CSE 421

NP-Completeness

Yin Tat Lee

Computational Complexity

Goal: Classify problems according to the amount of computational resources used by the best algorithms that solve them

Here we focus on time complexity

Recall: worst-case running time of an algorithm

- **max** # steps algorithm takes on any input of size **n**

Relative Complexity of Problems

- Want a notion that allows us to compare the complexity of problems
- Want to be able to make statements of the form

"If we could solve problem **B** in polynomial time then we can solve problem **A** in polynomial time"

"Problem **B** is at least as hard as problem **A**"

Polynomial Time Reduction

Def $A \leq_P B$: if there is an algorithm for problem A using a 'black box' (subroutine) that solve problem B s.t.,

- Algorithm uses only a polynomial number of steps
- Makes only a polynomial number of calls to a subroutine for **B**

So,

B is Polynomial time solvable → A is Polynomial time solvable

Conversely,

No efficient Algorithm for A → No efficient Algorithm for B

In words, B is as hard as A (it can be even harder)

$\leq_p^1$ Reductions

Here, we often use a restricted form of polynomial-time reduction often called Karp reduction.

$A \leq_p^1 B$: if and only if there is an algorithm for A given a black box solving B that on input x

- Runs for polynomial time computing an input f(x) of B
- Makes one call to the black box for B for input f(x)
- Returns the answer that the black box gave

We say that the function f(.) is the reduction

Let A = bipartite matching. Let B = maxflow.

$$A \leq_p B$$

$$B \leq_p A$$

$$A \leq_p^1 B$$

$$B \leq_p^1 A$$

Total Results: 0

Answer

Let A = bipartite matching. Let B = maxflow.

We know how to solve bipartite matching by calling maxflow once.

So, it may look like the answer is

Both $A \leq_p B$ and $A \leq_p^1 B$.

However, since both problems can be solved in polynomial time, one valid reduction would be simply doing nothing.

Hence, all statements are true. So, $\leq_p$ is mainly to distinguish if a problem is in P or not.

Fine-Grained Complexity

There are recent work on distinguishing different polytime.

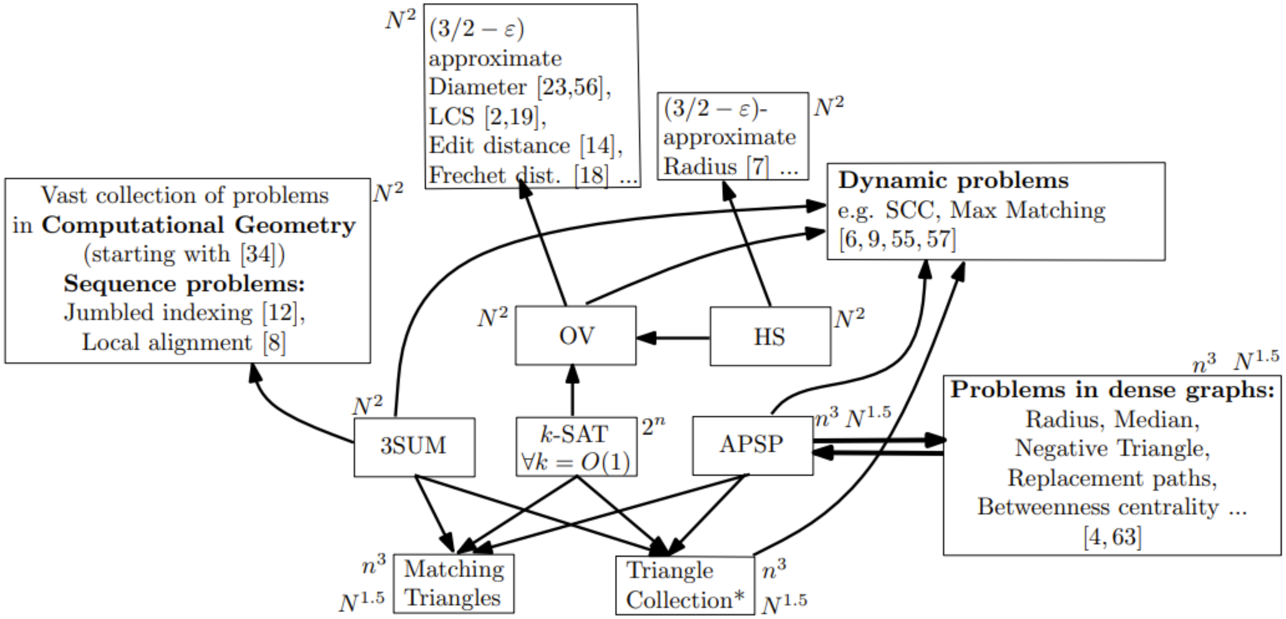


Figure 1 Partial summary of the implications of the main conjectures. An arrow from problem A to problem B, where A has $a(n)$ next to it, B has $b(n)$ next to it, implies that $A \leq_{a,b} B$. When the inputs are graphs, n stands for the number of nodes. N always stands for the total input size. When both n and N are present for a problem, we assume that $N = n^2$; the meaning is that the reductions are only for dense graphs in which case the input size is quadratic in n. For k-SAT, n denotes the number of variables. For the dynamic problems, different key problems can be reduced to different key problems, and the update/query time tradeoffs vary. References are not comprehensive.

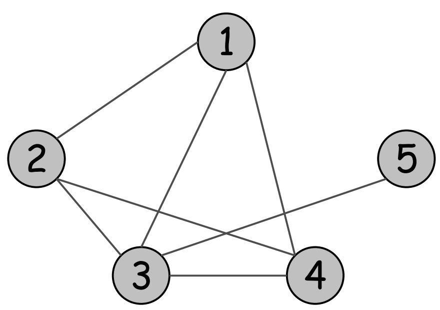
Example 1: Indep Set $\leq_p$ Clique

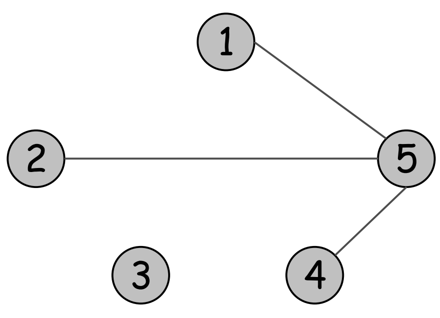
Indep Set: Given G=(V,E) and an integer k, is there $S \subseteq V$ s.t. $|S| \geq k$ and no two vertices in S are joined by an edge?

Clique: Given G=(V,E) and an integer k, is there $S \subseteq V$, |U| ≥ k s.t., every pair of vertices in S is joined by an edge?

Claim: Indep Set $\leq_p$ Clique

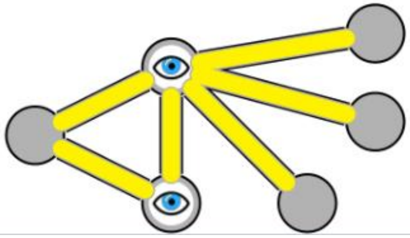
Pf: Given $G = (V, E)$ and instance of indep Set. Construct a new graph $G' = (V, E')$ where $\{u, v\} \in E'$ if and only if $\{u, v\} \notin E$.





S is an indep set in G ⟺ S is a clique in G'

Example 2: Vertex Cover $\leq_p$ Indep Set



Vertex Cover: Given G=(V,E) and an integer k, is there a vertex cover of size at most k?

Claim: For any graph $G = (V, E)$, S is an independent set iff $V - S$ is a vertex cover

Pf: =>

Let S be an independent set of G

Then, S has at most one endpoint of every edge of G

So, $V - S$ has at least one endpoint of every edge of G

So, $V - S$ is a vertex cover.

<= Suppose $V - S$ is a vertex cover

Then, there is no edge between vertices of S (otherwise, $V - S$ is not a vertex cover)

So, S is an independent set.

Example 3: Vertex Cover $\leq_p$ Set Cover

Set Cover: Given a set U, collection of subsets $S_1, \ldots, S_m$ of U and an integer k, is there a collection of k sets that contain all elements of U?

Claim: Vertex Cover $\leq_p$ Set Cover

Pf:

Given $(G = (V, E), k)$ of vertex cover we construct a set cover input $f(G, k)$

- $U = E$
- For each $v \in V$ we create a set S_v of all edges connected to v

This clearly is a polynomial-time reduction

So, we need to prove it gives the right answer

Example 3: Vertex Cover $\leq_p$ Set Cover

Claim: Vertex Cover $\leq_p$ Set Cover

Pf: Given $(G = (V, E), k)$ of vertex cover we construct a set cover input $f(G, k)$

- $U = E$
- For each $v \in V$ we create a set S_v of all edges connected to v

Vertex-Cover (G,k) is yes => Set-Cover f(G,k) is yes

If a set $W \subseteq V$ covers all edges, just choose S_v for all $v \in W$, it covers all U.

Set-Cover f(G,k) is yes => Vertex-Cover (G,k) is yes

If $(S_{v_1}, \ldots, S_{v_k})$ covers all U, the set $\{v_1, \ldots, v_k\}$ covers all edges of G.

Decision Problems

A decision problem is a computational problem where the answer is just yes/no

Here, we study computational complexity of decision Problems.

Why?

- Simpler to deal with
- Decision version is not harder than Search version, so it gives a lower bound for Decision version
- usually, you can use decider multiple times to find an answer.

Polynomial Time

Define P (polynomial-time) to be the set of all decision problems solvable by algorithms whose worst-case running time is bounded by some polynomial in the input size.

Do we understand P?

- We can prove that a problem is in P by exhibiting a polynomial time algorithm
- It is in most cases very hard to prove a problem is not in P.

Beyond P?

We have seen many problems that seem hard

- Independent Set
- 3-coloring
- Vertex Cover
- 3-SAT

The independent set S

The 3-coloring

The vertex cover S

The T/F assignment

Given a formula $(x_1 \vee \overline{x_2} \vee x_9) \wedge (\overline{x_2} \vee x_3 \vee x_7) \wedge \cdots$, is there a satisfying assignment?

Common Property: If the answer is yes, there is a "short" proof (a.k.a., certificate), that allows you to verify (in polynomial-time) that the answer is yes.

- The proof may be hard to find

Decision Problems

A decision problem is a computational problem where the answer is just yes/no.

We can define a problem by a set $X \subset \{0,1\}^n$.
The answer for the input s is YES iff $s \in X$.

Certifier: Algorithm C(s, t) is a certifier for problem A if
$s \in X$ if and only if (There is a t such that $C(s,t) = YES$))

NP: Set of all decision problems for which there exists a poly-time certifier.

Co-NP: $X \in co - NP$ if and only if $\overline{X} \in NP$.

Example: 3SAT is in NP

Given a 3-CNF formula, is there a satisfying assignment?

(conjunctive normal form (CNF) is AND of ORs)

Certificate: An assignment of truth values to the n boolean variables.

Verifier: Check that each clause has at least one true literal.

Ex: $(x_1 \vee \overline{x_3} \vee x_4) \wedge (x_2 \vee \overline{x_4} \vee x_3) \wedge (x_2 \vee \overline{x_1} \vee x_3)$

Certificate: $x_1 = T, x_2 = F, x_3 = T, x_4 = F$

Conclusion: 3-SAT is in NP

Question: Is Maxflow is in NP?

Decision problem: Is the maximum flow value = k?

Answer 1:

Certificate: A flow f and a cut $(S, \overline{S})$

Verifier: Check if $val(f) = cap(S, \overline{S})$

Answer 2:

Certificate: None

Verifier: Any polynomial time maxflow algo.

What do we know about NP?

- Nobody knows if all problems in NP can be done in polynomial time, i.e. does P=NP?
  - one of the most important open questions in all of science.
  - Huge practical implications specially if answer is yes
- Every problem in P is in NP

  one doesn't even need a certificate for problems in **P** so just ignore any hint you are given
- Every problem in NP is in exponential time
- Some problems in NP seem really hard
  - nobody knows how to prove that they are really hard to solve, i.e. $P \neq NP$

NP Completeness

Complexity Theorists Approach: We don’t know how to prove any problem in NP is hard. So, let’s find hardest problems in NP.

NP-hard: A problem B is NP-hard iff for any problem $A \in NP$, we have $A \leq_p B$

NP-Completeness: A problem B is NP-complete iff B is NP-hard and $B \in NP$.

Motivations:

- If $P \neq NP$, then every NP-Complete problems is not in P. So, we shouldn’t try to design Polytime algorithms
- To show $P = NP$, it is enough to design a polynomial time algorithm for just one NP-complete problem.

Cook-Levin Theorem

Theorem (Cook 71, Levin 73): 3-SAT is NP-complete, i.e., for all problems $A \in NP$, $A \leq_p$ 3-SAT.

Pf (Draft. Take CSE 431 for more.):

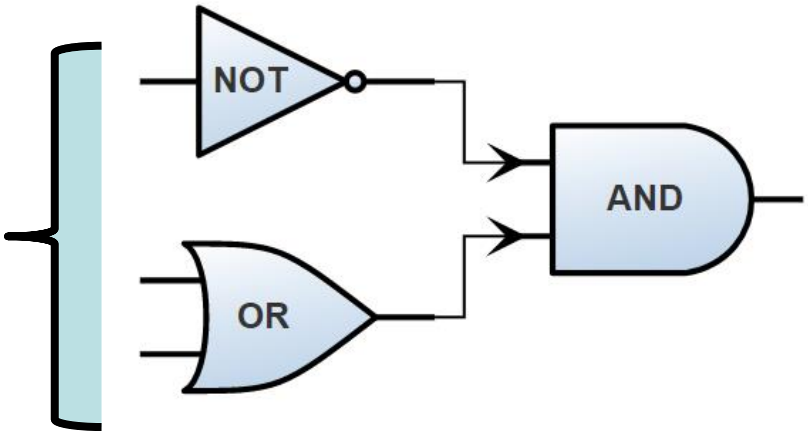
Since $A \in NP$, there is a polytime certifier C such that

$$s \in A \text{ iff } C(s,t) = 1 \text{ for some } t$$

To solve the problem A, it suffices to find t.

Since C is polytime, we can

- convert C to a poly size circuit (of AND OR NOT)
- Some input are the given s.
- Some input are t.
- Our goal is to find t to make the output is TRUE.



Cook-Levin Theorem

Theorem (Cook 71, Levin 73): 3-SAT is NP-complete, i.e., for all problems $A \in NP$, $A \leq_p$ 3-SAT.

Pf (Draft. Take CSE 431 for more.):

To find an input such that output is true,

we convert the circuit to 3CNF $(x_1 \vee \overline{x_2} \vee x_9) \wedge (\overline{x_2} \vee x_3 \vee x_7) \wedge \cdots$

Example:

- An OR gate with input a,b and output c can be represented by
$$(a \vee b \vee \bar{c}) \wedge (\bar{a} \vee c) \wedge (\bar{b} \vee c)$$
- A NOT gate with input a and output c can be represented by
$$(a \vee c) \wedge (\bar{a} \vee \bar{c})$$

Suppose the circuit gate $C_1, C_2, \cdots, C_q$ with final output Z

Then, the 3CNF is $\overline{C}_1 \wedge \overline{C}_2 \wedge \cdots \wedge \overline{C}_q \wedge Z$ where $\overline{C}_i$ are the 3CNF version of C_i.

Cook-Levin Theorem

Theorem (Cook 71, Levin 73): 3-SAT is NP-complete, i.e., for all problems $A \in NP, A \leq_p$ 3-SAT.

- So, 3-SAT is the hardest problem in NP.

What does this say about other problems of interest? Like Independent set, Vertex Cover, …

Fact: If $A \leq_p B$ and $B \leq_p C$ then, $A \leq_p C$

Pf idea: Just compose the reductions from A to B and B to C

So, if we prove 3-SAT $\leq_p$ Independent set, then Independent Set, Clique, Vertex cover, Set cover are all NP-complete

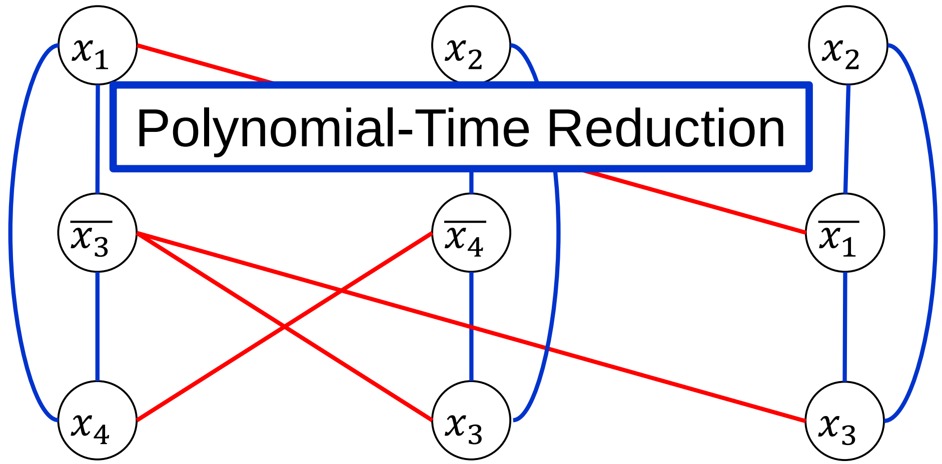
3-SAT $\leq_p$ Independent Set $\leq_p$ Vertex Cover $\leq_p$ Set Cover

3-SAT $\leq_p$ Independent Set

Map a 3-CNF to (G,k). Say m is number of clauses

- Create a vertex for each literal
- Joint two literals if
  - They belong to the same clause (blue edges)
  - The literals are negations, e.g., $x_i, \overline{x_i}$ (red edges)
- Set k be the # of clauses.

$$(x_1 \vee \overline{x_3} \vee x_4) \wedge (x_2 \vee \overline{x_4} \vee x_3) \wedge (x_2 \vee \overline{x_1} \vee x_3)$$



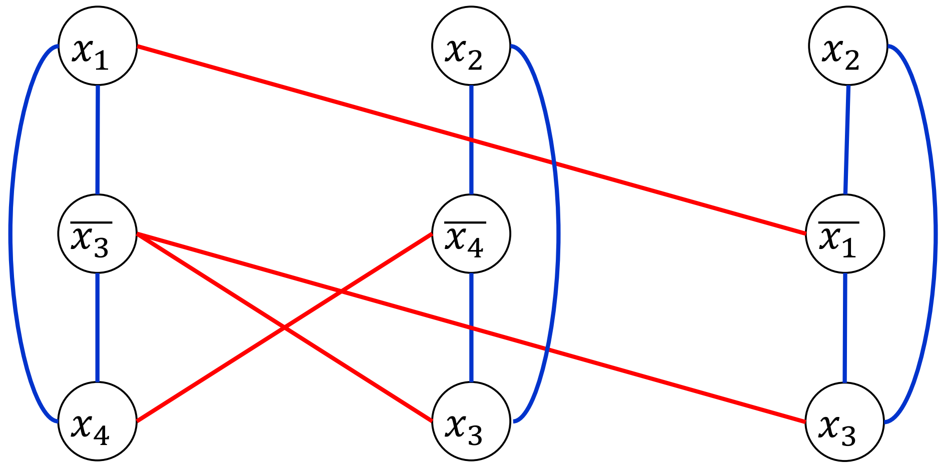
Correctness of 3-SAT $\leq_p$ Indep Set

F satisfiable => An independent of size k

Given a satisfying assignment, Choose one node from each clause where the literal is satisfied

$$(x_1 \vee \overline{x_3} \vee x_4) \wedge (x_2 \vee \overline{x_4} \vee x_3) \wedge (x_2 \vee \overline{x_1} \vee x_3)$$

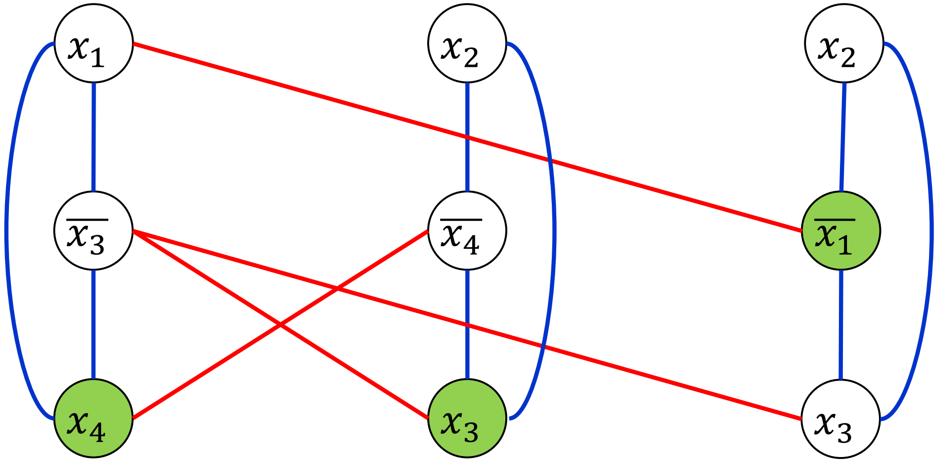
Satisfying assignment: $x_1 = T, x_2 = F, x_3 = T, x_4 = F$



- S has exactly one node per clause => No blue edges between S
- S follows a truth-assignment => No red edges between S
- S has one node per clause => |S|=k

Correctness of 3-SAT $\leq_p$ Indep Set

<u>An independent set of size k => A satisfying assignment</u>
Given an independent set S of size k.
S has exactly one vertex per clause (because of blue edges)
S does not have $x_i, \overline{x_i}$ (because of red edges)
So, S gives a satisfying assignment

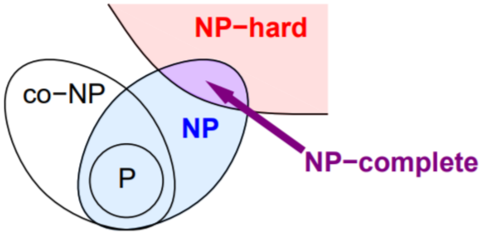


Satisfying assignment: $x_1 = F, x_2 =?, x_3 = T, x_4 = T$

$(x_1 \vee \overline{x_3} \vee x_4) \wedge (x_2 \vee \overline{x_4} \vee x_3) \wedge (x_2 \vee \overline{x_1} \vee x_3)$

Summary

- If a problem is NP-hard it does not mean that all instances are hared, e.g., Vertex-cover has a polynomial-time algorithm in trees
- We learned the crucial idea of polynomial-time reduction. This can be even used in algorithm design, e.g., we know how to solve max-flow so we reduce image segmentation to max-flow
- NP-Complete problems are the hardest problem in NP
- NP-hard problems may not necessarily belong to NP.
- Polynomial-time reductions are transitive relations



More of what we *think* the world looks like.