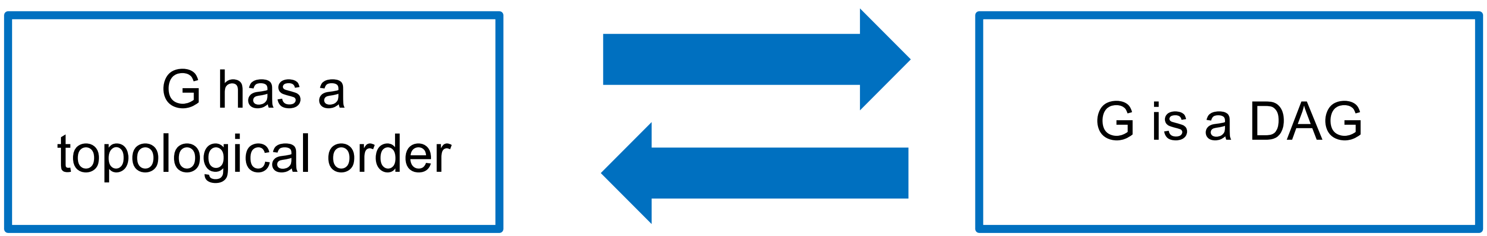
IH: Every DAG with n-1 vertices has a topological ordering.

IS: Given DAG with $n > 1$ nodes, find a source node v.
$G - \{ v \}$ is a DAG, since deleting v cannot create cycles.

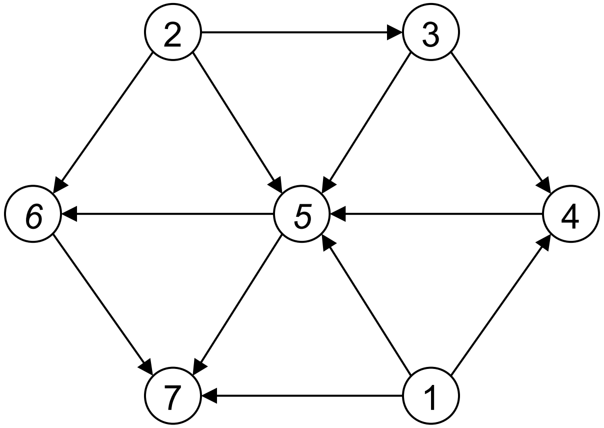
Reminder: Always remove vertices/edges to use IH

By IH, $G - \{ v \}$ has a topological ordering.
Place v first in topological ordering; then append nodes of G - { v } in topological order. This is valid since v has no incoming edges.

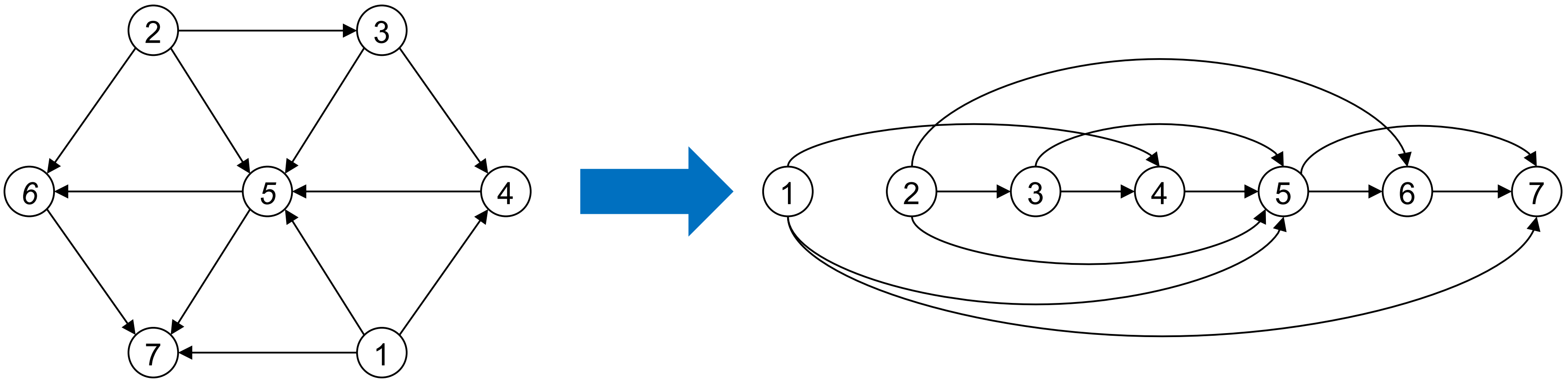
A Characterization of DAGs



Topological Order Algorithm: Example



Topological Order Algorithm: Example



Topological order: 1, 2, 3, 4, 5, 6, 7

Induction gives Algorithms!

Topological Sorting Algorithm

Maintain the following:

count[w] = (remaining) number of incoming edges to node w

S = set of (remaining) nodes with no incoming edges

Initialization:

count[w] = 0 for all w

count[w]++ for all edges (v,w) $\quad$ O(m + n)

S = S $\cup$ {w} for all w with count[w]=0

Main loop:

while S not empty

- remove some v from S
- make v next in topo order $\quad$ O(1) per node
- for all edges from v to some w $\quad$ O(1) per edge
  - decrement count[w]
  - add w to S if count[w] hits 0

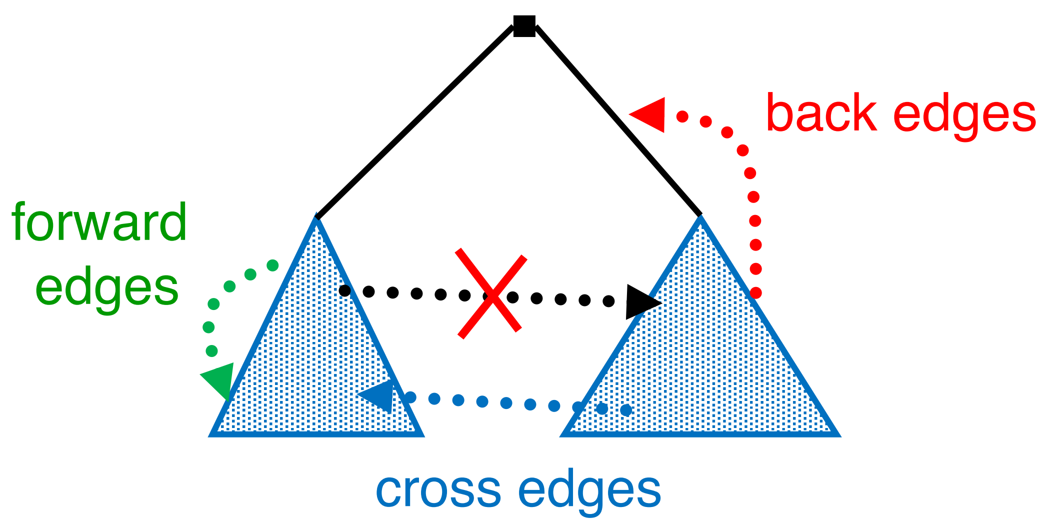
Correctness: clear, I hope

Time: O(m + n) (assuming edge-list representation of graph)

DFS on Directed Graphs

- Before DFS(s) returns, it visits all previously unvisited vertices reachable via directed paths from s

- Every cycle contains a back edge in the DFS tree



Summary

- Graphs: abstract relationships among pairs of objects
- Terminology: node/vertex/vertices, edges, paths, multi-edges, self-loops, connected
- Representation: Adjacency list, adjacency matrix
- Nodes vs Edges: $m = O(n^2)$, often less
- BFS: Layers, queue, shortest paths, all edges go to same or adjacent layer
- DFS: recursion/stack; all edges ancestor/descendant
- Algorithms: Connected Comp, bipartiteness, topological sort

Greedy Algorithms

Greedy Strategy

Goal: Given currency denominations: 1, 5, 10, 25, 100, give change to customer using *fewest* number of coins.

Ex: 34¢.

Cashier's algorithm: At each iteration, give the *largest* coin valued ≤ the amount to be paid.

Ex: $2.89.

Greedy is not always Optimal

Observation: Greedy algorithm is sub-optimal for US postal denominations: 1, 10, 21, 34, 70, 100, 350, 1225, 1500.

Counterexample. 140¢.

Greedy: 100, 34, 1, 1, 1, 1, 1, 1.

Optimal: 70, 70.

Lesson: Greedy is short-sighted. Always chooses the most attractive choice at the moment. But this may lead to a dead-end later.

Greedy Algorithms Outline

Pros

- Intuitive
- Often simple to design (and to implement)
- Often fast

Cons

- Often incorrect!

Proof techniques:

- Stay ahead
- Structural
- Exchange arguments