CSE 421

Divide and Conquer

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Quiz

Problem 4 (20 points).

Given an array of positive numbers $a = [a_1, a_2, \cdots, a_n]$. Give an $O(n \log n)$ time algorithm that find i and j (with $i \le j$) that maximize the subarray product $\prod_{k=i}^{j} a_k$. Prove the correctness and the runtime of the algorithm.

For example, in the array $a = [3, 0.2, 5, 7, 0.4, 4, 0.01]$, the sub-array from $i = 3$ to $j = 6$ has the product $5 \times 7 \times 0.4 \times 4 = 56$ and no other sub-array contains elements that product to a value greater than 56. So, the answer for this input is $i = 3, j = 6$.

Hints: Divide and Conquer.

Lecture 10 Quiz #95

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a day ago in Lectures

PIN STAR WATCHING 55 VIEWS

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I was wrong about the maximum product subarray. You can solve using a sliding window.

Remark: I can be wrong. So feel free to debate with me after the lecture if you think you have a better algo!

Here is the algorithm provided by Adam Wang and Alan Wu:

L = 1, R = 1. Answer = x_L. AnswerIdx = [L,R]

For R = 1, 2, ... N

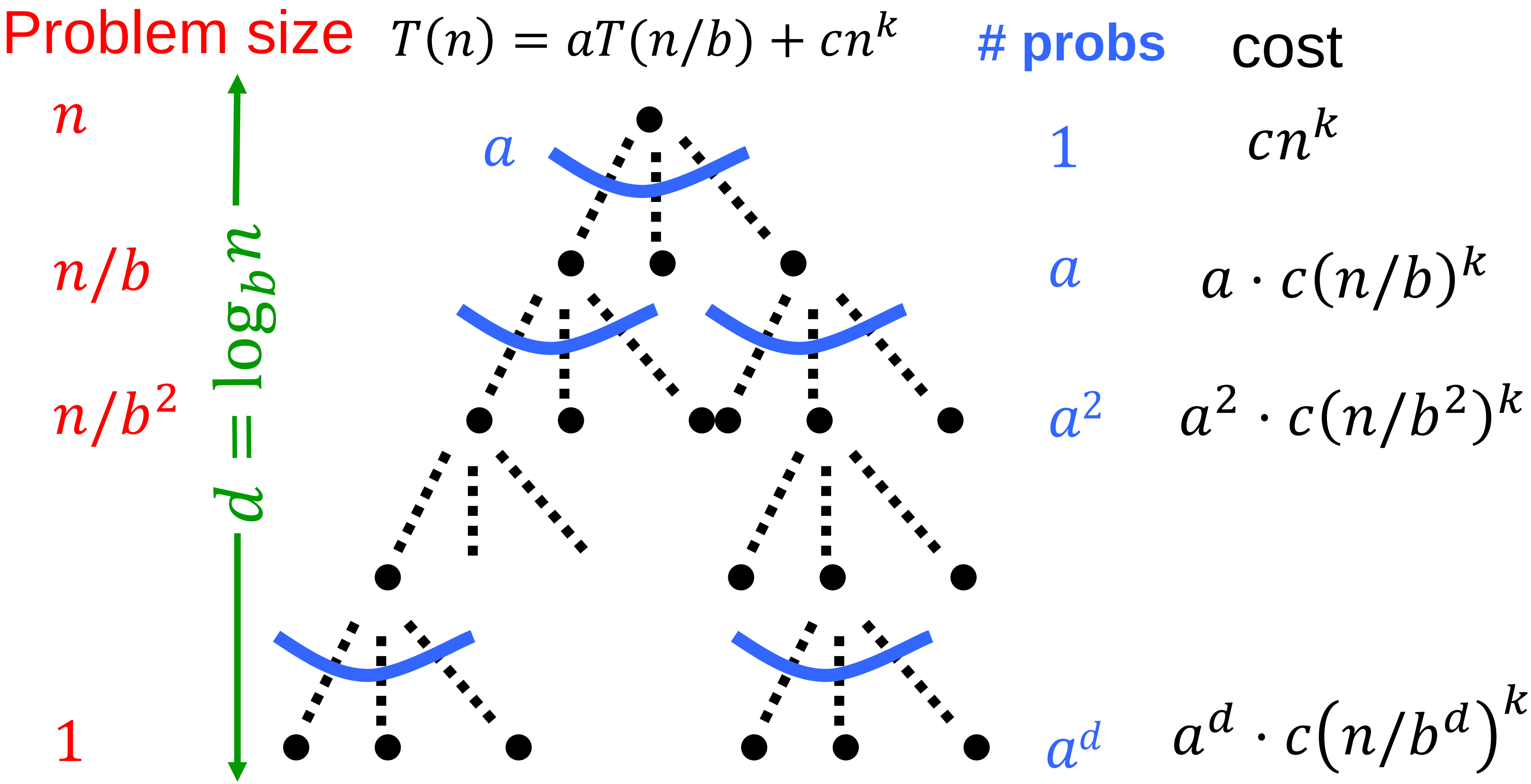
- Let P = $\prod_{i=L}^{R} x_i$.
- if P <= x_R
  - set L=R
- If Answer < P
  - set Answer = P. Set AnswerIdx = [L,R]

Output AnswerIdx

Runtime: O(n) by maintaining the product.

Master Theorem

Proving Master Theorem



$$T(n) = \sum_{i=0}^{d=\log_b n} a^i c \left(\frac{n}{b^i}\right)^k$$

Master Theorem

Suppose $T(n) = a\,T\left(\frac{n}{b}\right) + cn^k$ for all $n > b$. Then,

- If $a < b^k$ then $T(n) = \Theta\left(n^k\right)$

# of problems increases slower than the decreases of cost.
First term dominates.

- If $a = b^k$ then $T(n) = \Theta\left(n^k \log n\right)$

- If $a > b^k$ then $T(n) = \Theta\left(n^{\log_b a}\right)$

# of problems increases faster than the decreases of cost
Last term dominates.

A Useful Identity

Theorem: $1 + x + x^2 + \cdots + x^d = \frac{x^{d+1}-1}{x-1}$

Proof: Let $S = 1 + x + x^2 + \cdots + x^d$

Then, $xS = x + x^2 + \cdots + x^{d+1}$

So, $xS - S = x^{d+1} - 1$

i.e., $S(x-1) = x^{d+1} - 1$

Therefore, $S = \frac{x^{d+1}-1}{x-1}$

Corollary:

O_x means the hidden constant depends on x

$$1 + x + x^2 + \cdots + x^d = \begin{cases} O_x(1) & \text{if } x < 1 \\ d+1 & \text{if } x = 1 \\ O_x\left(x^{d+1}\right) & \text{if } x > 1 \end{cases}$$

Solve: $T(n) = aT\left(\frac{n}{b}\right) + cn^k$

Corollary:

$$1 + x + x^2 + \cdots + x^d = \begin{cases} \Theta_x(1) & \text{if } x < 1 \\ \Theta(d) & \text{if } x = 1 \\ \Theta_x\left(x^{d+1}\right) & \text{if } x > 1 \end{cases}$$

Going back, we have

$$T(n) = \sum_{i=0}^{d=\log_b n} a^i c \left(\frac{n}{b^i}\right)^k = cn^k \sum_{i=0}^{d=\log_b n} \left(\frac{a}{b^k}\right)^i$$

Hence, we have

$$T(n) = \Theta\left(n^k\right) \begin{cases} 1 & \text{if } a < b^k \\ \log_b n & \text{if } a = b^k \\ \left(\frac{a}{b^k}\right)^{\log_b n} & \text{if } a > b^k \end{cases}$$

constant depends on a, b, c

Solve: $T(n) = aT\left(\frac{n}{b}\right) + cn^k$

$$T(n) = \Theta(n^k)\begin{cases} 1 & \text{if } a < b^k \\ \log_b n & \text{if } a = b^k \\ \left(\frac{a}{b^k}\right)^{\log_b n} & \text{if } a > b^k \end{cases}$$

For $a < b^k$, we simply have $T(n) = \Theta(n^k)$.

For $a = b^k$, we have $T(n) = \Theta(n^k \log_b n) = \Theta(n^k \log n)$.

For $a > b^k$, we have $T(n) = \Theta\left(n^k \left(\frac{a}{b^k}\right)^{\log_b n}\right) = \Theta(n^{\log_b a})$.

$$\boxed{\begin{aligned} & b^{k \log_b n} \\ & = \left(b^{\log_b n}\right)^k \\ & = n^k \end{aligned}}$$

$$\boxed{\begin{aligned} & a^{\log_b n} \\ & = (b^{\log_b a})^{\log_b n} \\ & = (b^{\log_b n})^{\log_b a} \\ & = n^{\log_b a} \end{aligned}}$$

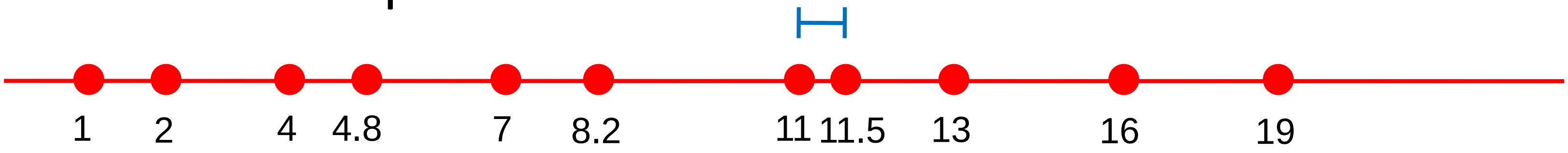
Finding the Closest Pair of Points

Closest Pair of Points (1-dimension)

Given n points on the real line, find the closest pair,

e.g., given 11, 2, 4, 19, 4.8, 7, 8.2, 16, 11.5, 13, 1

find the closest pair



Fact: Closest pair is adjacent in ordered list

So, first sort, then scan adjacent pairs.

Time $O(n \log n)$ to sort, if needed, Plus $O(n)$ to scan adjacent pairs

Key point: do *not* need to calculate distances between all pairs: exploit geometry + ordering

Closest Pair of Points (2-dimensions)

Given n points in the plane, find a pair with smallest Euclidean distance between them.

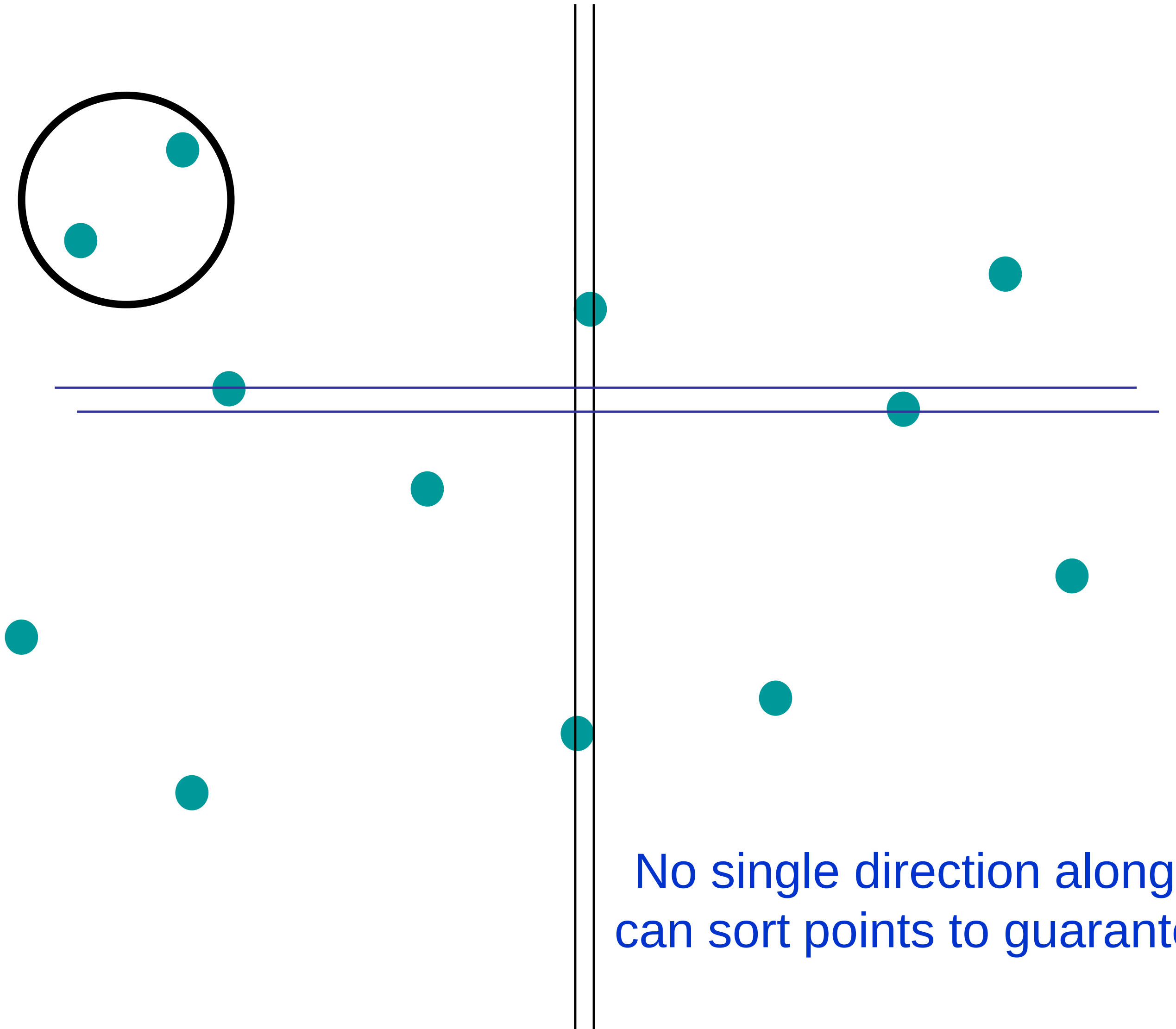
Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

Brute force: Check all pairs of points in $\Theta(n^2)$ time.

Assumption: No two points have same x coordinate.

Closest Pair of Points (2-dimensions)



No single direction along which one can sort points to guarantee success!

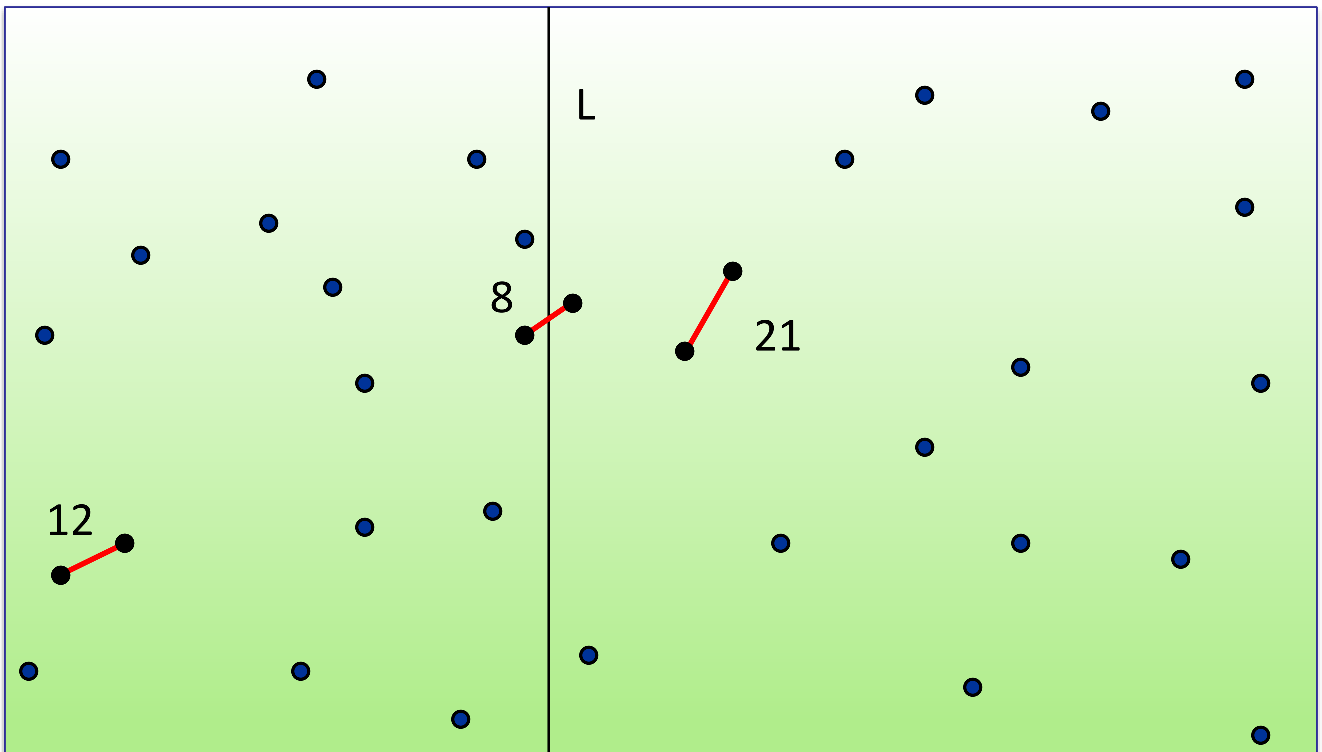
Divide & Conquer

Divide: draw vertical line L with $\approx n/2$ points on each side.

Conquer: find closest pair on each side, recursively.

Combine to find closest pair overall ← How ?

Return best solutions



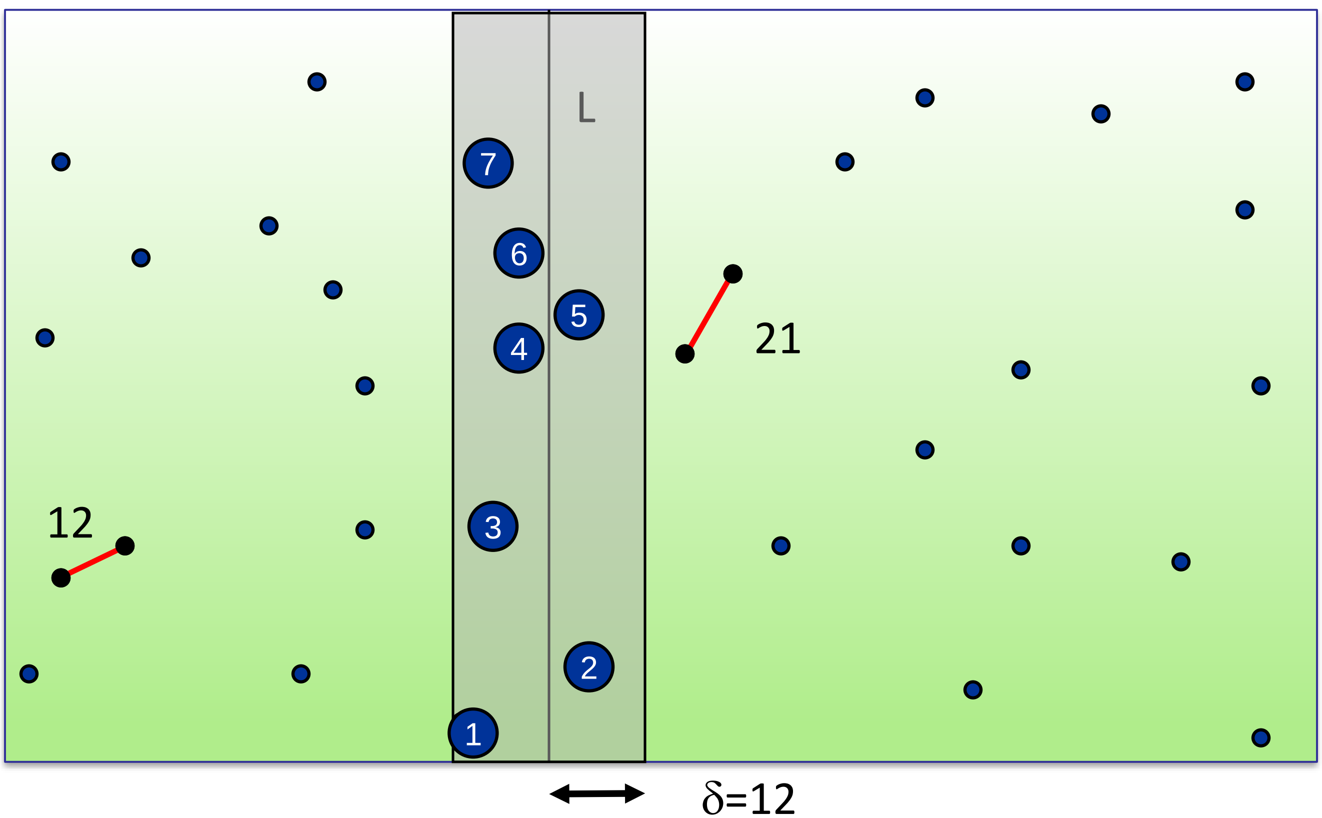
Key Observation

Suppose δ is the minimum distance of all pairs in left/right of L.

$$\delta = \min(12{,}21) = 12.$$

Key Observation: suffices to consider points within δ of line L.

Almost the one-D problem again: Sort points in 2δ-strip by their y coordinate.



Almost 1D Problem

Partition each side of L into $\frac{\delta}{2} \times \frac{\delta}{2}$ squares

Claim: No two points lie in the same $\frac{\delta}{2} \times \frac{\delta}{2}$ box.

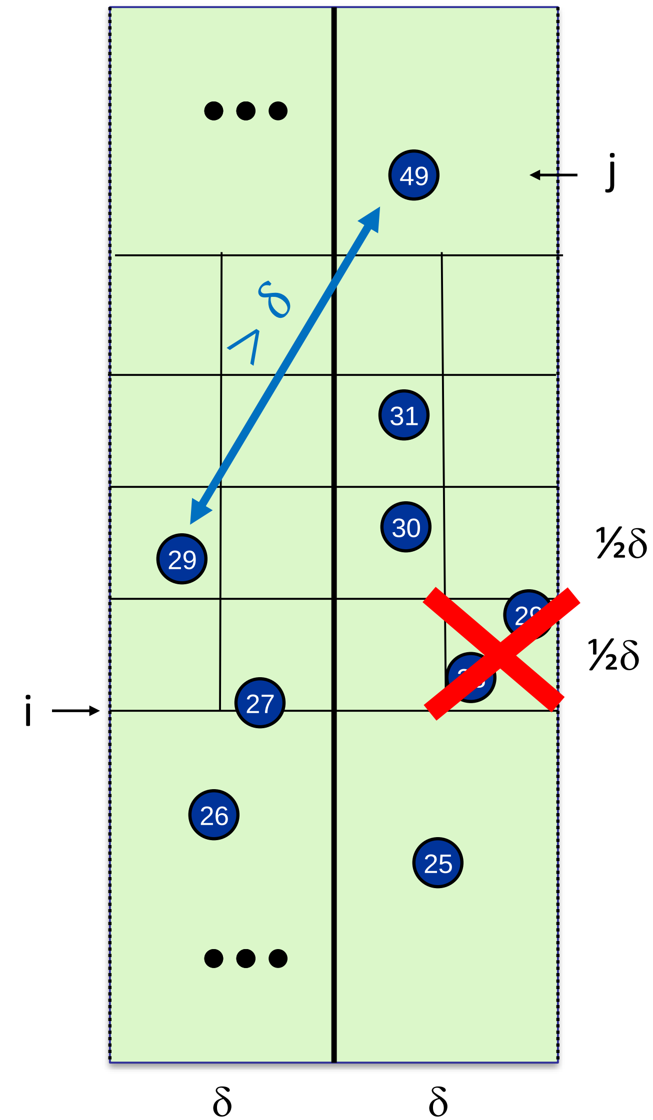
Proof: Such points would be within

$$\sqrt{\left(\frac{\delta}{2}\right)^2 + \left(\frac{\delta}{2}\right)^2} = \delta\sqrt{\frac{1}{2}} \approx 0.7\delta < \delta$$

Let s_i have the i^{th} smallest y-coordinate among points in the 2δ-width-strip.

Claim: If $|i - j| > 11$, then the distance between s_i and s_j is $> \delta$.

Proof: only 11 boxes within δ of $y(s_i)$.



Where is the bottleneck?

Closest Pair (2 dimension)

```
Closest-Pair(p_1,p_2,⋯,p_n) {
   if(n ≤ 2) return |p_1 − p_2|

   Compute separation line L such that half the points
   are on one side and half on the other side.

   δ_1 = Closest-Pair(left half)
   δ_2 = Closest-Pair(right half)
   δ  = min(δ_1,δ_2)

   Delete all points further than δ from separation line L

   Sort remaining points p[1]…p[m] by y-coordinate.

   for i = 1,2,⋯,m
      for k = 1,2,⋯,11
        if i + k ≤ m
            δ = min(δ, distance(p[i], p[i+k]));

   return δ.
}
```

Closest Pair Analysis

Let $D(n)$ be the number of pairwise distance calculations in the Closest-Pair Algorithm

$$D(n) \le \begin{cases} 1 & \text{if } n = 1 \\ 2D\left(\frac{n}{2}\right) + 11\,n & \text{o.w.} \end{cases} \Rightarrow D(n) = O(n\log n)$$

BUT, that's only the number of *distance calculations*

What if we counted running time?

$$T(n) \le \begin{cases} 1 & \text{if } n = 1 \\ 2T\left(\frac{n}{2}\right) + O(n\log n) & \text{o.w.} \end{cases} \Rightarrow T(n) = O(n\log^2 n)$$

Closest Pair (2 dimension) Improved

```
Closest-Pair(p_1, p_2, ..., p_n) {
    if(n ≤ 2) return |p_1 - p_2|

    Compute separation line L such that half the points
    are on one side and half on the other side.

   (δ_1, p_1) = Closest-Pair(left half)
   (δ_2, p_2) = Closest-Pair(right half)
    δ         = min(δ_1, δ_2)
    p_sorted  = merge(p_1, p_2)   (merge sort it by y-coordinate)

    Let q be points (ordered as p_sorted) that is δ from line L.

    for i = 1, 2, ..., m
        for k = 1, 2, ..., 11
          if i + k ≤ m
              δ = min(δ, distance(q[i], q[i+k]));

    return δ and p_sorted.
}
```

$$T(n) \le \begin{cases} 1 & \text{if } n = 1 \\ 2T\left(\frac{n}{2}\right) + O(n) & \text{o.w.} \end{cases}$$

$$\Rightarrow T(n) = O(n \log n)$$

Quiz

How to solve closest pair in 3 dimension?

```
Closest-Pair(p_1, p_2, ..., p_n) {
   if(n ≤ 2) return |p_1 − p_2|

   Compute separation line L such that half the points
   are on one side and half on the other side.

   δ_1 = Closest-Pair(left half)
   δ_2 = Closest-Pair(right half)
   δ  = min(δ_1, δ_2)

   Delete all points further than δ from separation line L

   Put points into δ/2 × δ/2 × δ/2 cubes (via hash table)

   for i = 1, 2, ..., m
      Let (a,b,c) be the cube for p[i].
      for x,y,z = -3,-2,1,0,1,2,3
         check the cube (a+x,b+y,c+z)
         if there is a point q in the cube,
            δ = min(δ, distance(p[i], q));

   return δ.
}
```

In d dimension, the runtime is

$$T(n) = 2^{O(d)} n \log n$$

Median

Selecting k-th smallest

Problem: Given numbers $x_1, \dots, x_n$ and an integer $1 \leq k \leq n$ output the k-th smallest number

$$\mathrm{Sel}(\{x_1, \dots, x_n\}, k)$$

A simple algorithm: Sort the numbers in time $O(n \log n)$ then return the k-th smallest in the array.

Can we do better?

Yes, in time $O(n)$ if $k = 1$ or $k = 2$.

Can we do $O(n)$ for all possible values of k?

An Idea

Choose a number w from $x_1, \dots, x_n$

Define

$$\left.\begin{array}{l} \bullet\ S_<(w) = \{x_i : x_i < w\} \\ \bullet\ S_=(w) = \{x_i : x_i = w\} \\ \bullet\ S_>(w) = \{x_i : x_i > w\} \end{array}\right\}$$

Can be computed in linear time

Solve the problem recursively as follows:

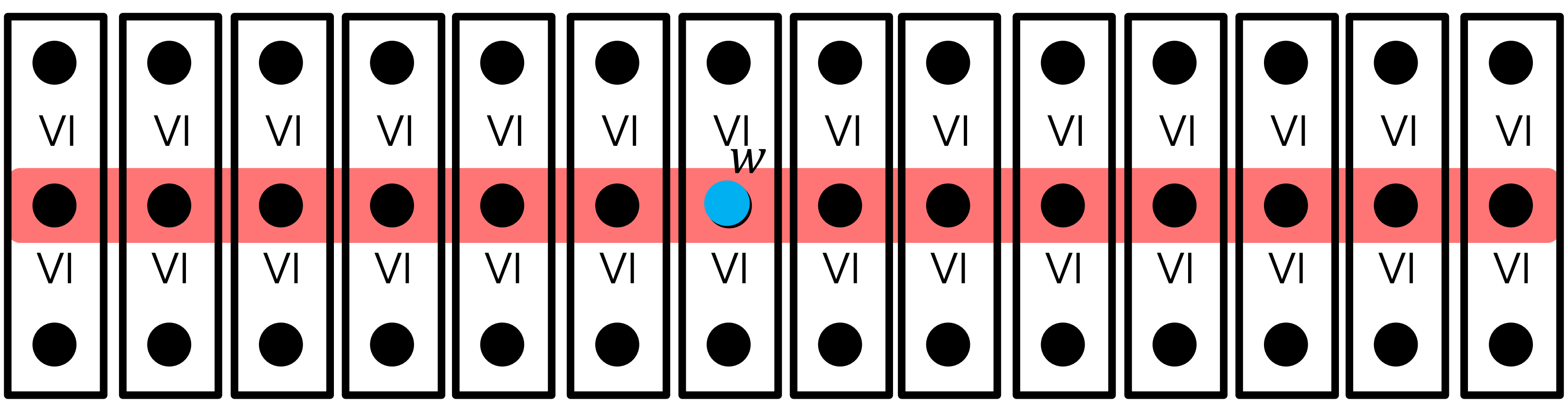
- If $k \leq |S_<(w)|$, output $Sel(S_<(w), k)$
- Else if $k \leq |S_<(w)| + |S_=(w)|$, output w
- Else output $Sel(S_>(w), k - |S_<(w)| - |S_=(w)|)$

Ideally want $|S_<(w)|, |S_>(w)| \leq n/2$. In this case ALG runs in $O(n) + O\left(\frac{n}{2}\right) + O\left(\frac{n}{4}\right) + \cdots + O(1) = O(n)$.

How to choose w?

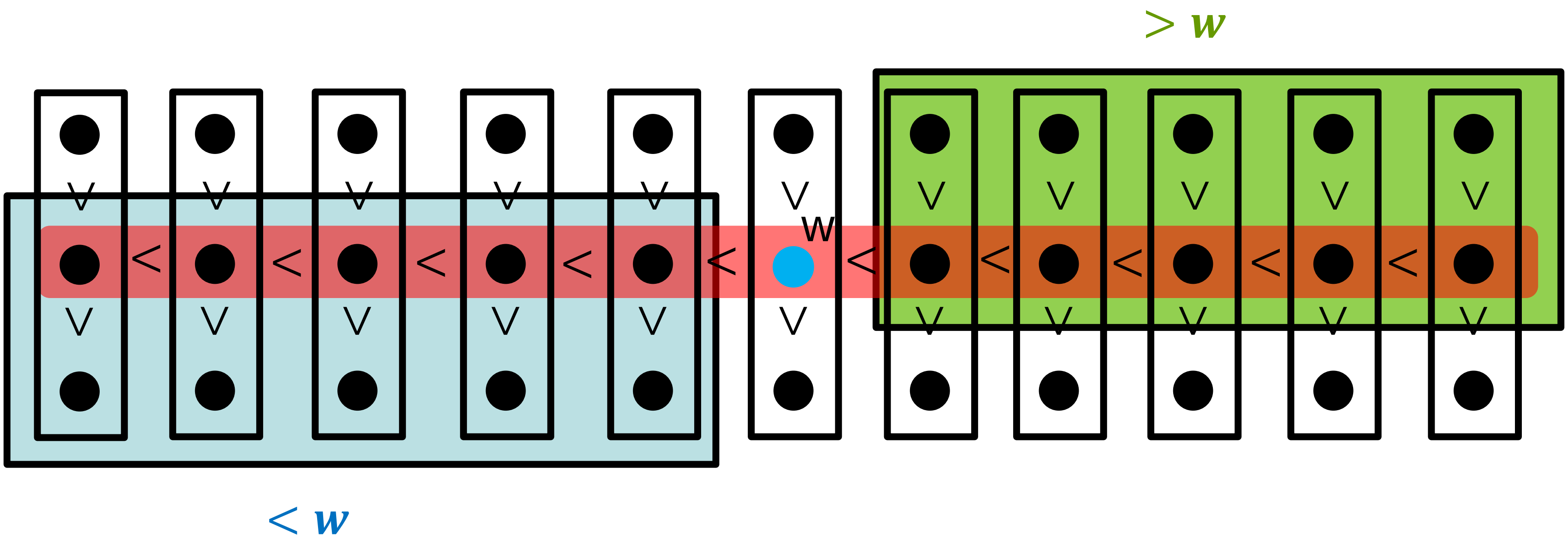
Suppose we choose w uniformly at random
similar to the pivot in quicksort.
Then, $\mathbb{E}[|S_<(w)|] = \mathbb{E}[|S_>(w)|] = n/2$. Algorithm runs in $O(n)$ in expectation.
Can we get $O(n)$ running time deterministically?

- Partition numbers into sets of size 3.
- Sort each set (takes O(n))
- $w = Sel(midpoints, n/6)$



Assume all numbers are distinct for simplicity.

How to lower bound $|S_<(w)|, |S_>(w)|$?



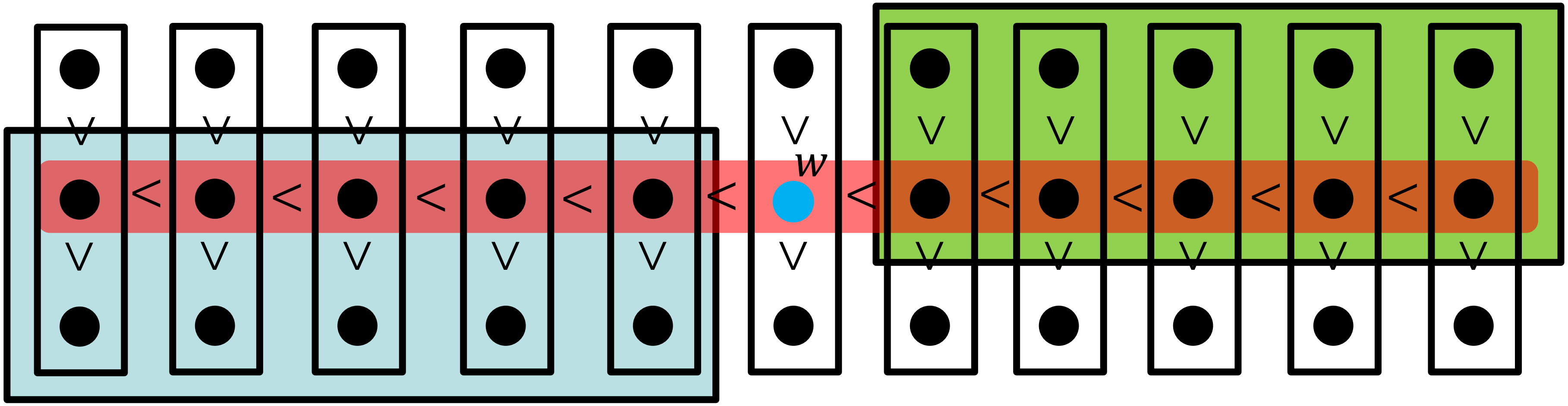
- $|S_<(w)| \geq 2\left(\frac{n}{6}\right) = \frac{n}{3}$
- $|S_>(w)| \geq 2\left(\frac{n}{6}\right) = \frac{n}{3}$.

$$\frac{n}{3} \leq |S_<(w)|, |S_>(w)| \leq \frac{2n}{3}$$

So, what is the running time?

Assume all numbers are distinct for simplicity.

Asymptotic Running Time?



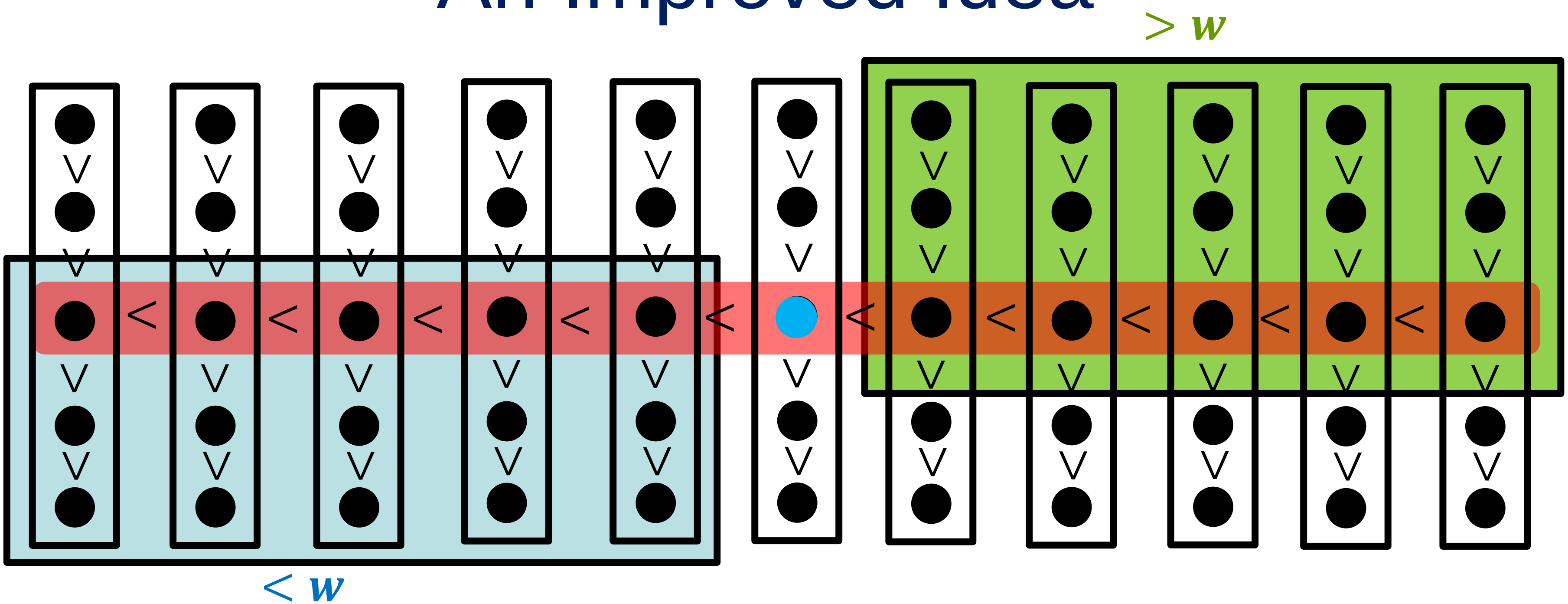
- If $k \leq |S_<(w)|$, output $Sel(S_<(w), k)$
- Else if $k \leq |S_<(w)| + |S_=(w)|$, output w
- Else output $Sel(S_>(w), k - |S_<(w)| - |S_=(w)|)$

$O(n \log n)$ again?
So, what is the point?

Where $\frac{n}{3} \leq |S_<(w)|, |S_>(w)| \leq \frac{2n}{3}$

$$T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + O(n) \Rightarrow T(n) = O(n \log n)$$

An Improved Idea



Partition into n/5 sets. Sort each set and set $w = Sel(\textcolor{red}{midpoints}, n/10)$

- $|S_<(w)| \geq 3\left(\frac{n}{10}\right) = \frac{3n}{10}$
- $|S_>(w)| \geq 3\left(\frac{n}{10}\right) = \frac{3n}{10}$

$$\frac{3n}{10} \leq |S_<(w)|, |S_>(w)| \leq \frac{7n}{10}$$

$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + O(n) \Rightarrow T(n) = O(n)$$

Can we do it even better?

Goal: Finding median.

Fix ϵ.

Randomly select T/ϵ^2 elements.

Output the median of these T/ϵ^2 elements.

One can prove that it will gives an element with rank

$$(1/2 - \epsilon)n \text{ and } (1/2 + \epsilon)n$$

with probability at least $1 - \exp(-T)$.

Think $\epsilon = 0.1$ and $T = 30$.

Then, we have almost median with high prob in $O(1)$ time.

Integer Multiplication

Integer Arithmetic

Add: Given two n-bit integers a and b, compute $a + b$.

Add

1	1	1	1	1	1	0	1	
	1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0

$O(n)$ bit operations.

Multiply: Given two n-bit integers a and b, compute $a \times b$.
The "grade school" method:

Multiply

								1	1	0	1	0	1	0	1
							*	0	1	1	1	1	1	0	1
								1	1	0	1	0	1	0	1
							0	0	0	0	0	0	0	0	
						1	1	0	1	0	1	0	1		
					1	1	0	1	0	1	0	1			
				1	1	0	1	0	1	0	1				
			1	1	0	1	0	1	0	1					
		1	1	0	1	0	1	0	1						
	0	0	0	0	0	0	0	0							
0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	1

$O(n^2)$ bit operations.

Divide and Conquer

Let x, y be two n-bit integers

Write $x = 2^{n/2}x_1 + x_0$ and $y = 2^{n/2}y_1 + y_0$
  where x_0, x_1, y_0, y_1 are all $n/2$-bit integers.

$$\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right) \\
&= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot (x_1 y_0 + x_0 y_1) + x_0 y_0
\end{aligned}$$

Therefore,

$$T(n) = 4T\left(\frac{n}{2}\right) + \Theta(n)$$

We only need 3 values
$x_1 y_1, x_0 y_0, x_1 y_0 + x_0 y_1$
Can we find all 3 by only
3 multiplication?

So,

$$T(n) = \Theta(n^2).$$

Key Trick: 4 multiplies at the price of 3

$$x = 2^{n/2} \cdot x_1 + x_0$$
$$y = 2^{n/2} \cdot y_1 + y_0$$
$$xy = \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right)$$
$$= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot (x_1 y_0 + x_0 y_1) + x_0 y_0$$

$$\alpha = x_1 + x_0$$
$$\beta = y_1 + y_0$$
$$\alpha\beta = (x_1 + x_0)(y_1 + y_0)$$
$$= x_1 y_1 + (x_1 y_0 + x_0 y_1) + x_0 y_0$$
$$(x_1 y_0 + x_0 y_1) = \alpha\beta - x_1 y_1 - x_0 y_0$$

Key Trick: 4 multiplies at the price of 3

Theorem [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in $O(n^{1.585...})$ bit operations.

$$x = 2^{n/2} \cdot x_1 + x_0 \Rightarrow \alpha = x_1 + x_0$$
$$y = 2^{n/2} \cdot y_1 + y_0 \Rightarrow \beta = y_1 + y_0$$
$$xy = \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right)$$
$$= 2^n \cdot \underbrace{x_1 y_1}_{A} + 2^{n/2} \cdot \underbrace{(x_1 y_0 + x_0 y_1)}_{\alpha\beta - A - B} + \underbrace{x_0 y_0}_{B}$$

To multiply two n-bit integers:

Add two $n/2$ bit integers.

Multiply three $n/2$-bit integers.

Add, subtract, and shift $n/2$-bit integers to obtain result.

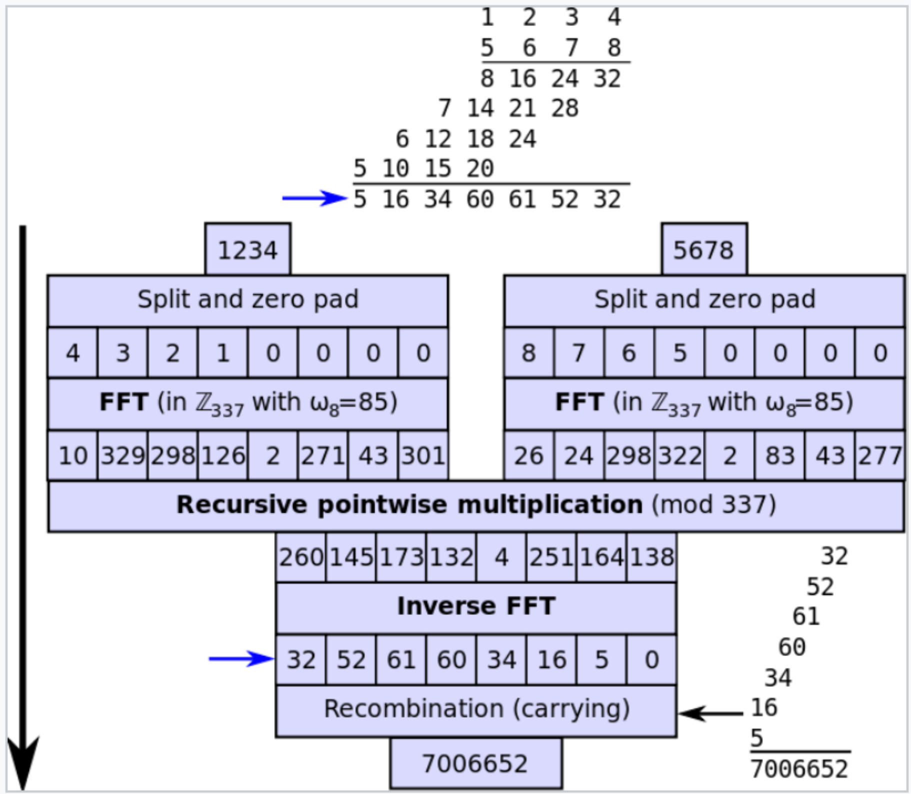
$$T(n) = 3T\left(\frac{n}{2}\right) + O(n) \Rightarrow T(n) = O\left(n^{\log_2 3}\right) = O(n^{1.585...})$$

Integer Multiplication (Summary)

- Exercise: generalize Karatsuba to do 5 size $n/3$ subproblems
  - This gives $\Theta(n^{1.46\ldots})$ time algorithm

Date	Authors	Time complexity
<3000 BC	Unknown	$O(n^2)$
1962	Karatsuba	$O(n^{\log 3/\log 2})$
1963	Toom	$O(n\,2^{5\sqrt{\log n/\log 2}})$
1966	Schönhage	$O(n\,2^{\sqrt{2\log n/\log 2}}\,(\log n)^{3/2})$
1969	Knuth	$O(n\,2^{\sqrt{2\log n/\log 2}}\,\log n)$
1971	Schönhage–Strassen	$O(n\log n\log\log n)$
2007	Fürer	$O\left(n\log n\,2^{O(\log^* n)}\right)$
2014	Harvey-Hoeven-Lecerf	$O(n\log n\,8^{\log^* n})$
2019	Harvey-Hoeven	$O(n\log n)$

Integer Multiplication (Summary)



Demonstration of multiplying 1234 × 5678 = 7006652 using fast Fourier transforms (FFTs). Number-theoretic transforms in the integers modulo 337 are used, selecting 85 as an 8th root of unity. Base 10 is used in place of base 2^W for illustrative purposes.

Matrix Multiplication

Multiplying Matrices

Let A be an $n \times m$ matrix, B be an $m \times p$ matrix.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{pmatrix}$$

Then, $C = AB$ is an $n \times p$ matrix

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{np} \end{pmatrix}$$

such that

$$c_{ij} = a_{i1}b_{1j} + \cdots + a_{im}b_{mj} = \sum_{k=1}^{m} a_{ik}b_{kj},$$

Question: Why matrix multiplication is defined in such way?

Multiplying Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} & \circ & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b_{44} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} & \circ & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34}+a_{24}b_{44} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}+a_{34}b_{41} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42} & \circ & a_{31}b_{14}+a_{32}b_{24}+a_{33}b_{34}+a_{34}b_{44} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31}+a_{44}b_{41} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32}+a_{44}b_{42} & \circ & a_{41}b_{14}+a_{42}b_{24}+a_{43}b_{34}+a_{44}b_{44} \end{bmatrix}$$

Multiplying Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} + a_{14}b_{42} & \circ & a_{11}b_{14} + a_{12}b_{24} + a_{13}b_{34} + a_{14}b_{44} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} + a_{24}b_{41} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} + a_{24}b_{42} & \circ & a_{21}b_{14} + a_{22}b_{24} + a_{23}b_{34} + a_{24}b_{44} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} + a_{34}b_{41} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} + a_{34}b_{42} & \circ & a_{31}b_{14} + a_{32}b_{24} + a_{33}b_{34} + a_{34}b_{44} \\ a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} + a_{44}b_{41} & a_{41}b_{12} + a_{42}b_{22} + a_{43}b_{32} + a_{44}b_{42} & \circ & a_{41}b_{14} + a_{42}b_{24} + a_{43}b_{34} + a_{44}b_{44} \end{bmatrix}$$

Multiplying Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \bullet \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} & \circ & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b_{44} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} & \circ & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34}+a_{24}b_{44} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}+a_{34}b_{41} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42} & \circ & a_{31}b_{14}+a_{32}b_{24}+a_{33}b_{34}+a_{34}b_{44} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31}+a_{44}b_{41} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32}+a_{44}b_{42} & \circ & a_{41}b_{14}+a_{42}b_{24}+a_{43}b_{34}+a_{44}b_{44} \end{bmatrix}$$

$$\begin{bmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{bmatrix}$$

Simple Divide and Conquer

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{pmatrix}$$

- $T(n) = 8T(n/2) + 4\left(\frac{n}{2}\right)^2 = 8T(n/2) + n^2$

So, $T(n) = \Theta\left(n^{\log_2 8}\right) = \Theta(n^3)$

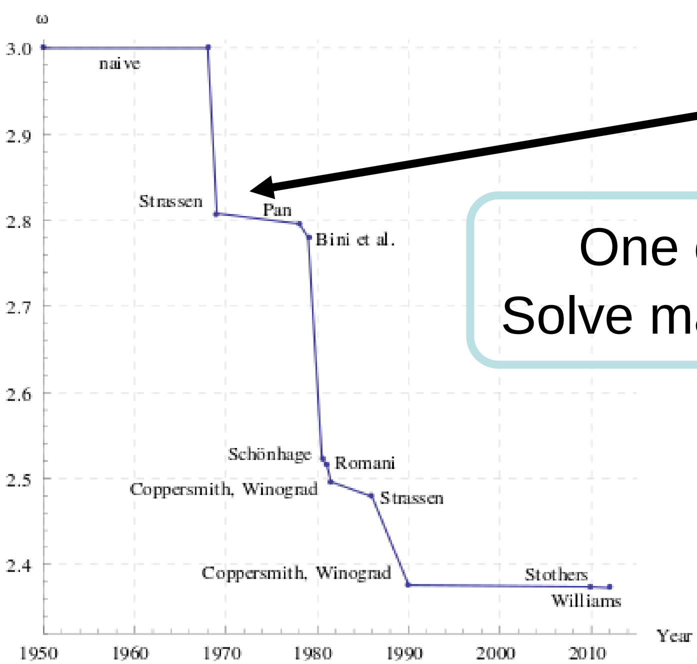
Strassen’s Divide and Conquer Algorithm

- Strassen’s algorithm

  Multiply $\mathbf{2} \times \mathbf{2}$ matrices using $\mathbf{7}$ instead of $\mathbf{8}$ multiplications (and 18 additions)

$$T(n) = 7T\left(\frac{n}{2}\right) + 18n$$

  Hence, we have $T(n) = O\left(n^{\log_2 7}\right)$.



Useful when $n \sim 500$.

One of the most important open problem:
Solve matrix multiplication in $O(n^2 \log^{O(1)} n)$ time

Strassen's Divide and Conquer Algorithm

Naive

$$\mathbf{C}_{1,1} = \mathbf{A}_{1,1}\mathbf{B}_{1,1} + \mathbf{A}_{1,2}\mathbf{B}_{2,1}$$
$$\mathbf{C}_{1,2} = \mathbf{A}_{1,1}\mathbf{B}_{1,2} + \mathbf{A}_{1,2}\mathbf{B}_{2,2}$$
$$\mathbf{C}_{2,1} = \mathbf{A}_{2,1}\mathbf{B}_{1,1} + \mathbf{A}_{2,2}\mathbf{B}_{2,1}$$
$$\mathbf{C}_{2,2} = \mathbf{A}_{2,1}\mathbf{B}_{1,2} + \mathbf{A}_{2,2}\mathbf{B}_{2,2}$$

Strassen

$$\mathbf{M}_1 := (\mathbf{A}_{1,1} + \mathbf{A}_{2,2})(\mathbf{B}_{1,1} + \mathbf{B}_{2,2})$$
$$\mathbf{M}_2 := (\mathbf{A}_{2,1} + \mathbf{A}_{2,2})\mathbf{B}_{1,1}$$
$$\mathbf{M}_3 := \mathbf{A}_{1,1}(\mathbf{B}_{1,2} - \mathbf{B}_{2,2})$$
$$\mathbf{M}_4 := \mathbf{A}_{2,2}(\mathbf{B}_{2,1} - \mathbf{B}_{1,1})$$
$$\mathbf{M}_5 := (\mathbf{A}_{1,1} + \mathbf{A}_{1,2})\mathbf{B}_{2,2}$$
$$\mathbf{M}_6 := (\mathbf{A}_{2,1} - \mathbf{A}_{1,1})(\mathbf{B}_{1,1} + \mathbf{B}_{1,2})$$
$$\mathbf{M}_7 := (\mathbf{A}_{1,2} - \mathbf{A}_{2,2})(\mathbf{B}_{2,1} + \mathbf{B}_{2,2})$$

$$\mathbf{C}_{1,1} = \mathbf{M}_1 + \mathbf{M}_4 - \mathbf{M}_5 + \mathbf{M}_7$$
$$\mathbf{C}_{1,2} = \mathbf{M}_3 + \mathbf{M}_5$$
$$\mathbf{C}_{2,1} = \mathbf{M}_2 + \mathbf{M}_4$$
$$\mathbf{C}_{2,2} = \mathbf{M}_1 - \mathbf{M}_2 + \mathbf{M}_3 + \mathbf{M}_6$$

How did Strassen come up with his matrix multiplication method?
Stackexchange: I've been told no-one really knows, anything would be mainly speculation.