CSE 421: Introduction to Algorithms

Application of BFS

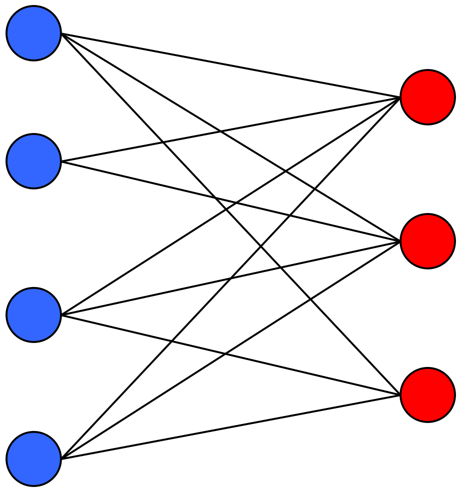
Yin Tat Lee

Bipartite Graphs

Definition: An undirected graph $G = (V, E)$ is bipartite
if you can partition the vertex set into 2 parts (say, blue/red or left/right) so that
all edges join vertices in different parts
i.e., no edge has both ends in the same part.

Application:

- Scheduling: machine=red, jobs=blue
- Stable Matching: men=blue, woman=red



a bipartite graph

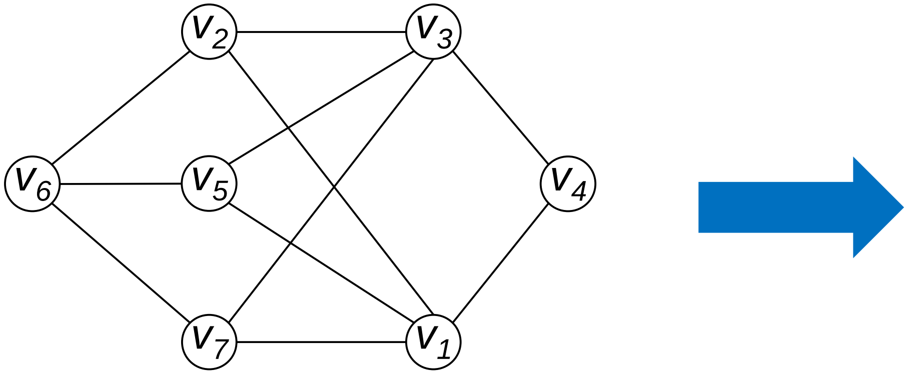
Testing Bipartiteness

Problem: Given a graph G, is it bipartite?

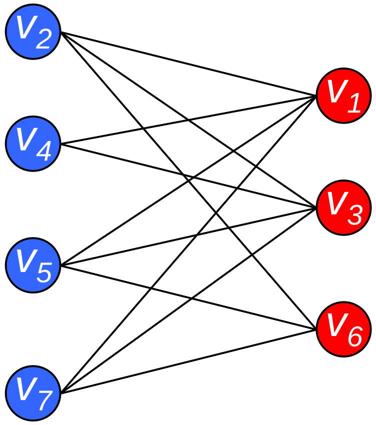
Many graph problems become:

- Easier/Tractable if the underlying graph is bipartite (matching)

Before attempting to design an algorithm, we need to understand structure of bipartite graphs.



a bipartite graph G

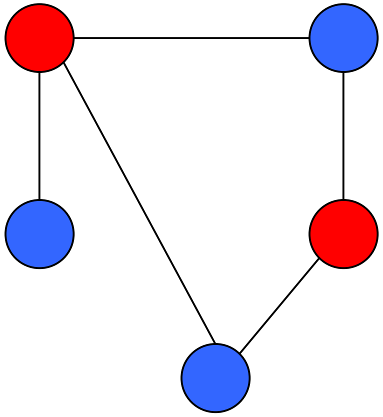


another drawing of G

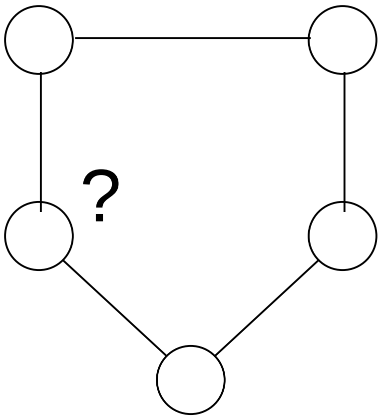
An Obstruction to Bipartiteness

Lemma: If G is bipartite, then it does not contain an odd length cycle.

Proof: We cannot 2-color an odd cycle, let alone G.



bipartite
(2-colorable)



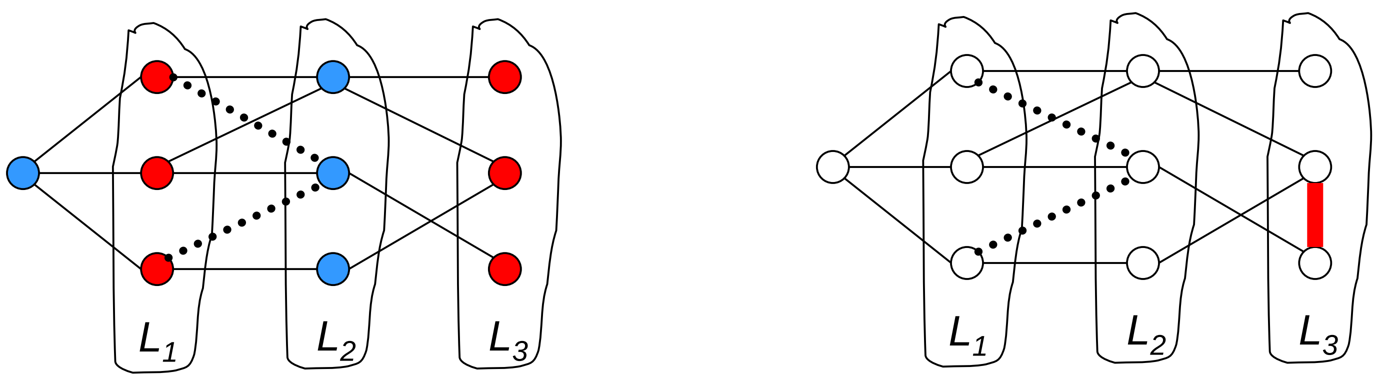
not bipartite
(not 2-colorable)

A Characterization of Bipartite Graphs

Lemma: Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS(s). Exactly one of the following holds.

(i) No edge of G joins two nodes of the same layer, and G is bipartite.

(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).



Case (i) *Case (ii)*

A Characterization of Bipartite Graphs

Lemma: Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS(s). Exactly one of the following holds.

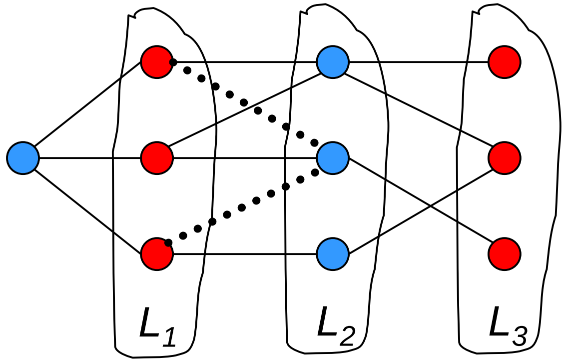
(i) No edge of G joins two nodes of the same layer, and G is bipartite.

(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).

Proof. (i)

Suppose no edge joins two nodes in the same layer.

By previous lemma, all edges join nodes on adjacent levels.



Case (i)

Bipartition:

blue = nodes on odd levels,

red = nodes on even levels.

A Characterization of Bipartite Graphs

Lemma: Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS(s). Exactly one of the following holds.

(i) No edge of G joins two nodes of the same layer, and G is bipartite.

(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).

Proof. (ii)

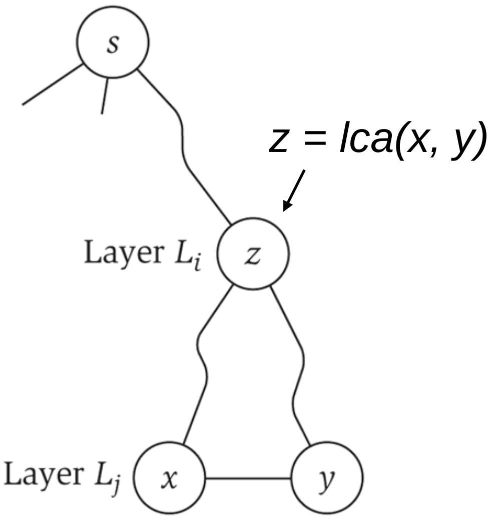
Suppose $\{x, y\}$ is an edge & x, y in same level L_j.

Let $z =$ their lowest common ancestor in BFS tree.

Let L_i be level containing z.

Consider cycle that takes edge from x to y, then tree from y to z, then tree from z to x.

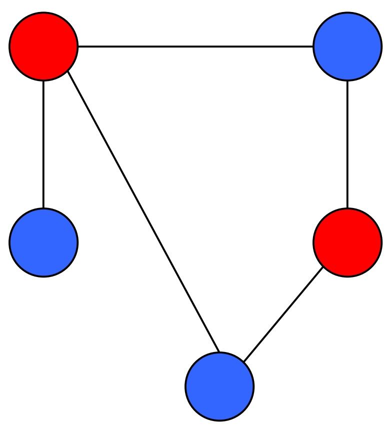
Its length is $1 + (j - i) + (j - i)$, which is odd.



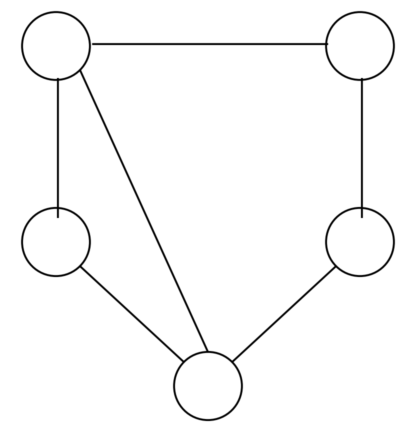
Obstruction to Bipartiteness

Corollary: A graph G is bipartite if and only if it contains no odd length cycles.

Furthermore, one can test bipartiteness using BFS.



bipartite
(2-colorable)



not bipartite
(not 2-colorable)

Summary

- BFS(s) implemented using queue.
- Edges into then-undiscovered vertices define a tree – the "Breadth First spanning tree" of G
- Level i in the tree are exactly all vertices v s.t., the shortest path (in G) from the root s to v is of length i
- All nontree edges join vertices on the same or adjacent layers of the tree
- Applications:
  - Shortest Path
  - Connected component
  - Test bipartiteness / 2-coloring

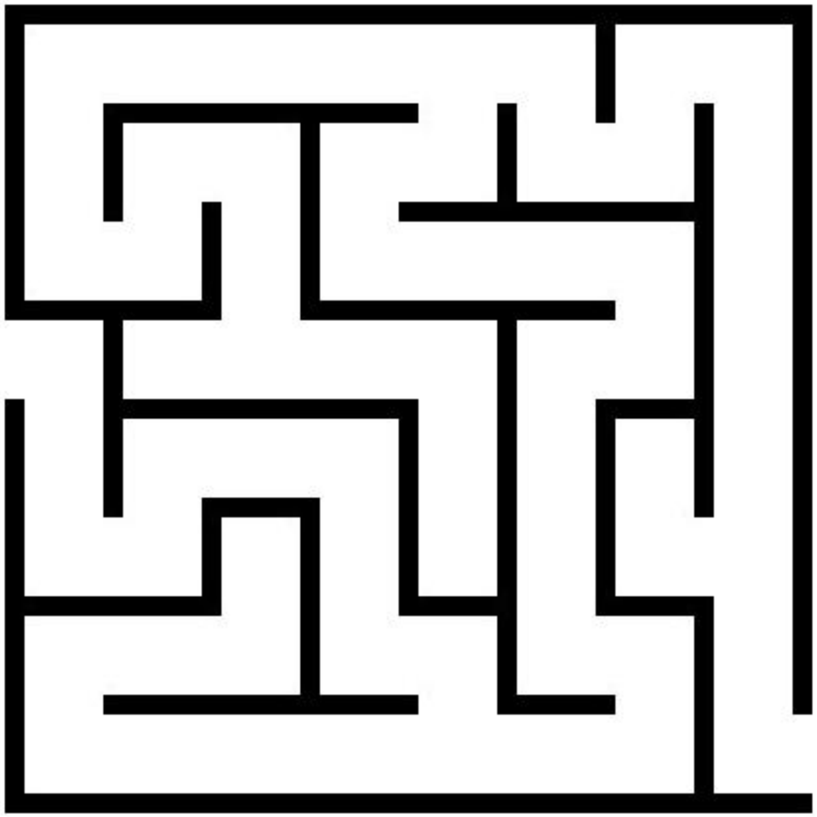
CSE 421

Depth First Search

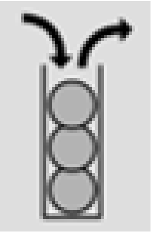
Yin Tat Lee

Depth First Search

Follow the first path you find as far as you can go; back up to last unexplored edge when you reach a dead end, then go as far you can



Naturally implemented using recursive calls or a stack



DFS(s) – Recursive version

Initialization: mark all vertices undiscovered

DFS(v)

Mark v discovered

for each edge $\{v, x\}$

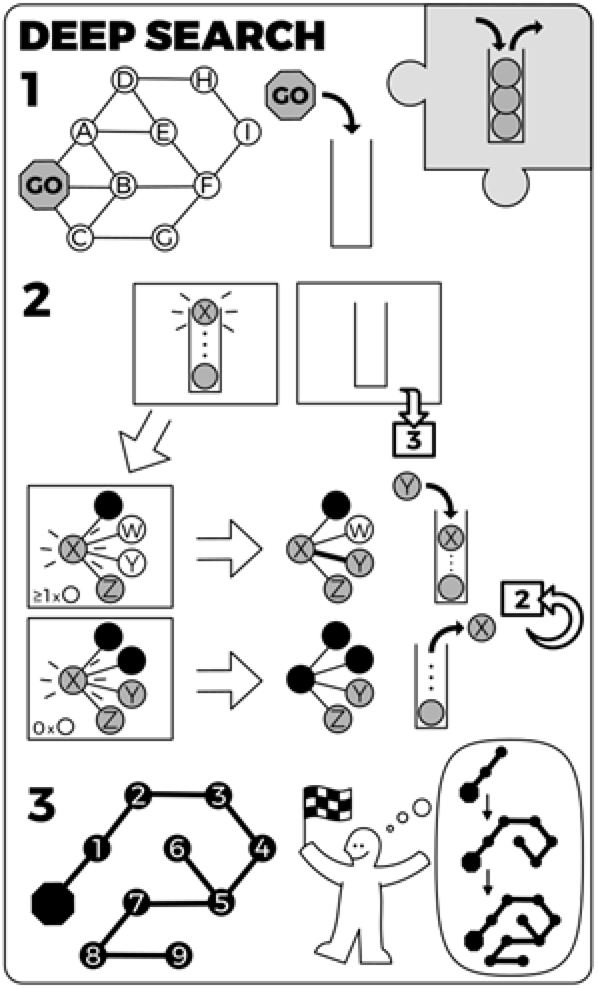
if (x is undiscovered)

Mark x discovered

$x \rightarrow \text{parent} = u$

DFS(x)

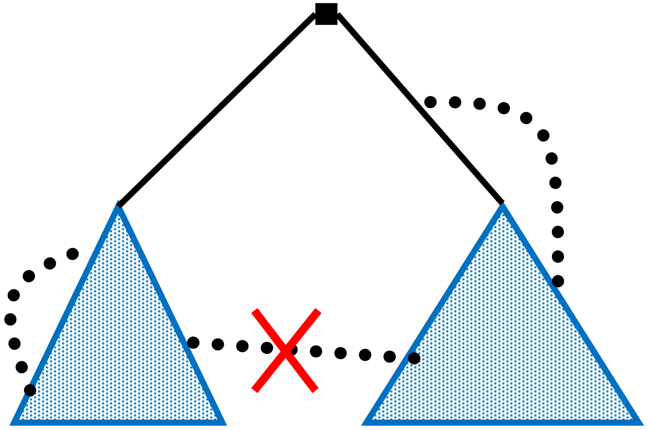
Mark v fully-discovered



Non-Tree Edges in DFS

BFS tree ≠ DFS tree, but, as with BFS, DFS has found a tree in the graph s.t. non-tree edges are "simple" in some way.

All non-tree edges join a vertex and one of its descendants/ancestors in the DFS tree



DFS(A)

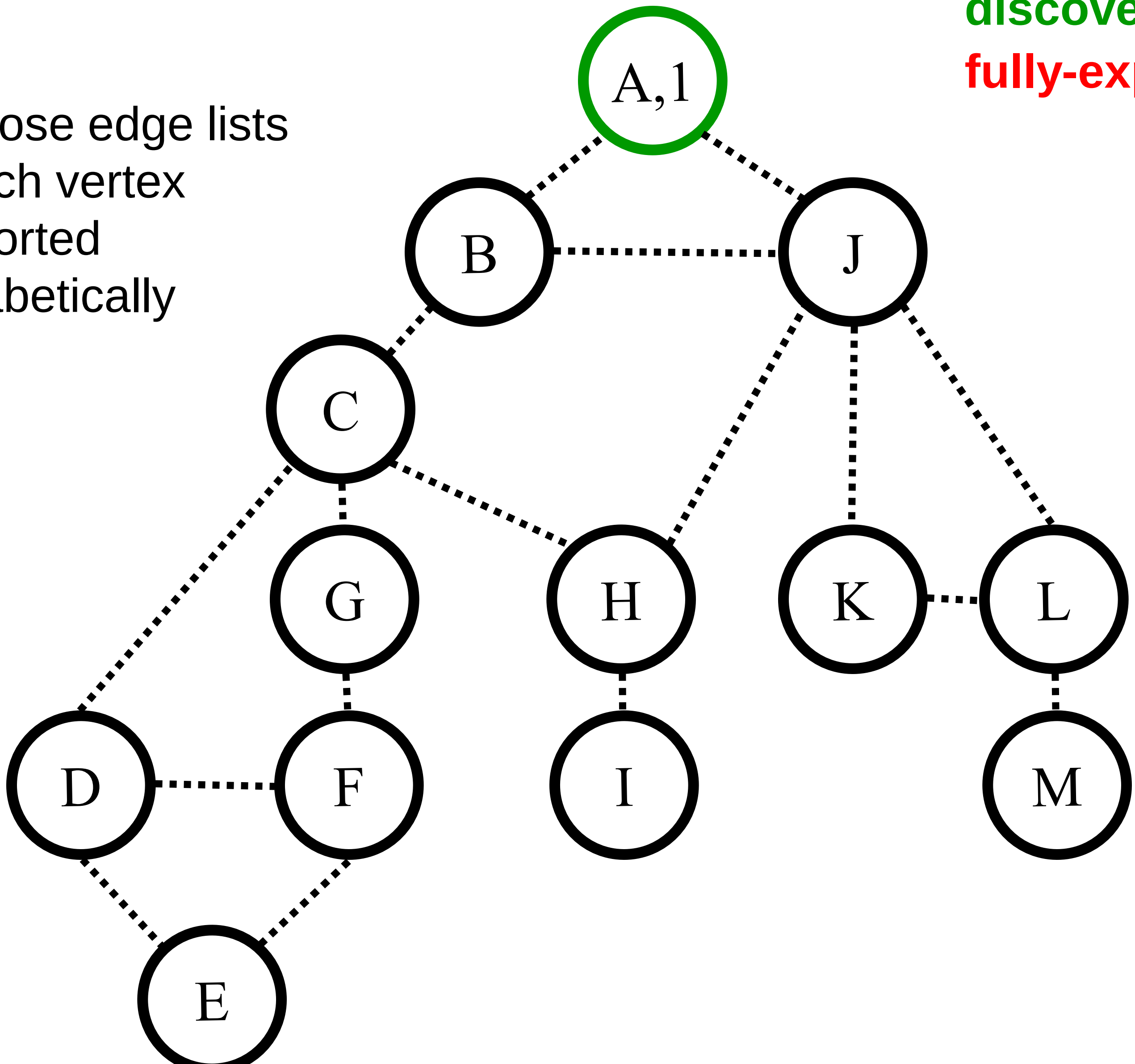
Color code:

undiscovered

discovered

fully-explored

Suppose edge lists at each vertex are sorted alphabetically



Call Stack (Edge list):

A (B,J)

st[] = {1}

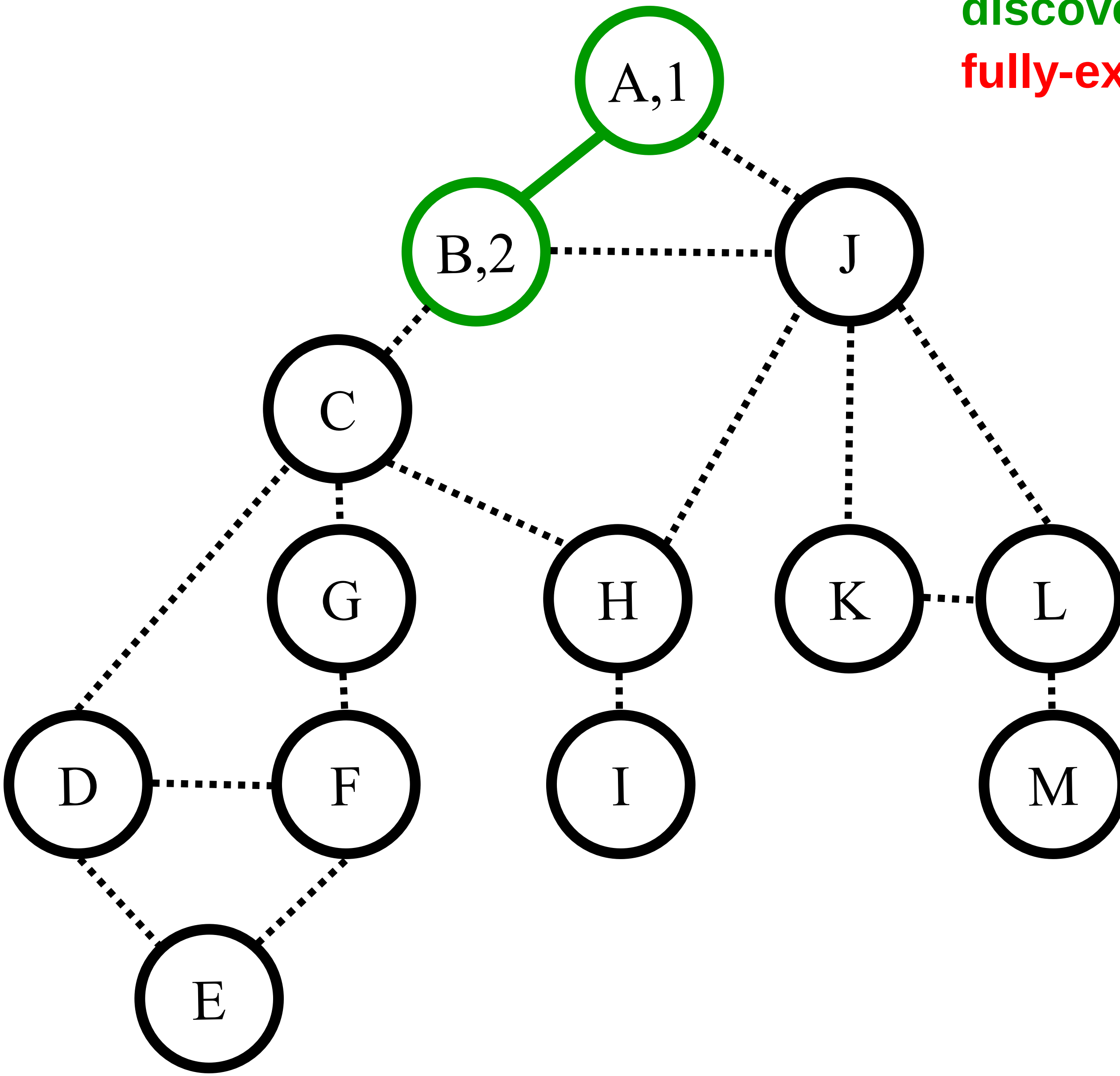
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (A,C,J)

st[] =
{1,2}

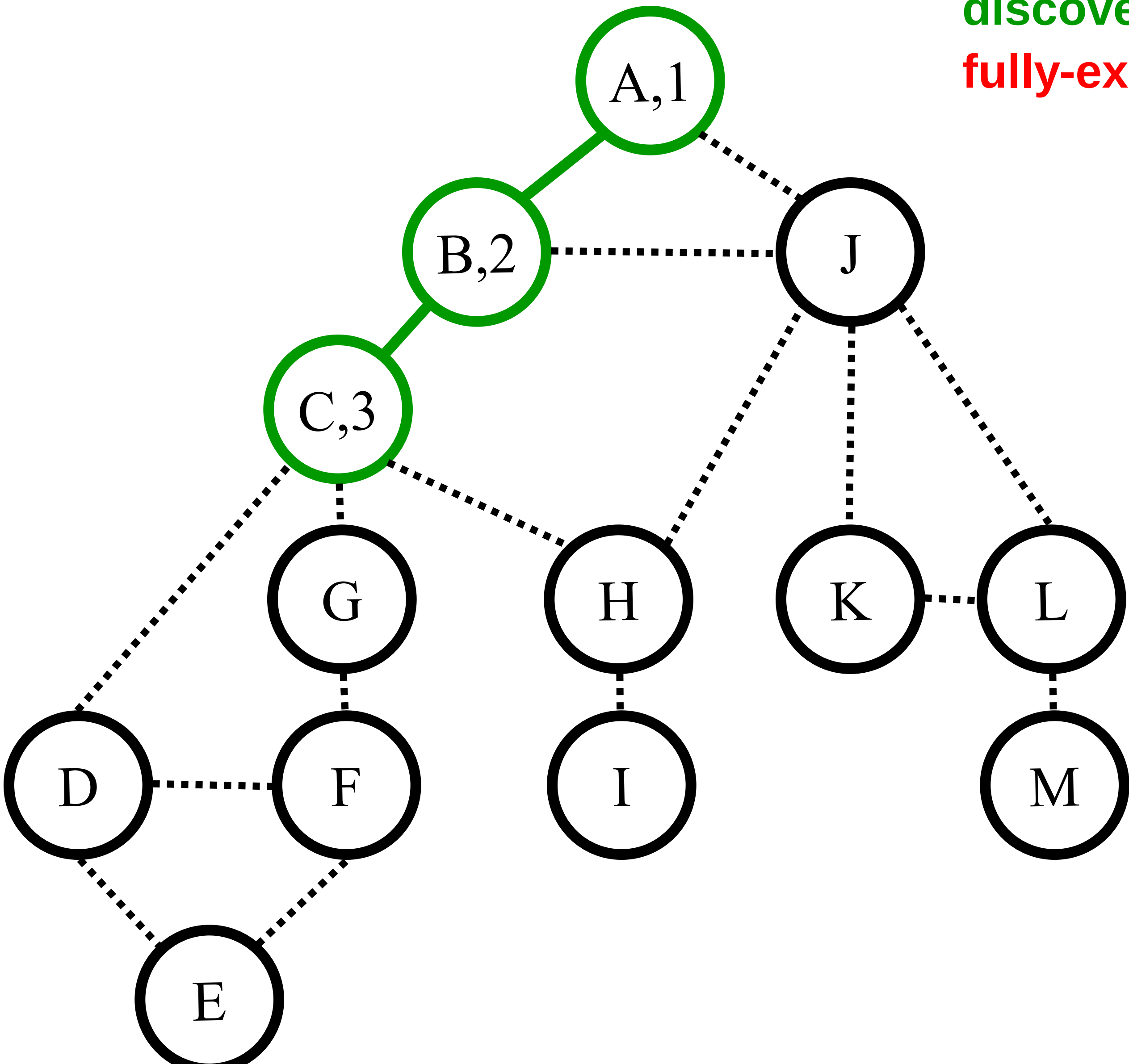
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (B,J)
B (A,C,J)
C (B,D,G,H)

st[] =
{1,2,3}

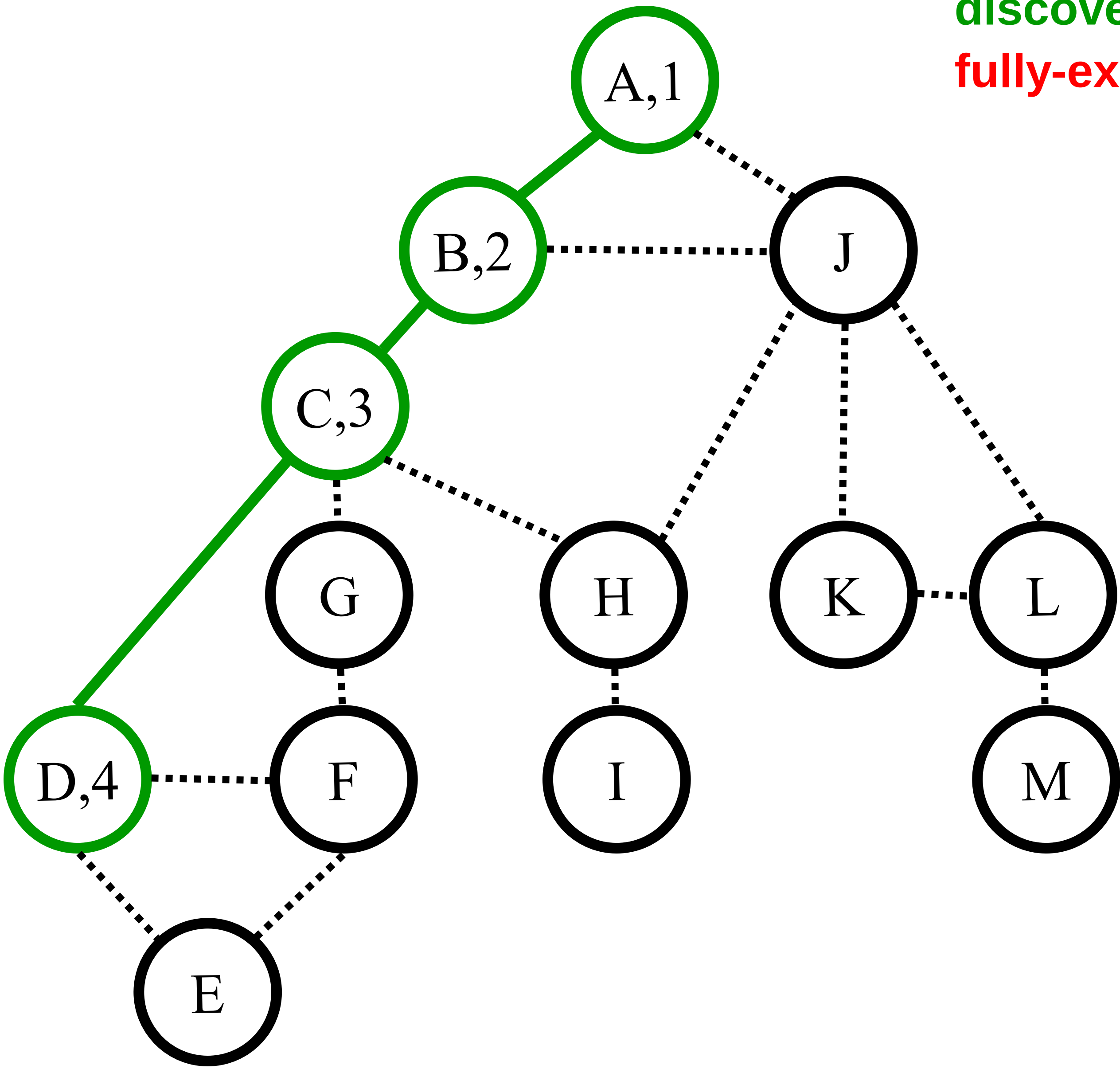
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

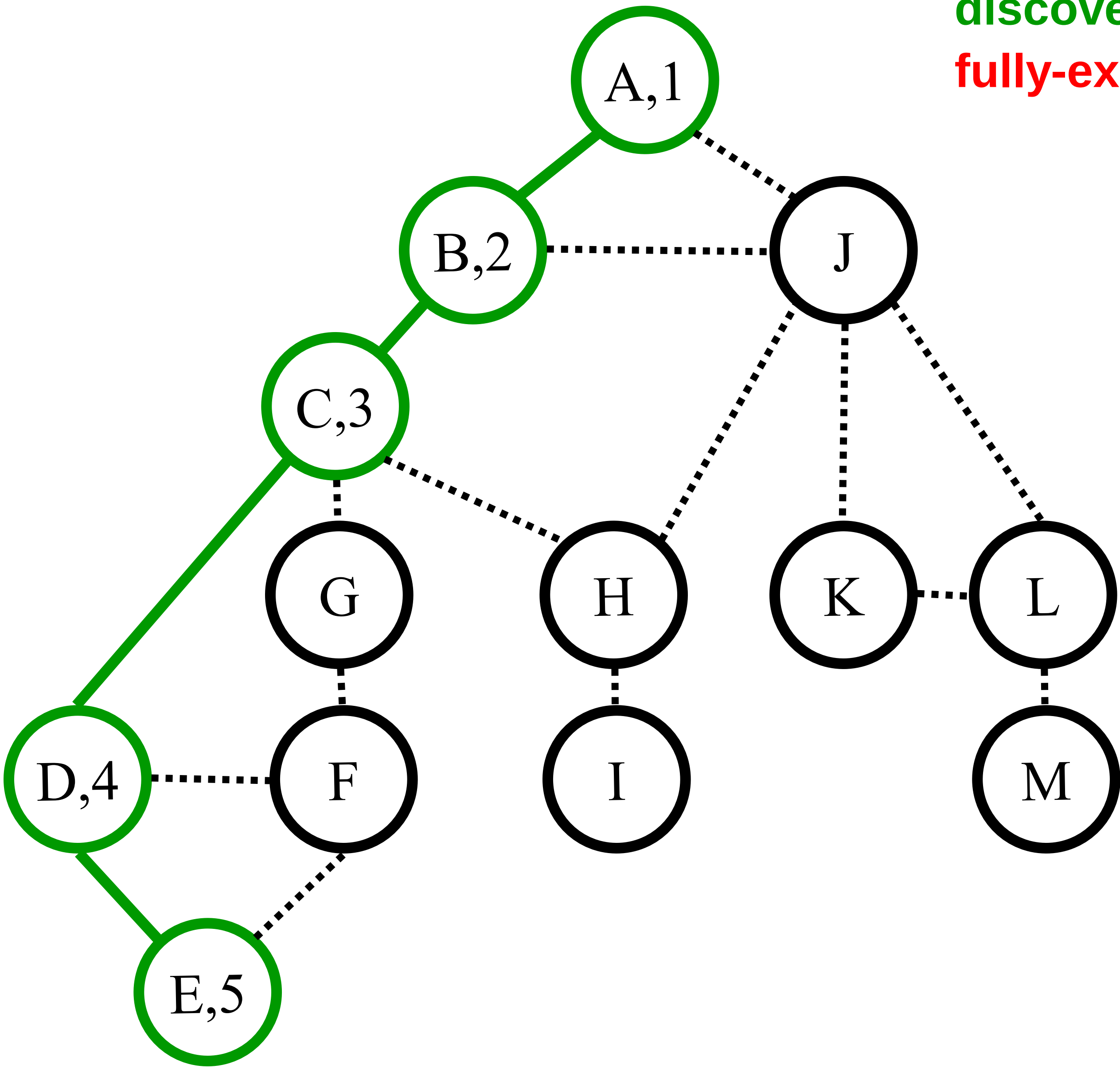
A (B,J)
B (A,C,J)
C (B,D,G,H)
D (C,E,F)

st[] =
{1,2,3,4}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (D,F)

st[] =
{1,2,3,4,5}

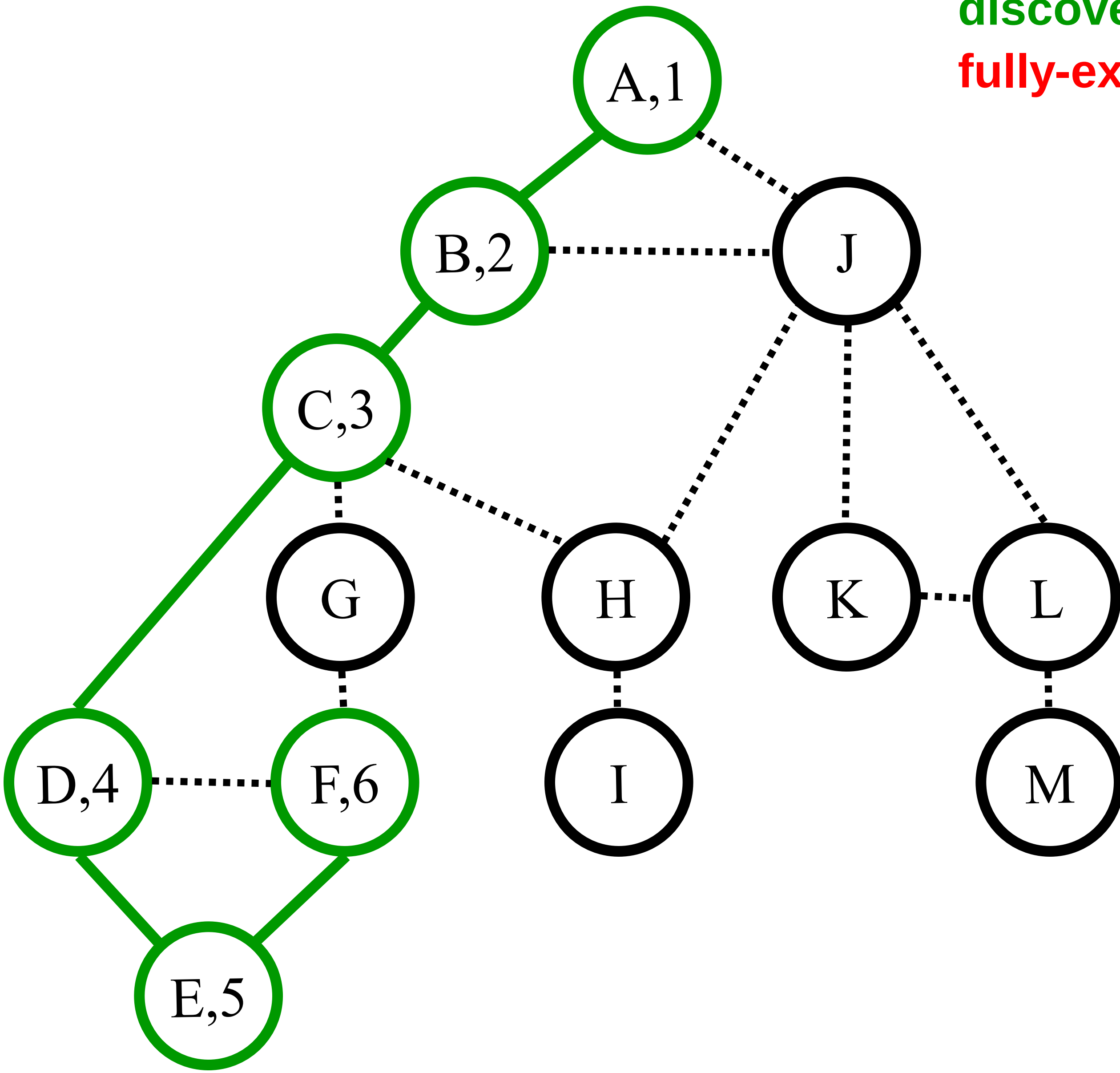
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (D,E,G)

st[] =
{1,2,3,4,5,
6}

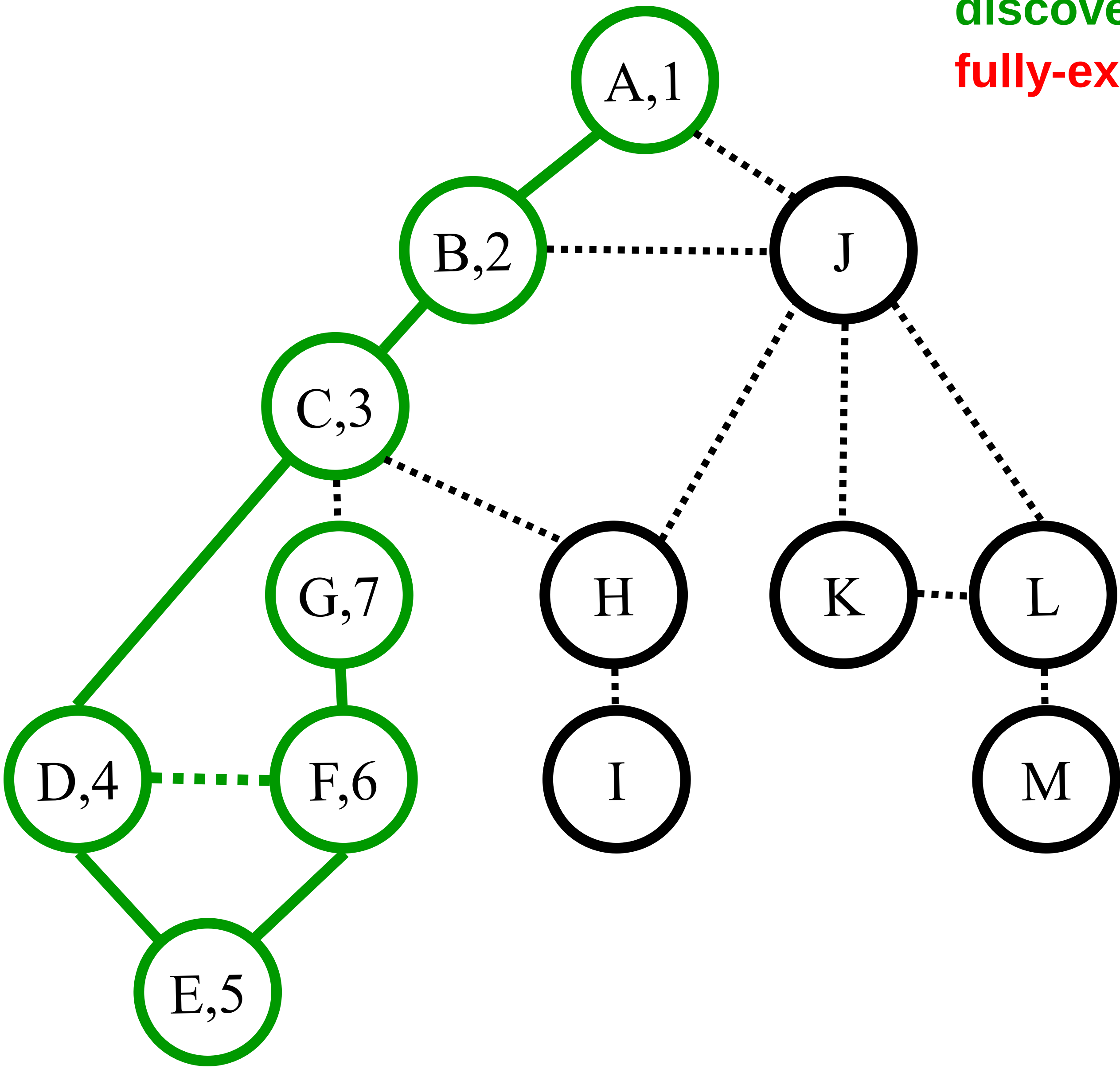
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

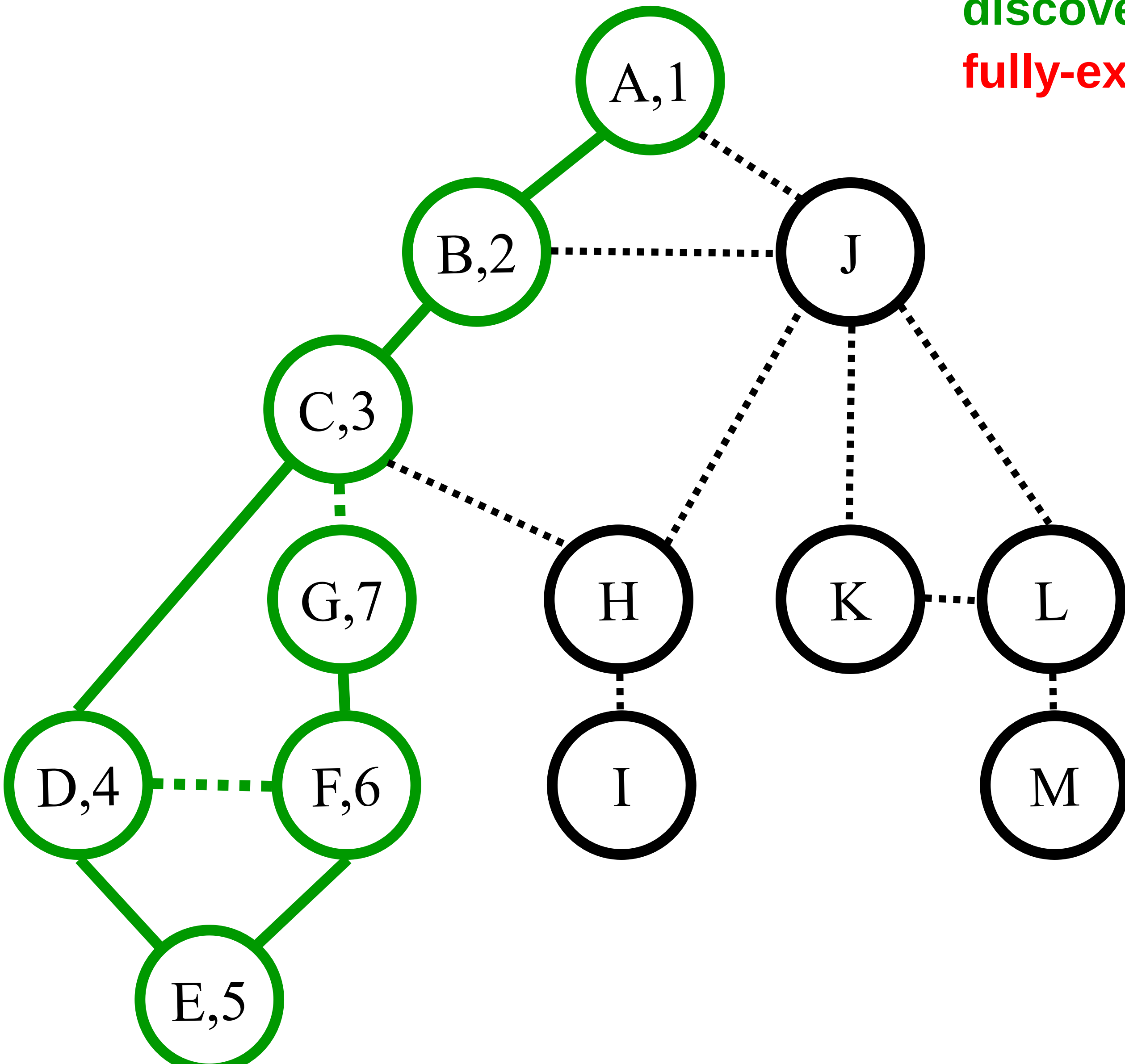
A (B,J)
B (A,C,J)
C (B,D,G,H)
D (C,E,F)
E (D,F)
F (D,E,G)
G (C,F)

st[] =
{1,2,3,4,5,
6,7}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)
G (~~C~~,~~F~~)

st[] =
{1,2,3,4,5,
6,7}

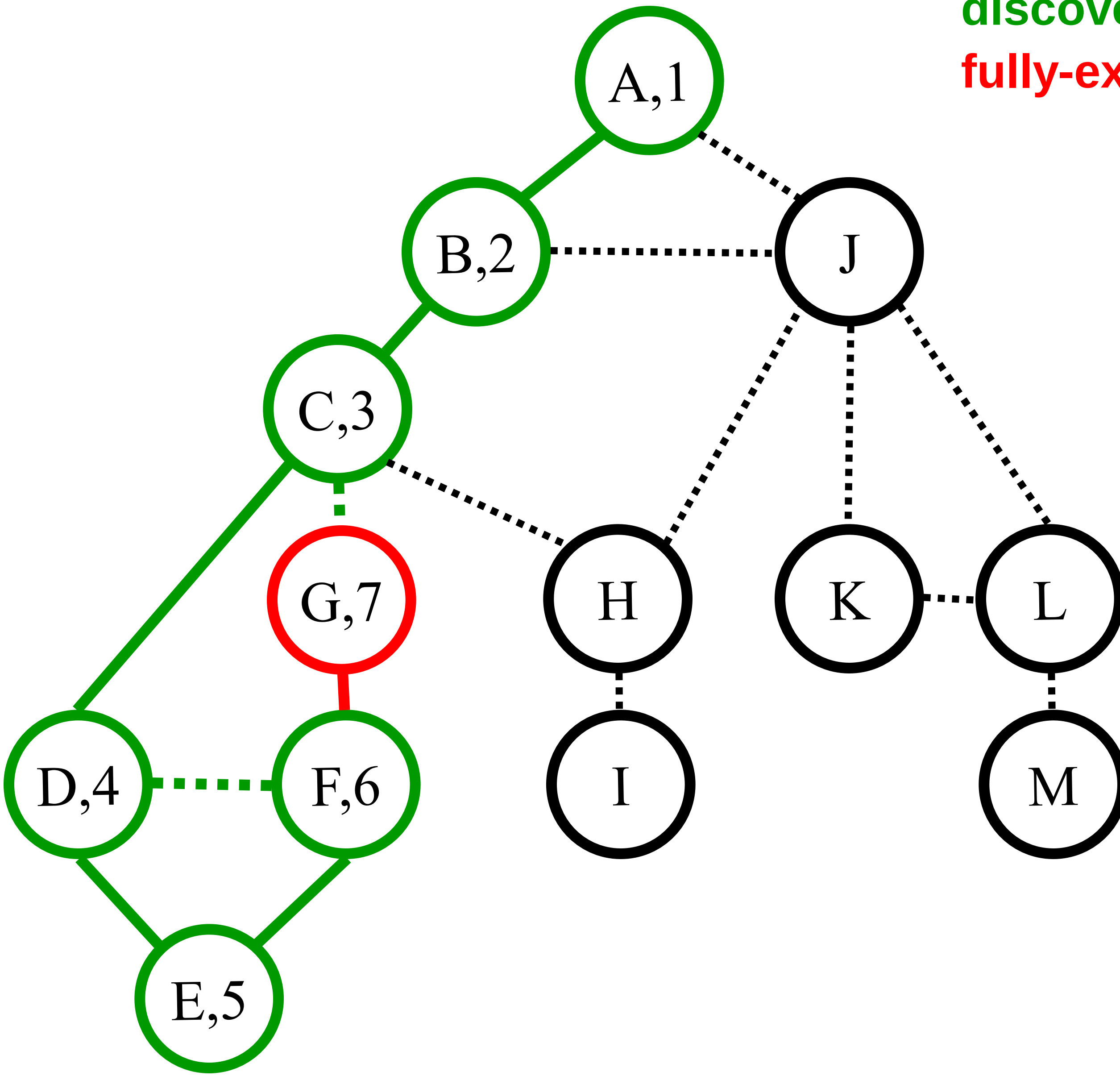
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)

st[] = {1,2,3,4,5, 6}

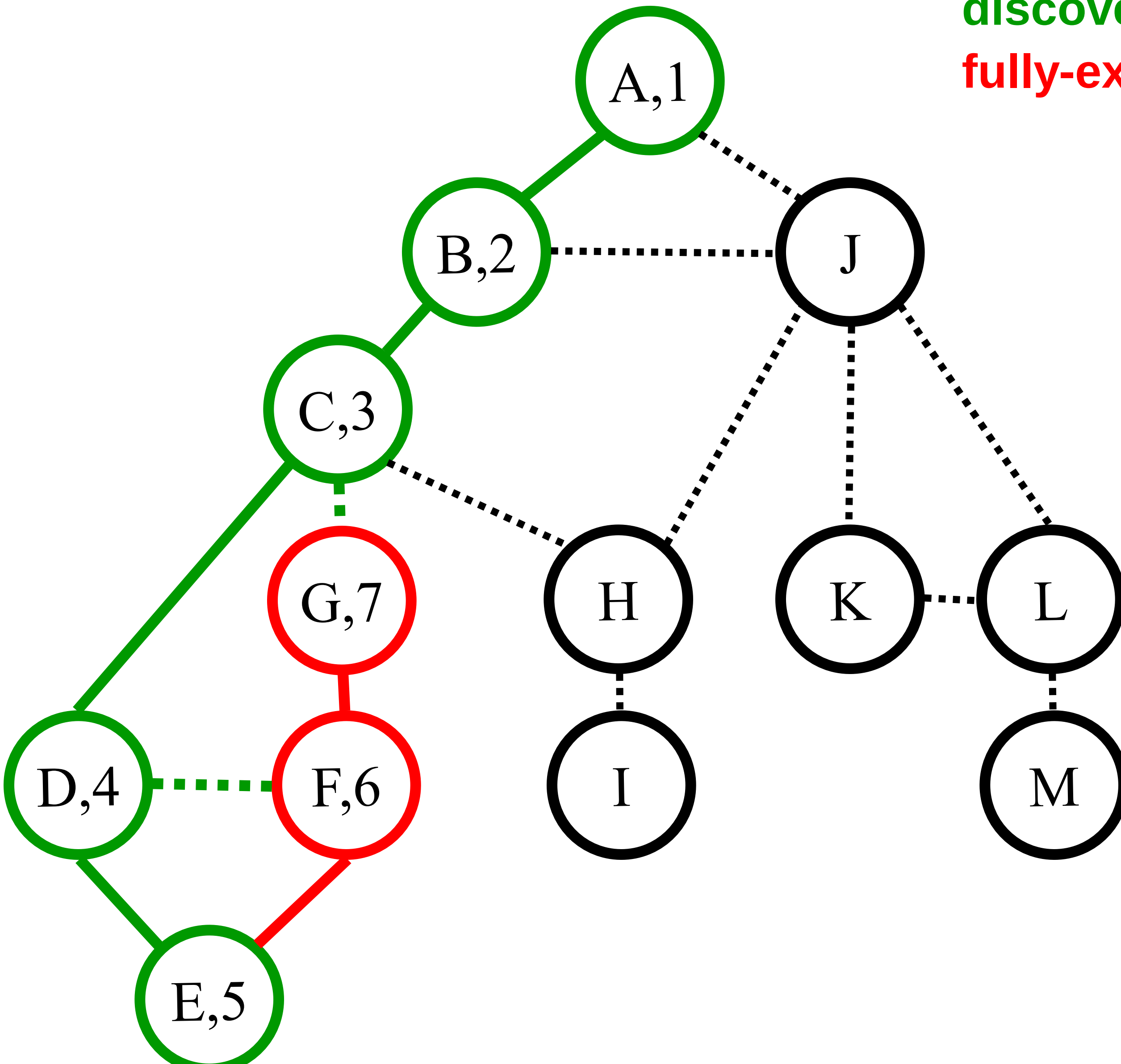
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)

st[] =
{1,2,3,4,5}

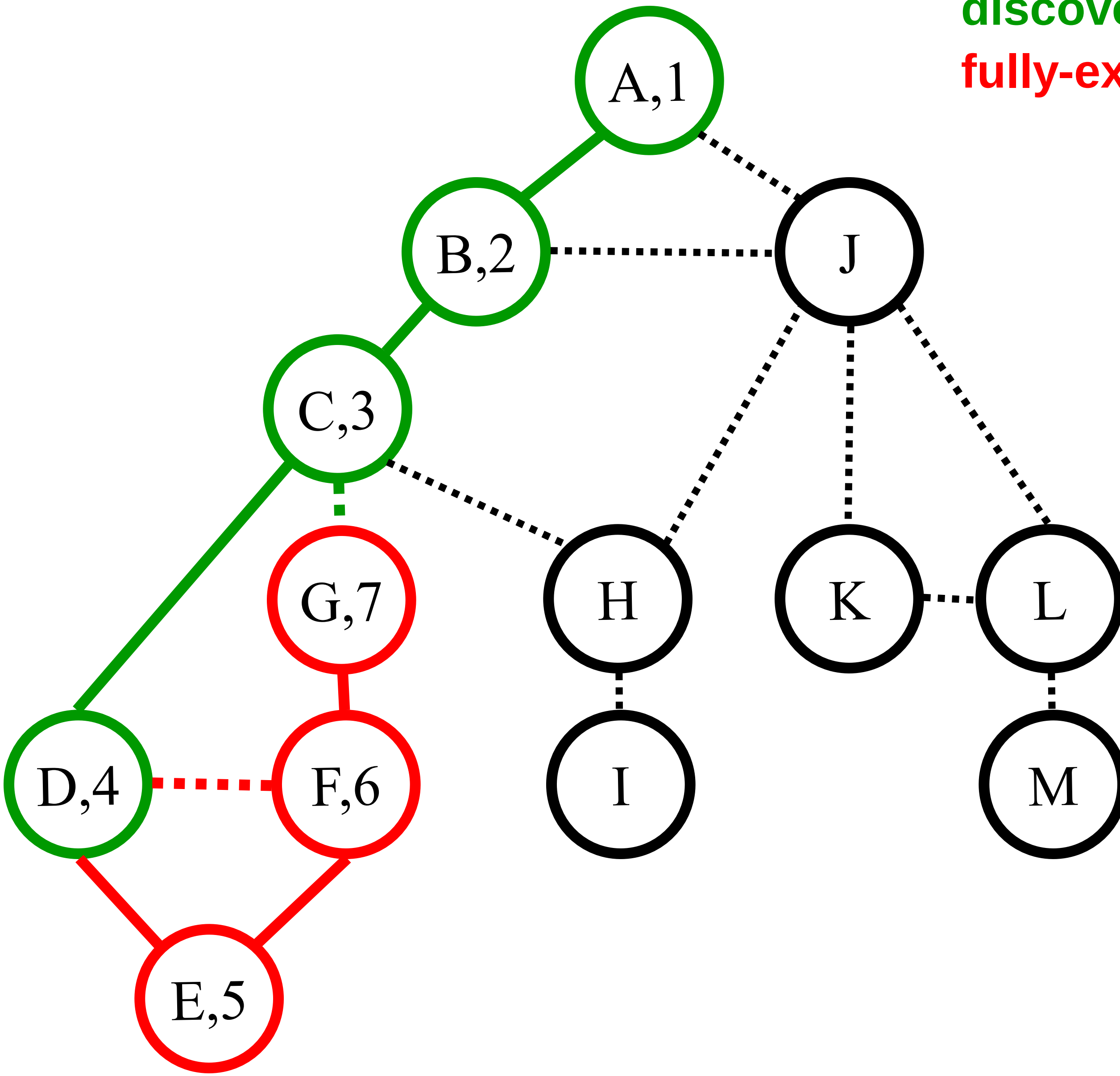
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

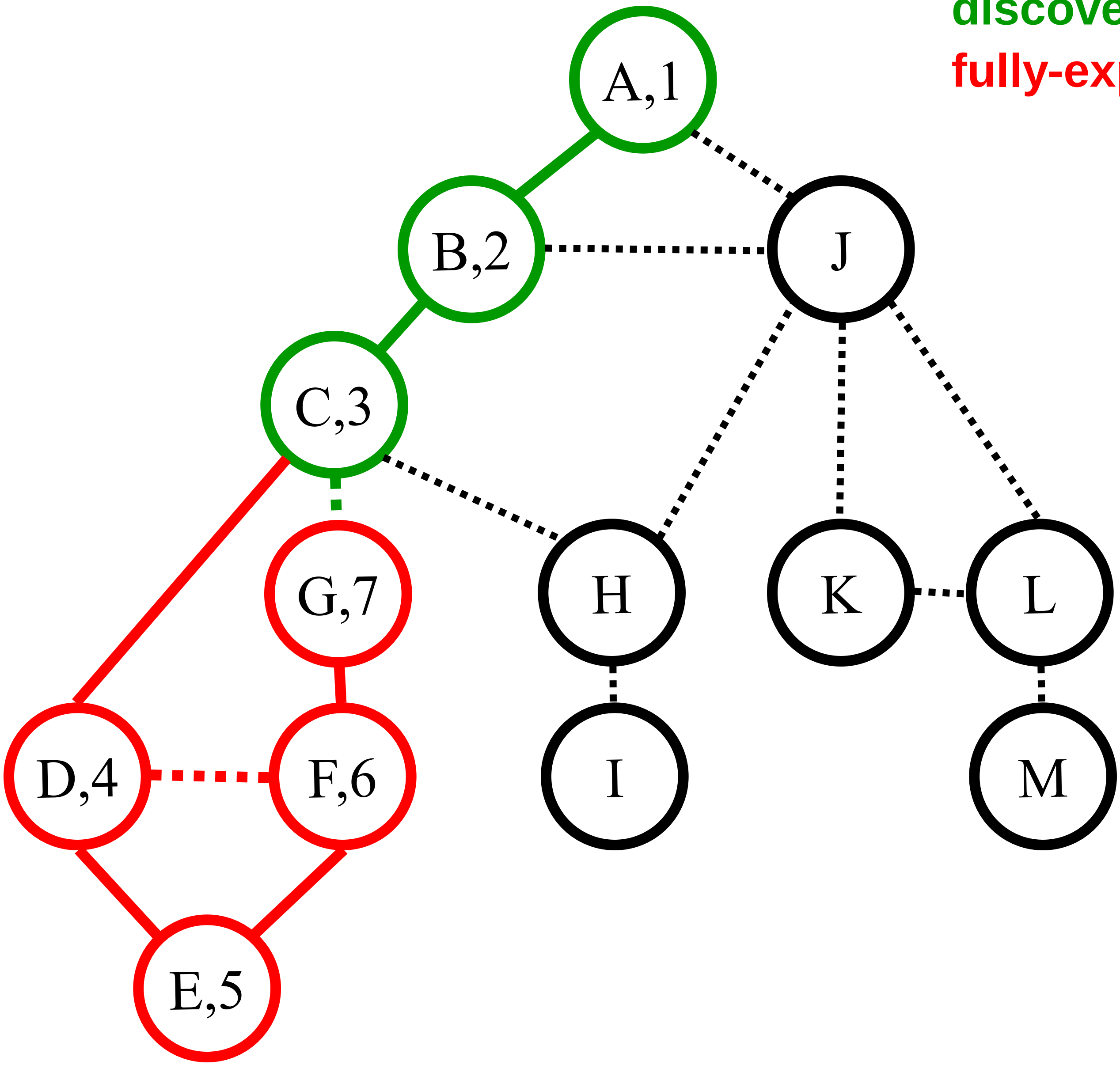
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,~~F~~)

st[] =
{1,2,3,4}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (B,J)
B (A,C,J)
C (B,D,G,H)

st[] =
{1,2,3}

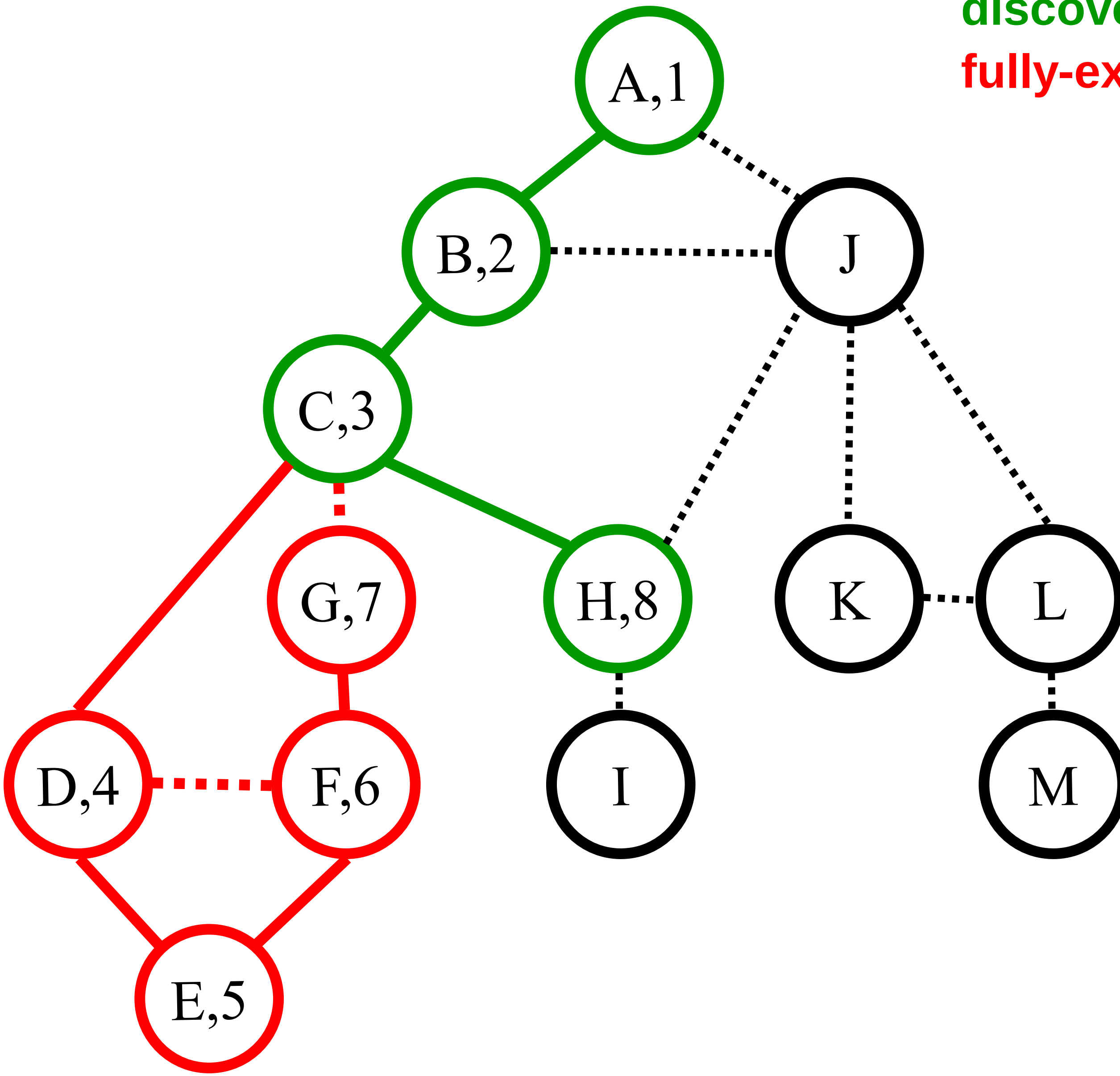
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (C,I,J)

st[] =
{1,2,3,8}

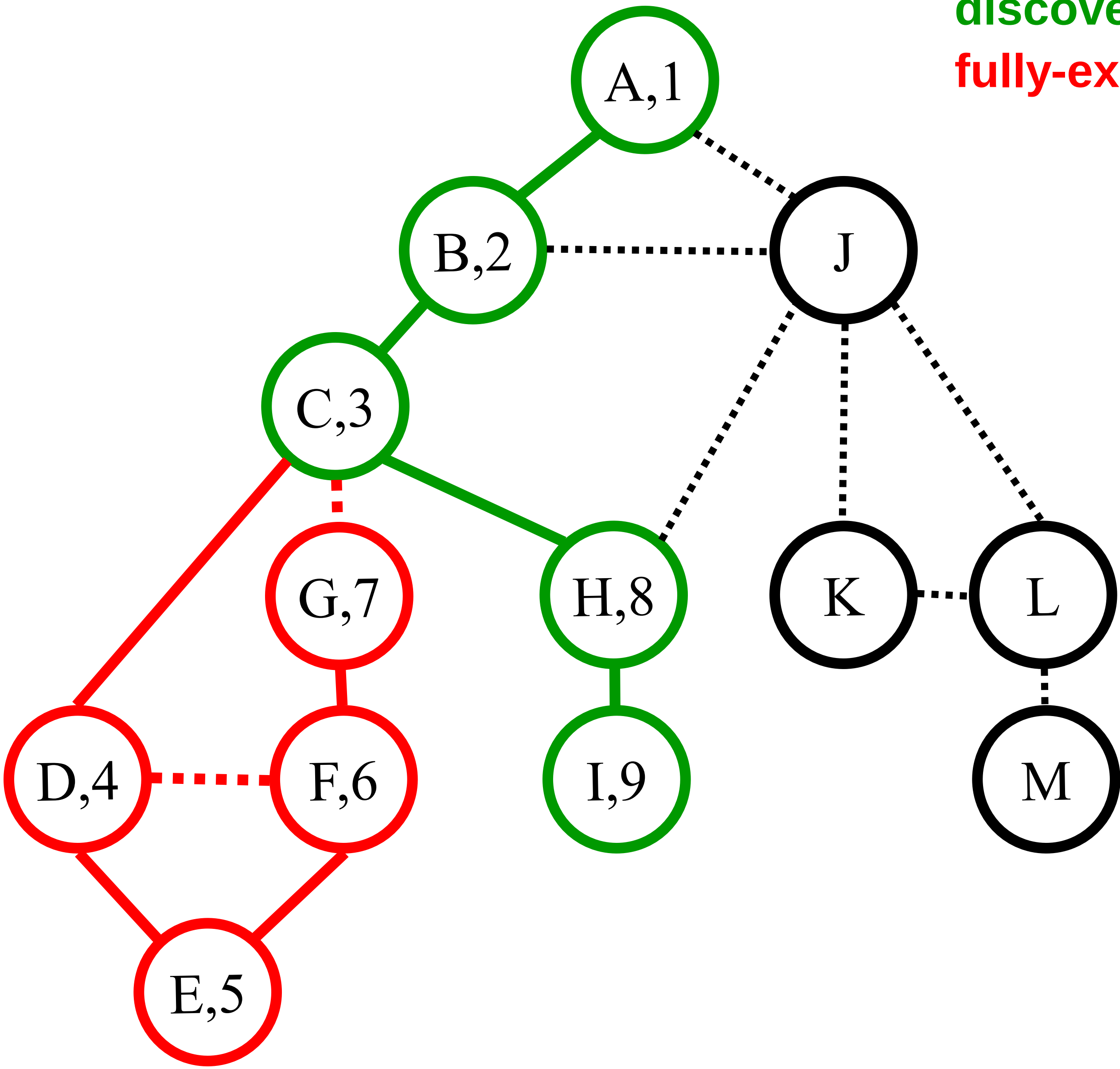
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

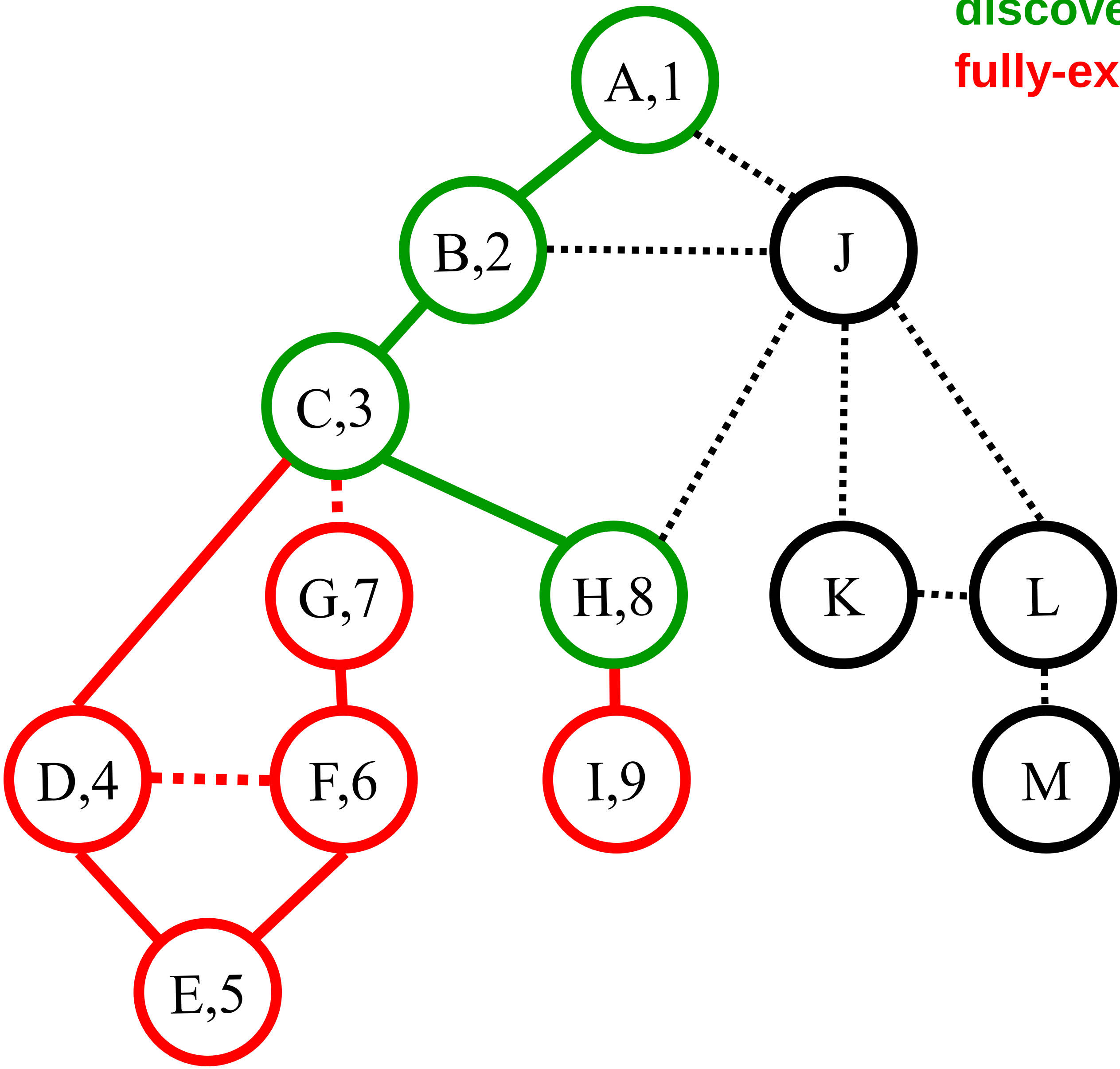
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,J)
I (H)

st[] =
{1,2,3,8,9}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,J)

st[] =
{1,2,3,8}

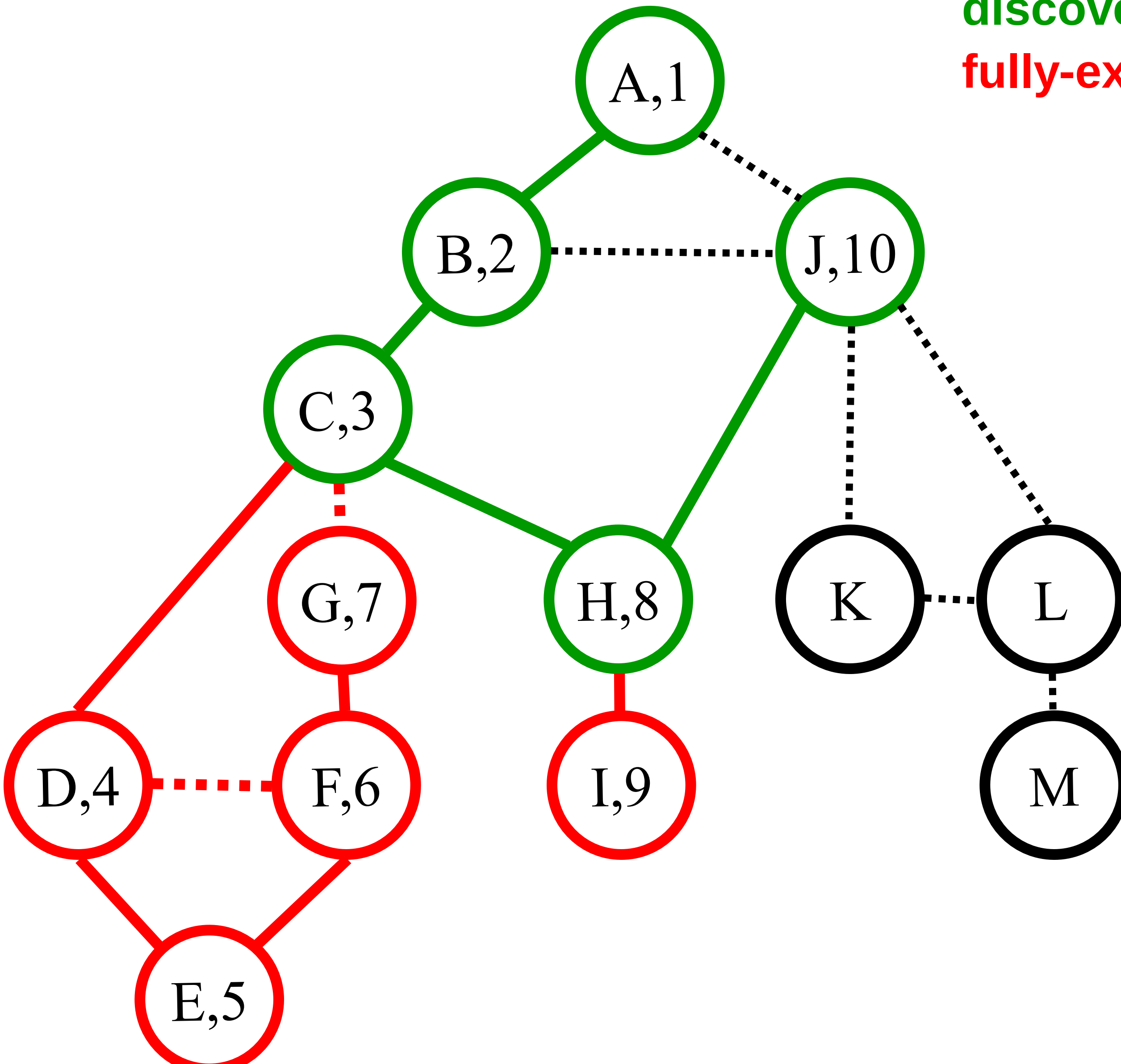
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (A,B,H,K,L)

st[] =
{1,2,3,8,
10}

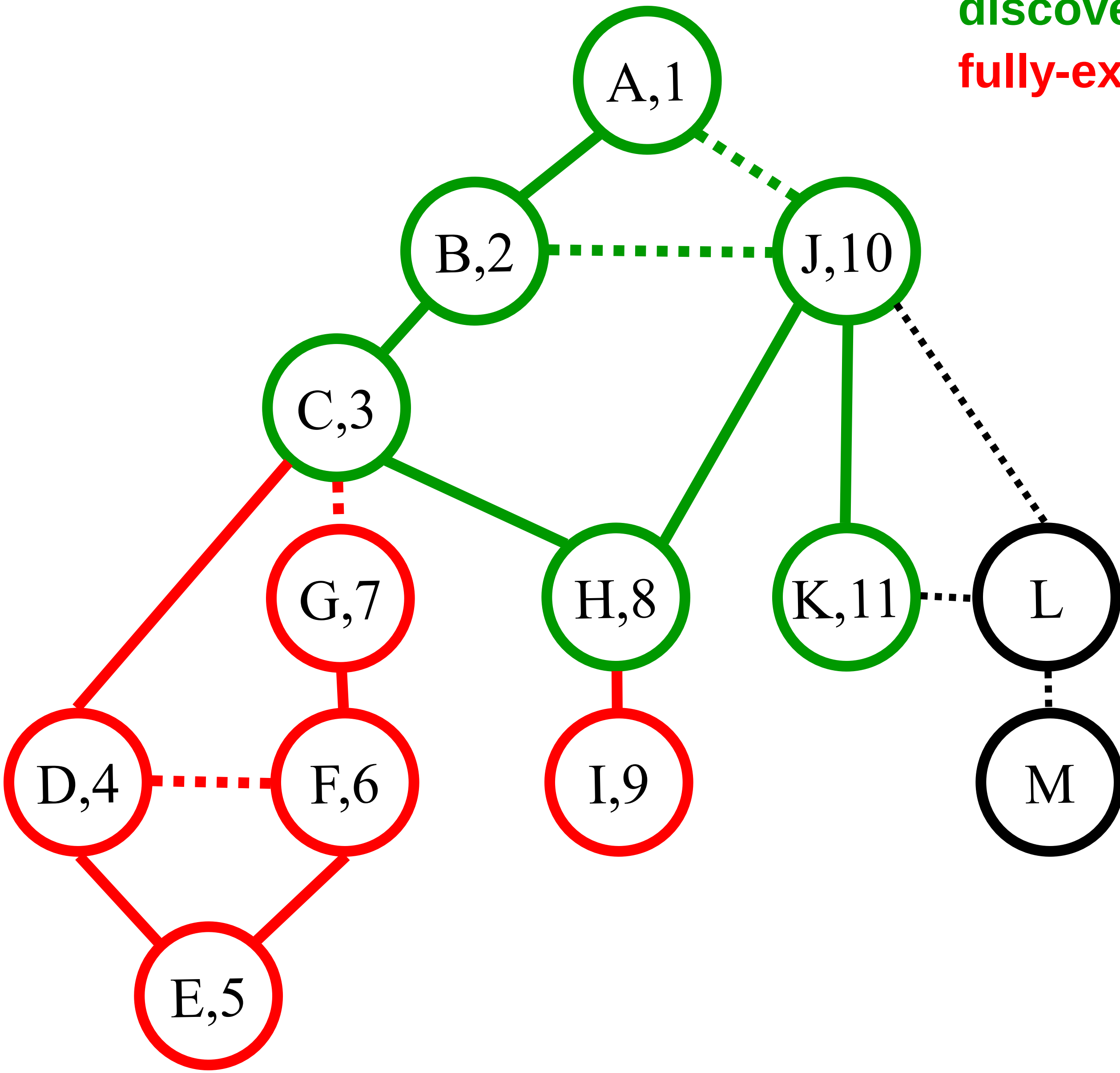
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

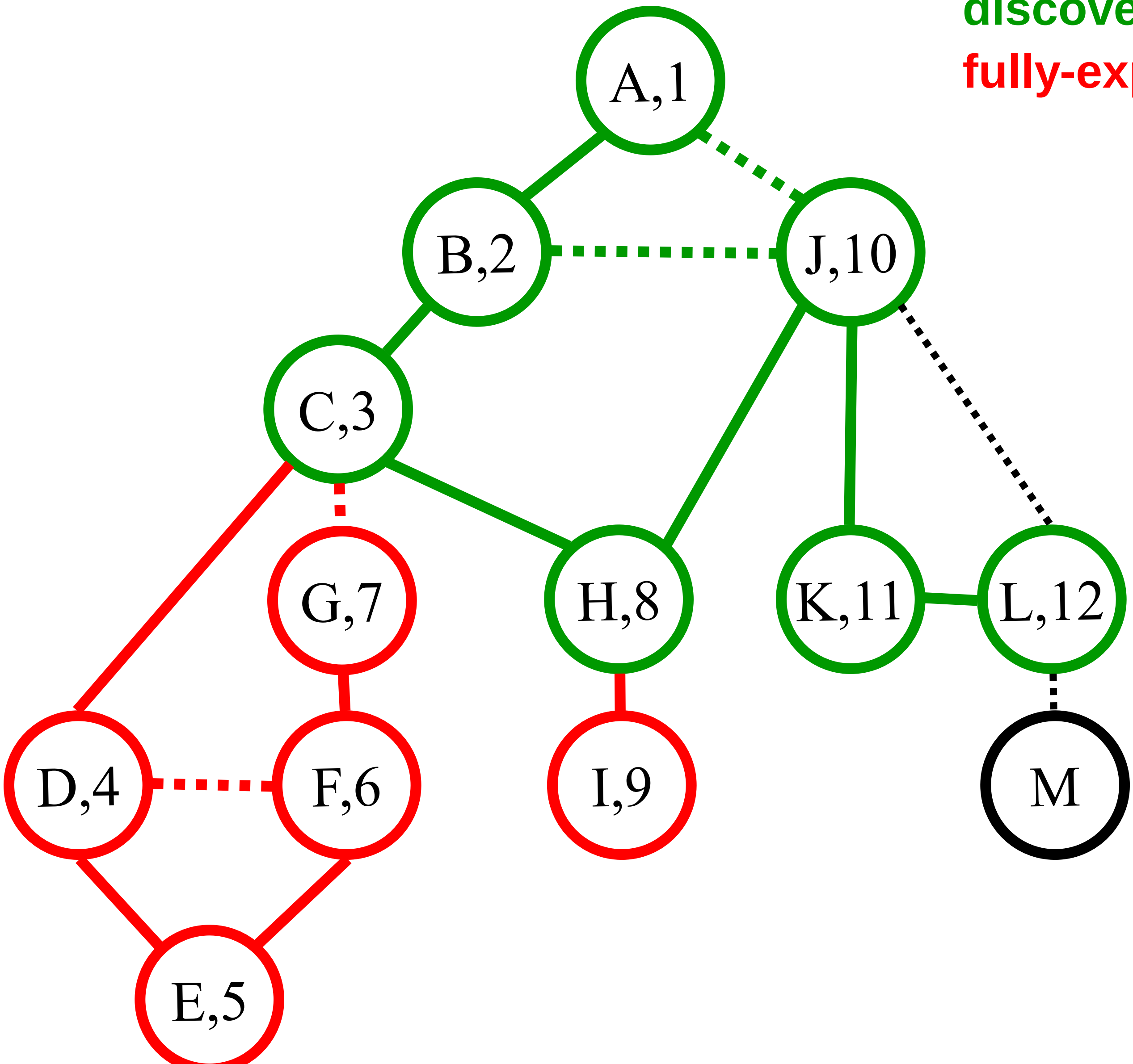
A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)
K (J,L)

st[] =
{1,2,3,8,10
,11}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)
K (J,L)
L (J,K,M)

st[] = {1,2,3,8,10,11,12}

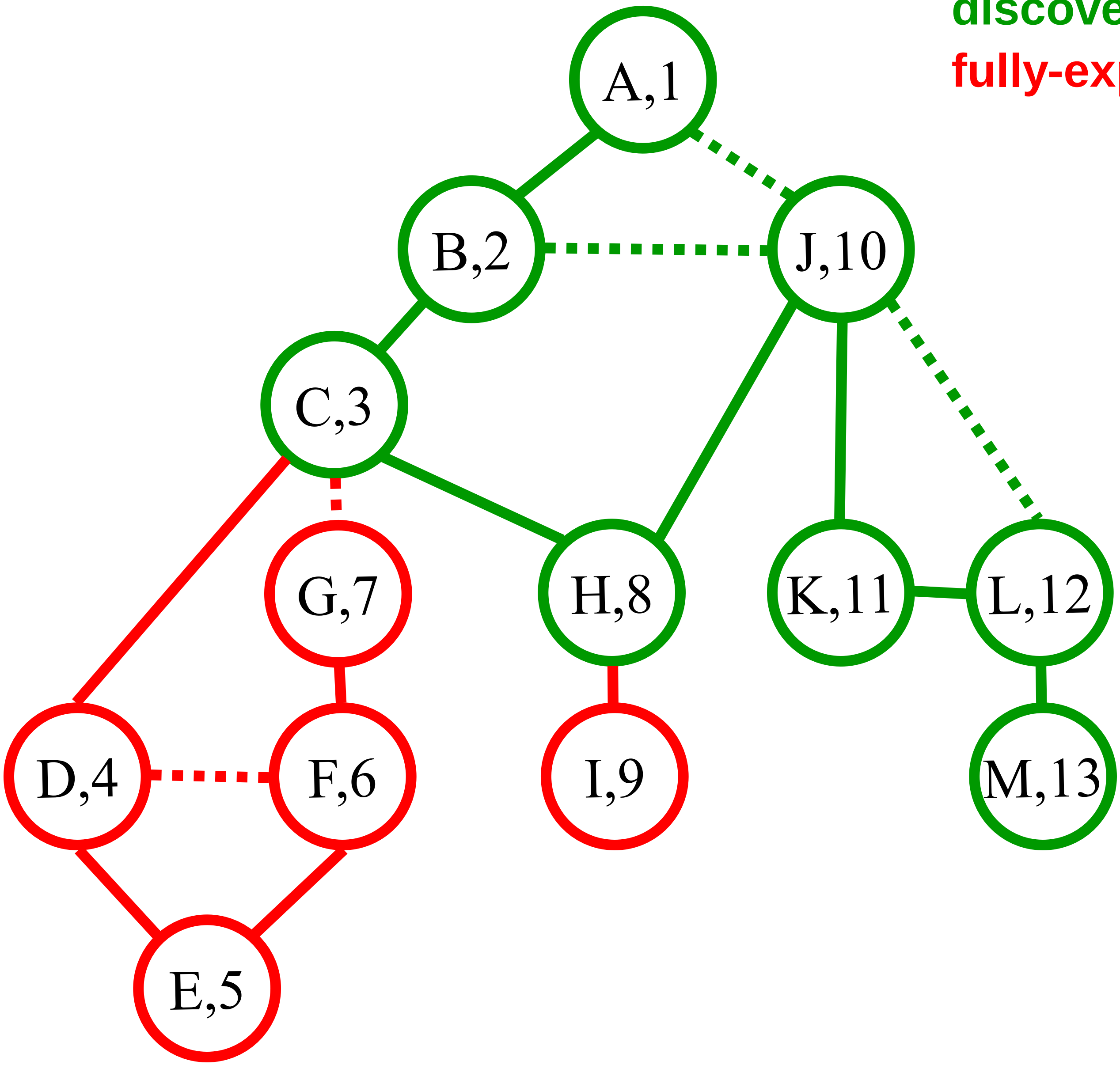
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)
K (J,L)
L (J,K,M)
M (L)

st[] = {1,2,3,8,10 ,11,12,13}

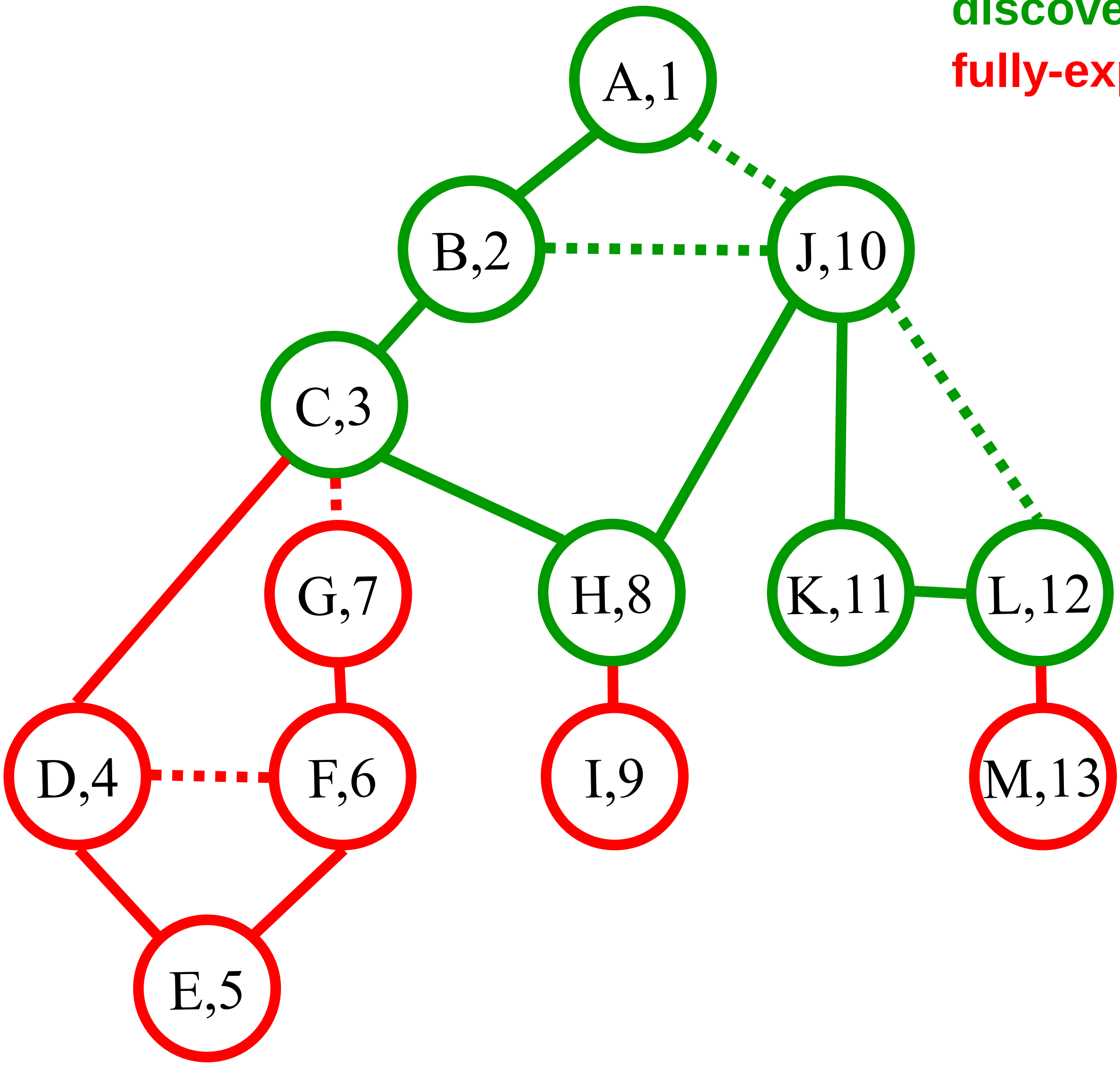
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)
K (J,L)
L (J,K,M)

st[] =
{1,2,3,8,10
,11,12}

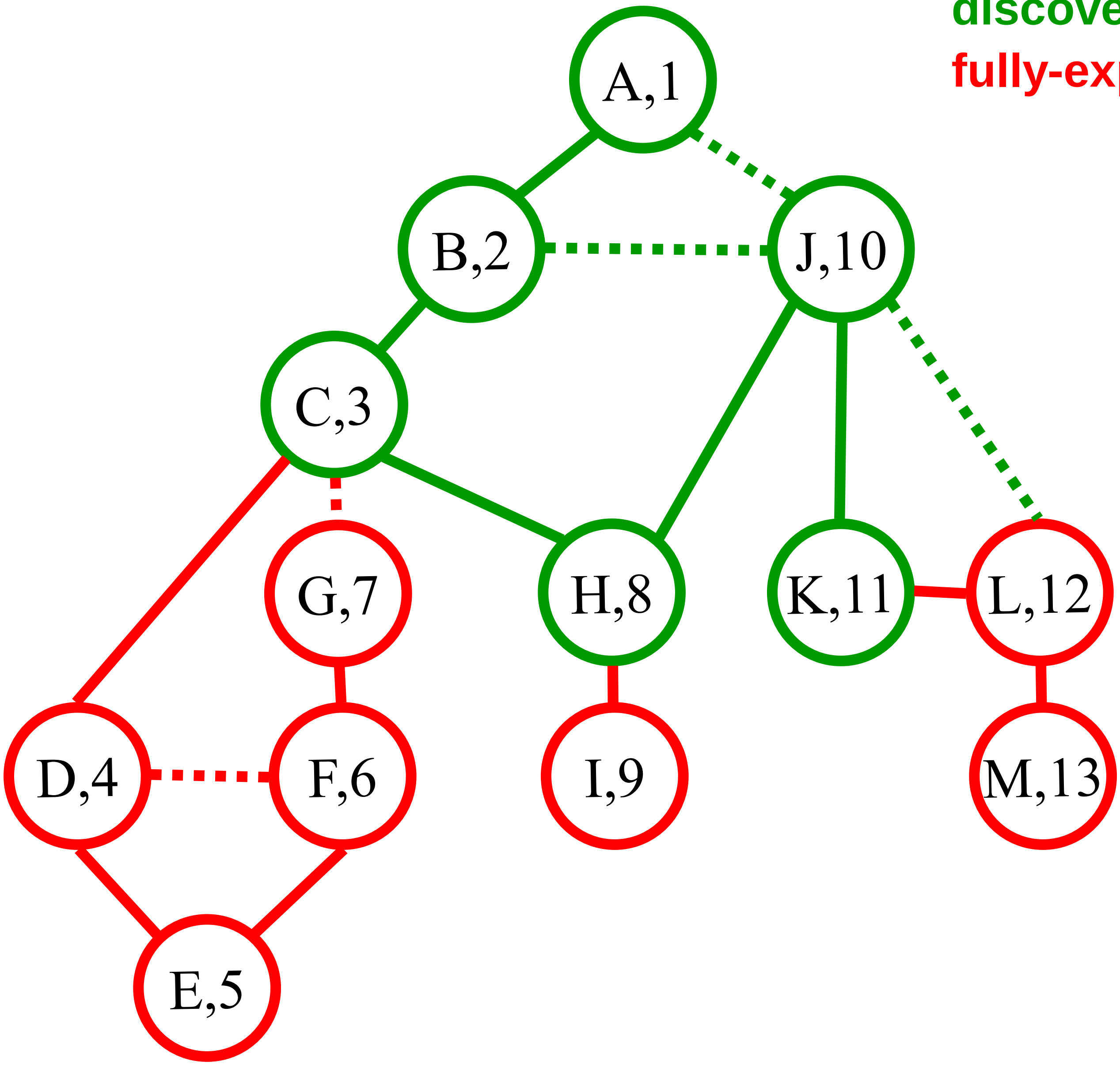
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)
K (J,L)

st[] =
{1,2,3,8,10
,11}

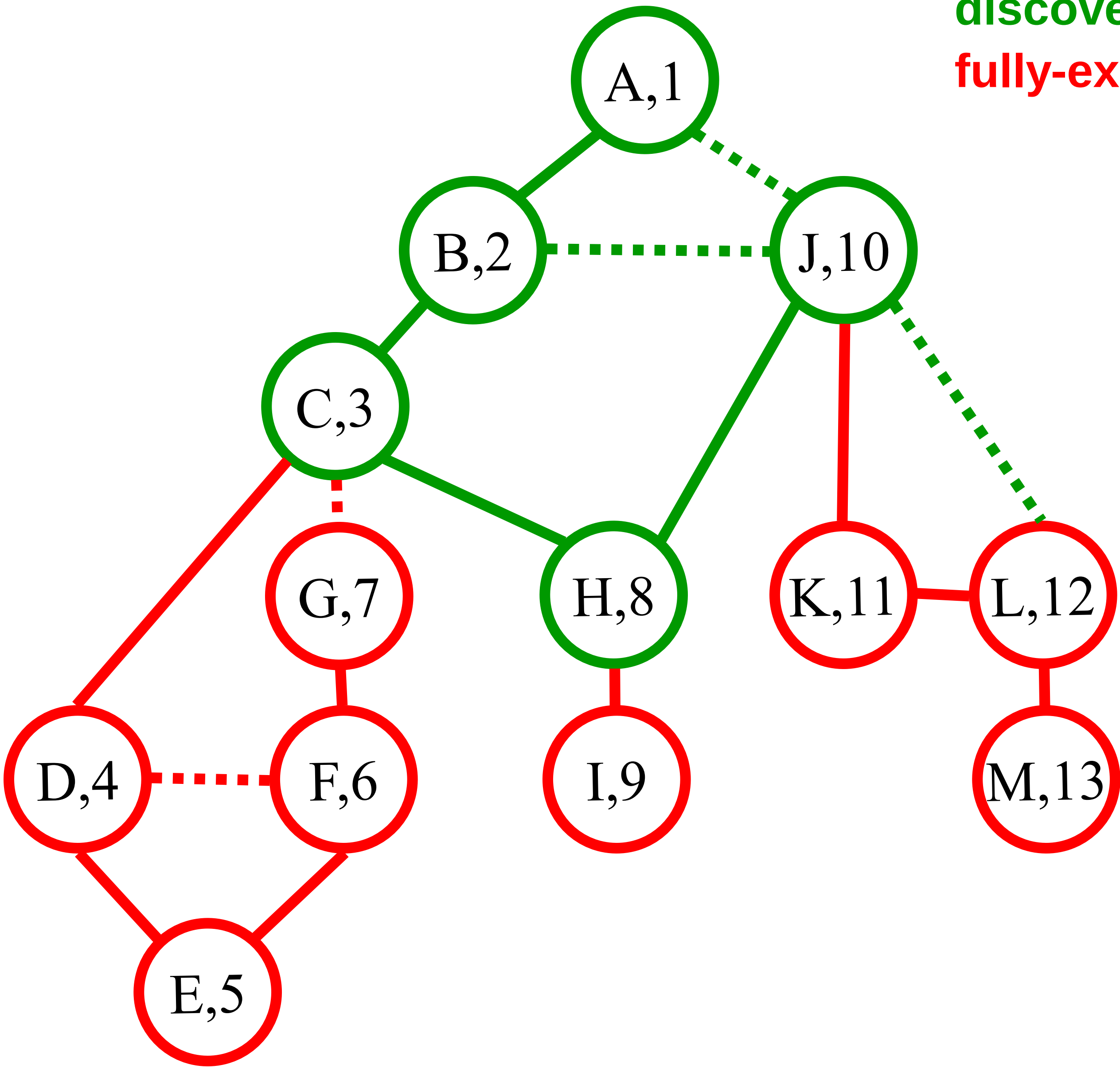
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)

st[] =
{1,2,3,8,
10}

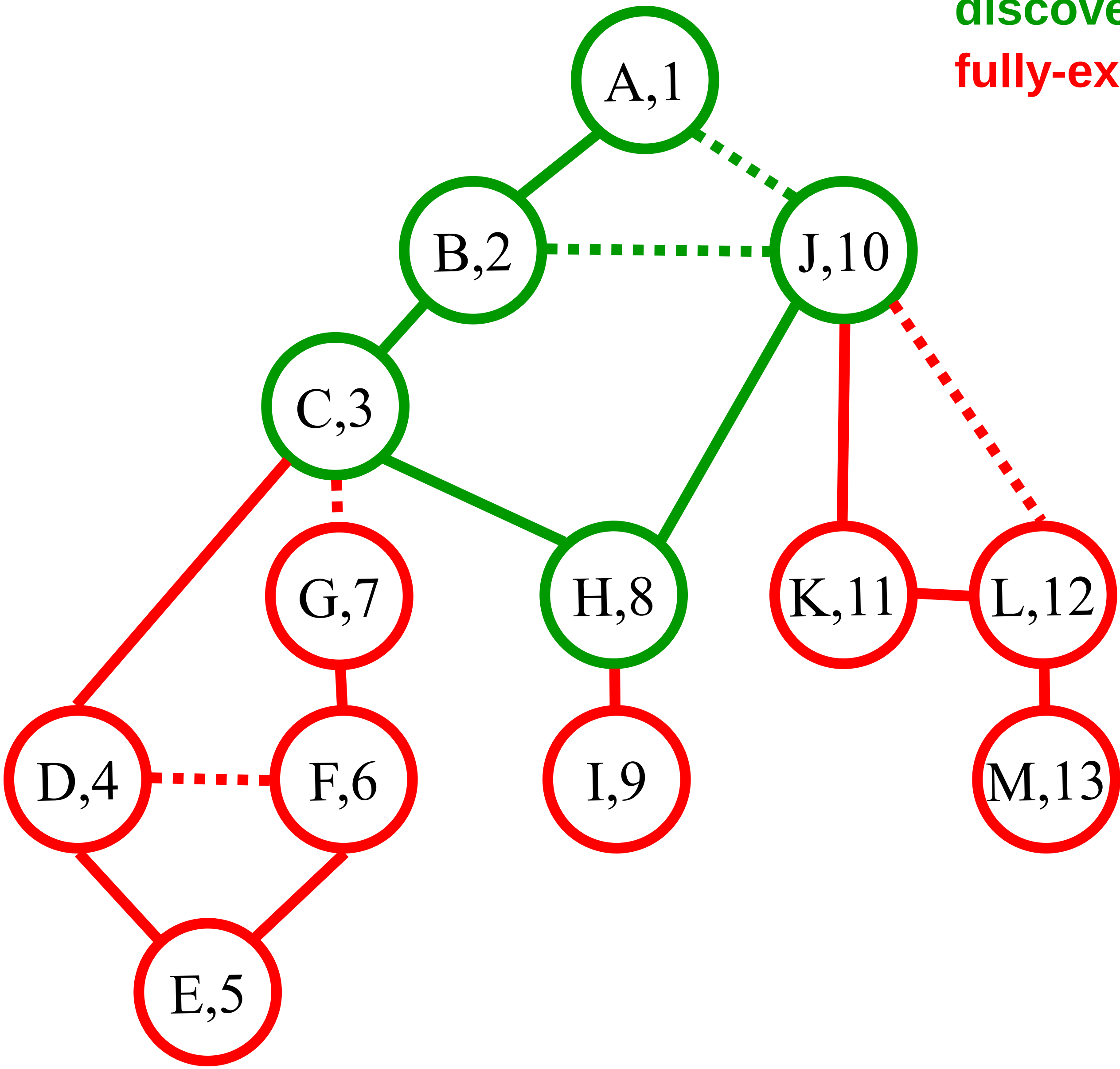
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)

st[] =
{1,2,3,8,
10}

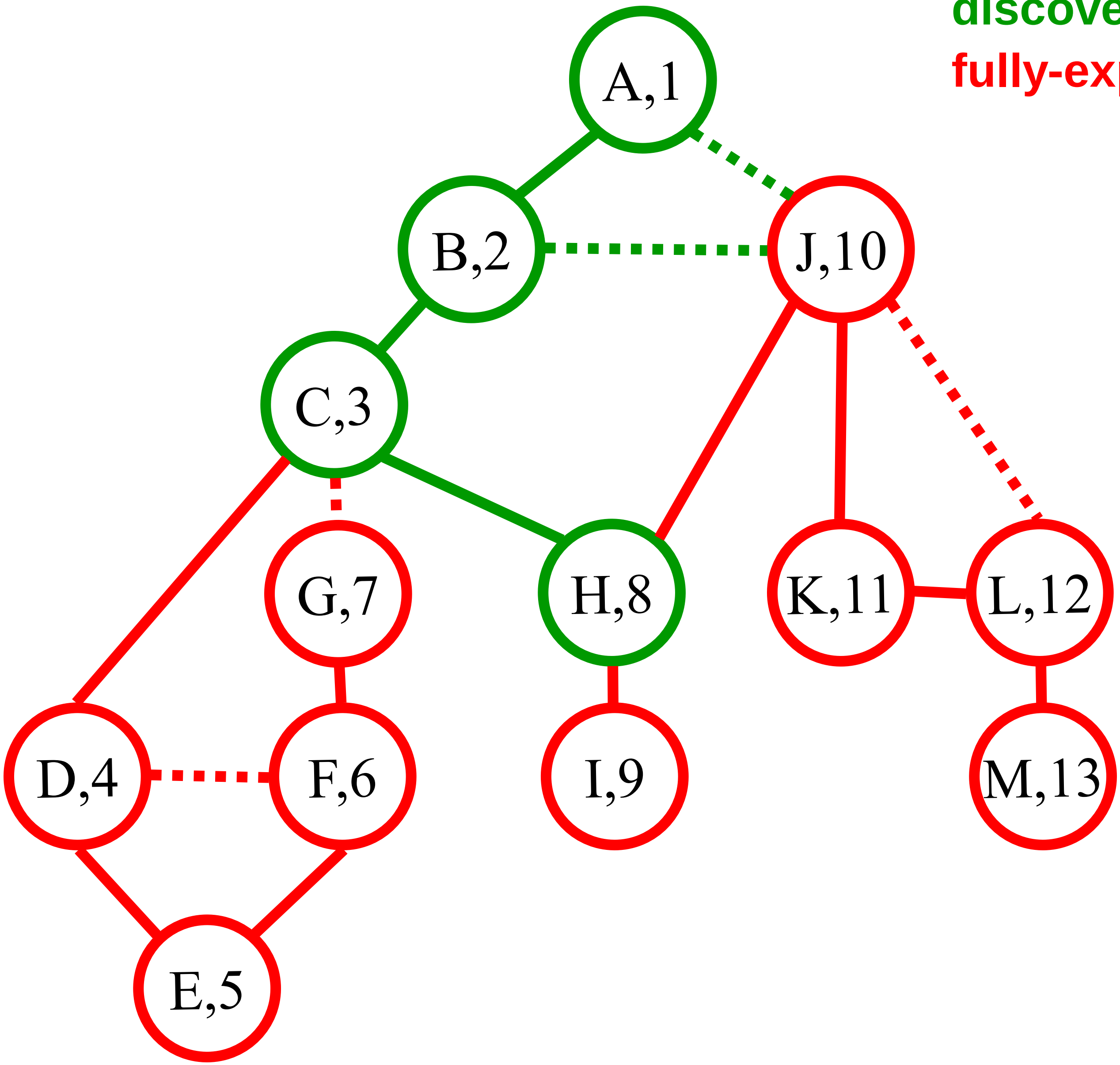
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

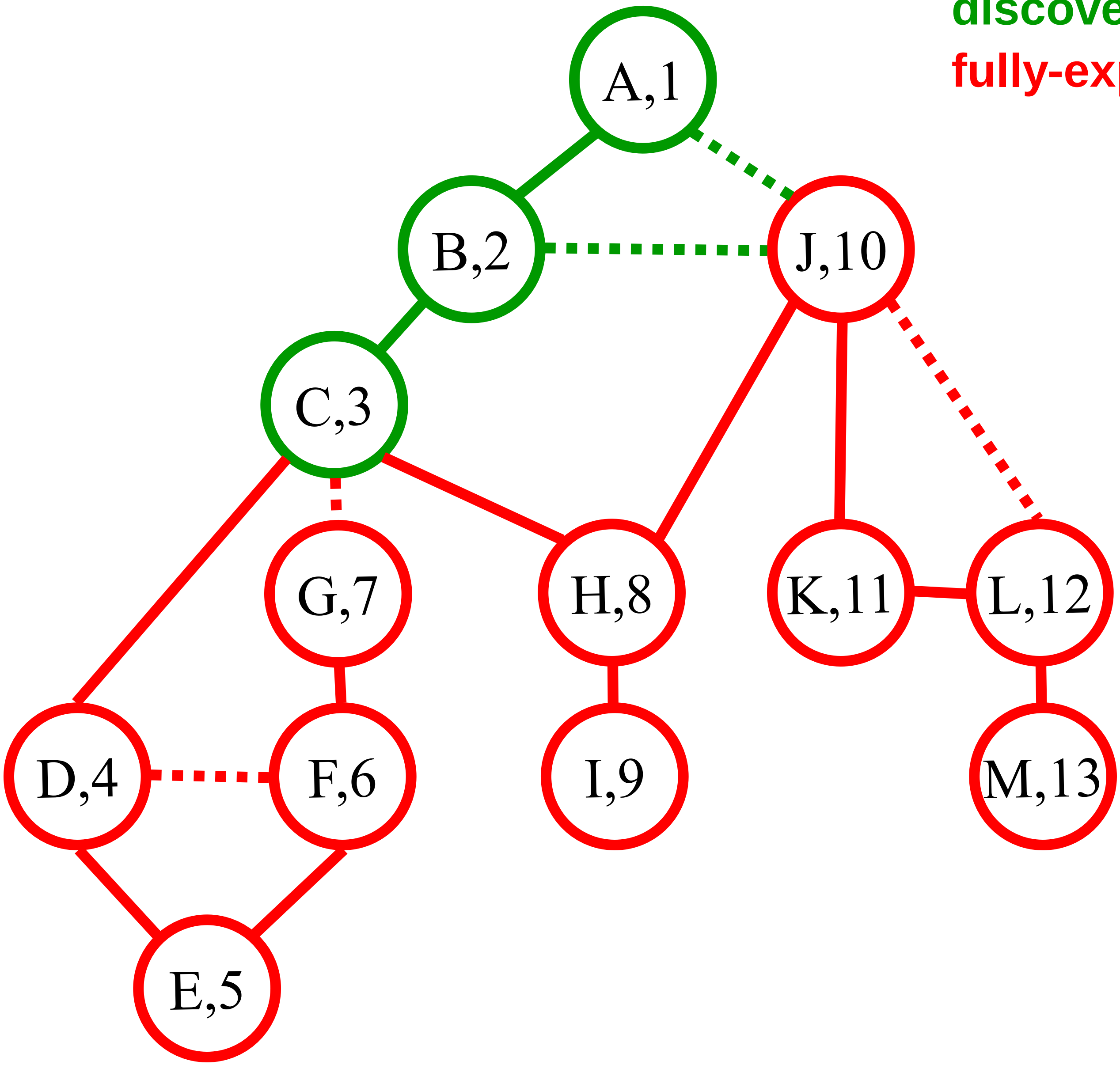
A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)

st[] =
{1,2,3,8}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (B,J)
B (A,C,J)
C (B,D,G,H)

st[] =
{1,2,3}

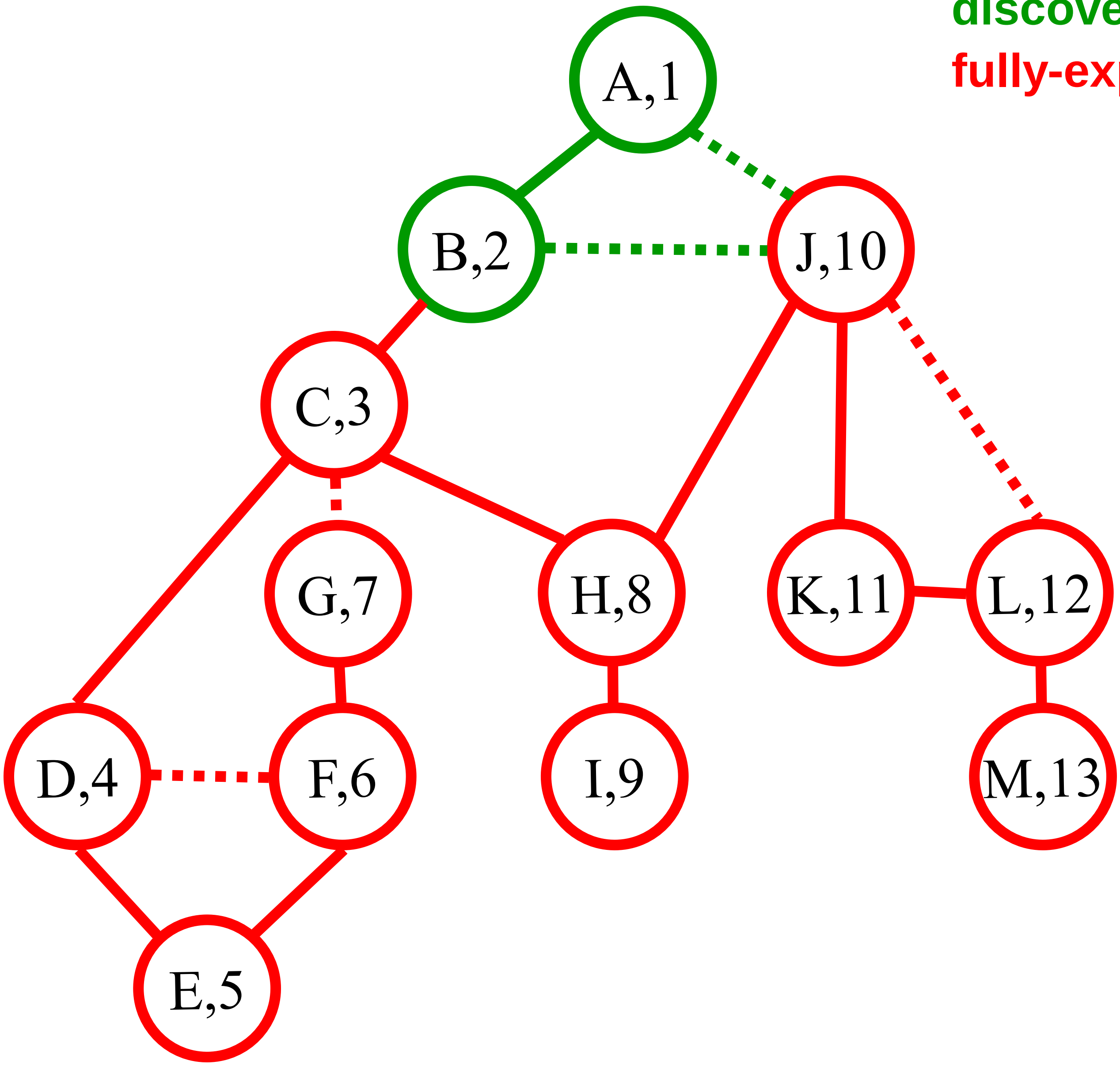
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)

st[] =
{1,2}

DFS(A)

Color code:

undiscovered

discovered

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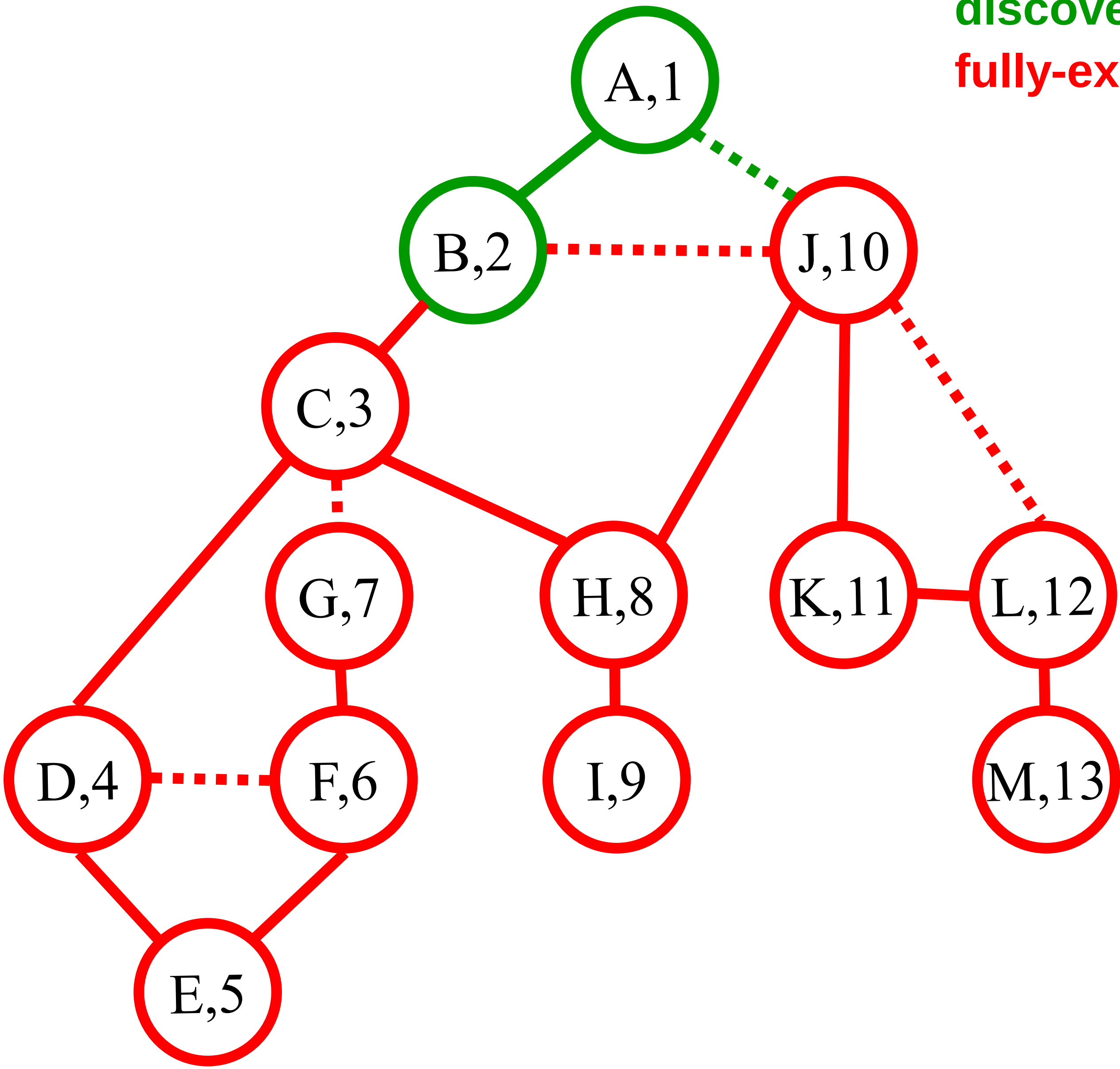

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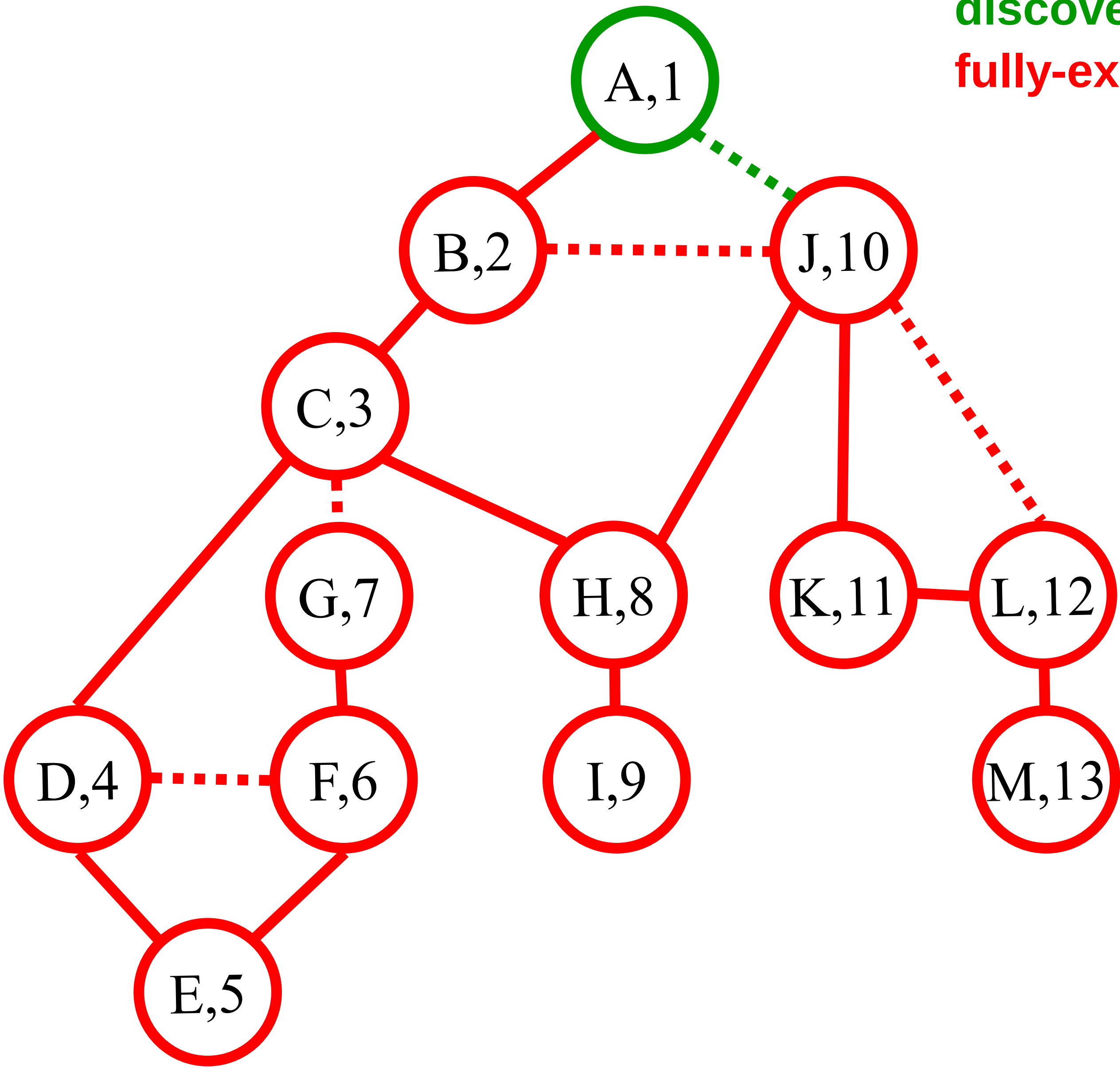

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DFS(A)

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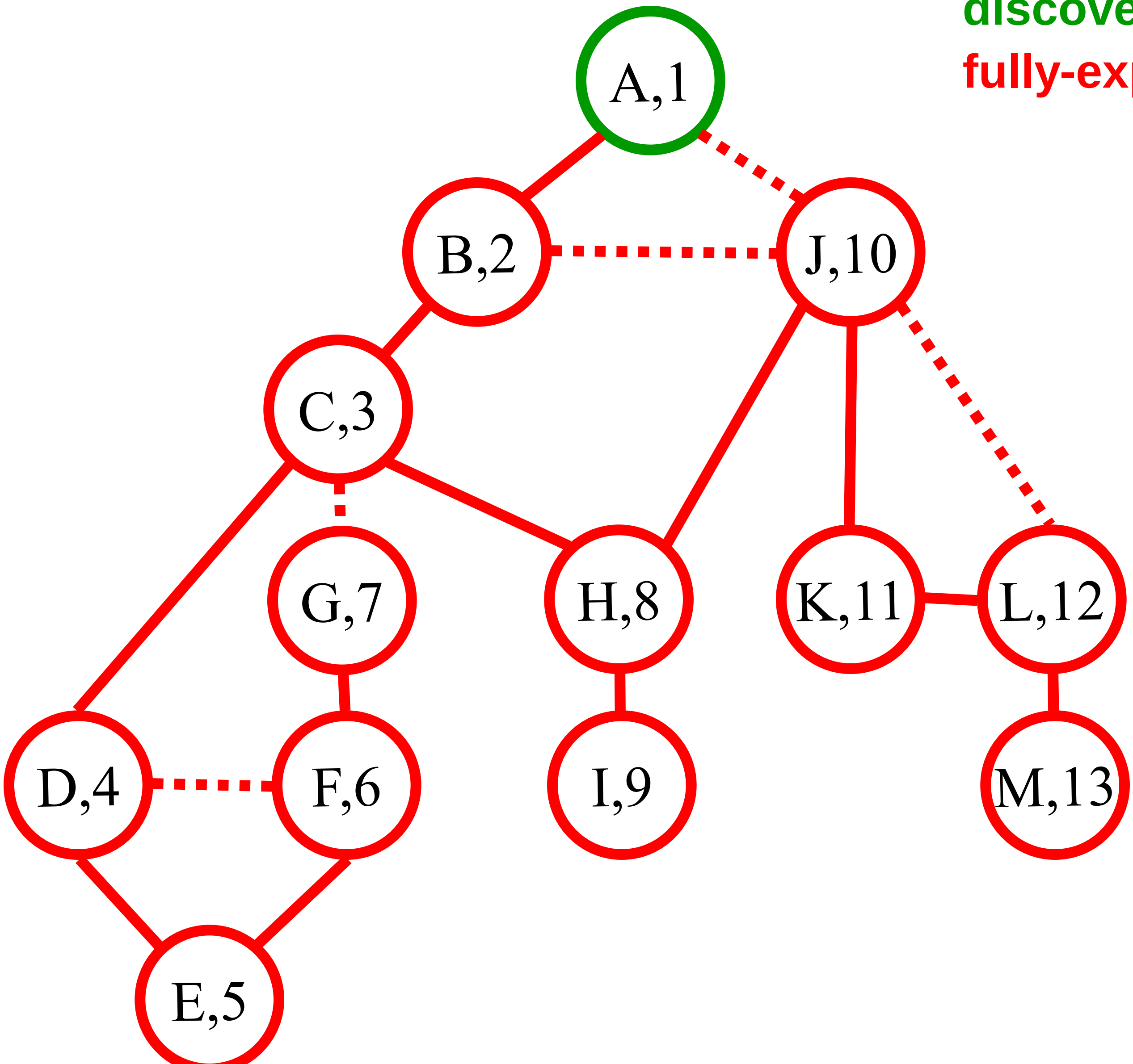

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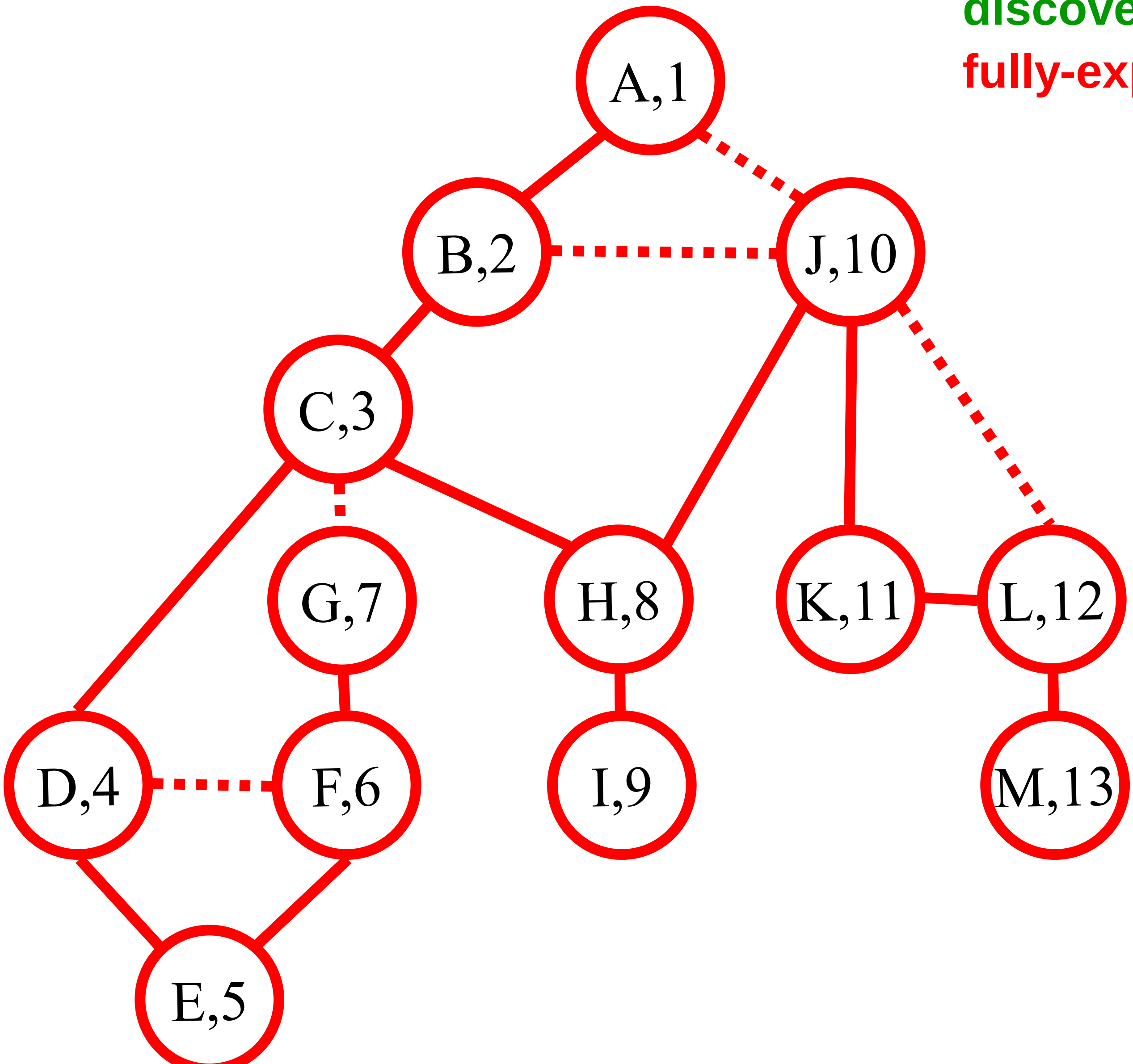
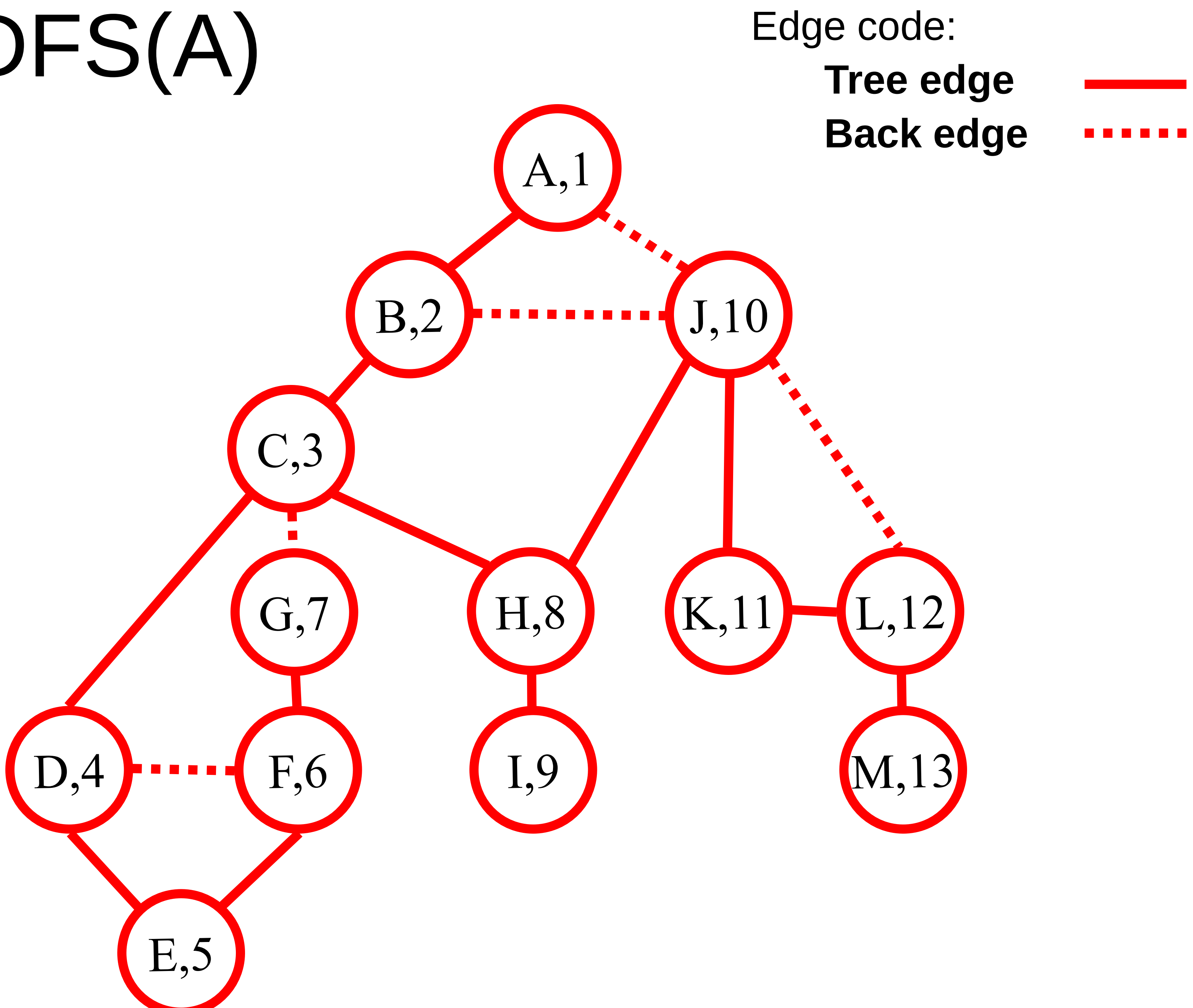


Call Stack:
(Edge list)

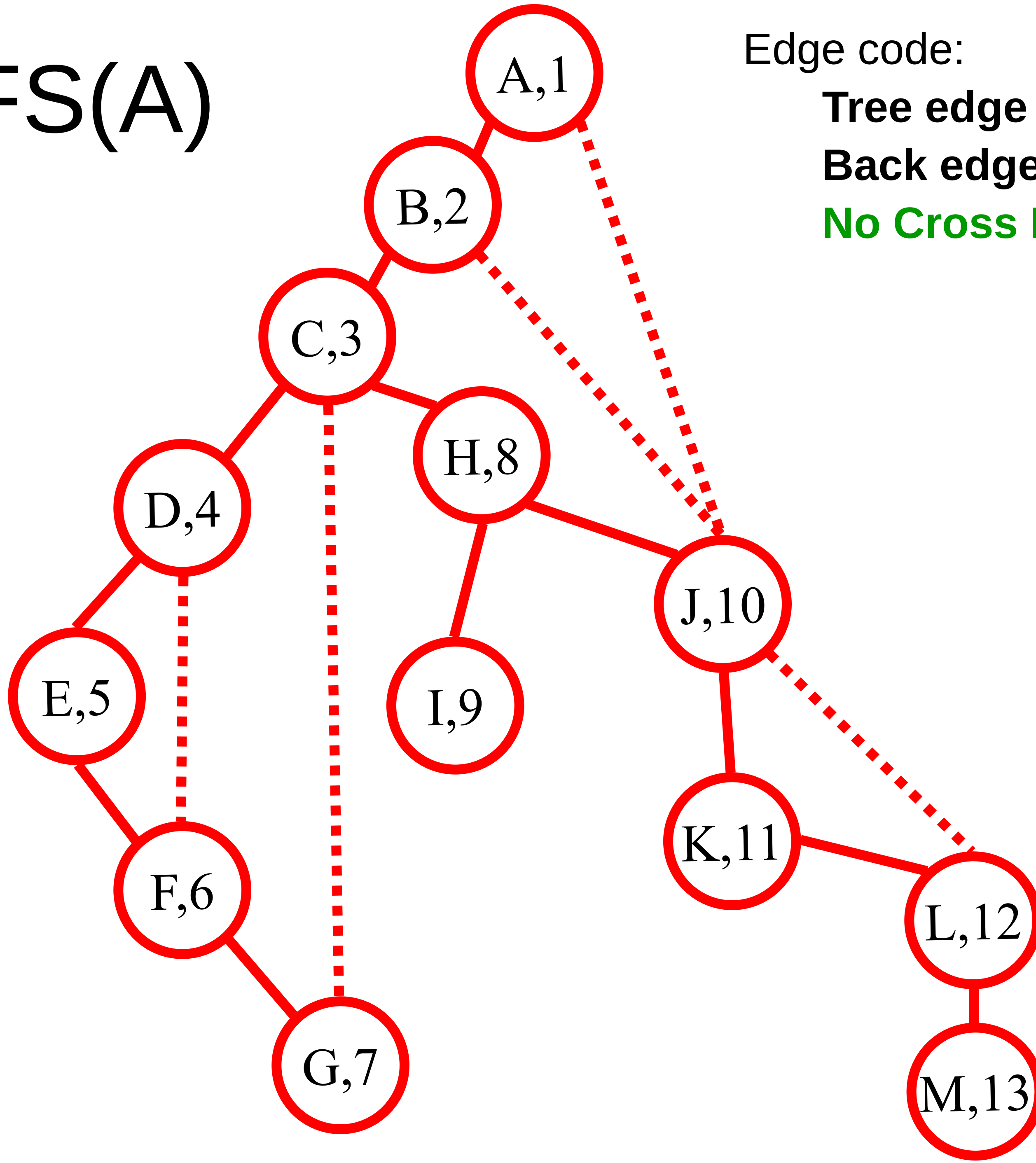
TA-DA!!

st[] = {}

DFS(A)


Edge code:
Tree edge
Back edge
A,1
B,2
J,10
C,3
G,7
H,8
K,11
L,12
D,4
F,6
I,9
M,13
E,5

DFS(A)



Edge code:

Tree edge ——

Back edge ······

No Cross Edges!

Properties of (undirected) DFS

Like BFS(s):

- DFS(s) visits x iff there is a path in G from s to x

  So, we can use DFS to find connected components
- Edges into then-undiscovered vertices define a ***tree*** – the "depth first spanning tree" of G

Unlike the BFS tree:

- The DF spanning tree isn't minimum depth
- Its levels don't reflect min distance from the root
- Non-tree edges never join vertices on the same or adjacent levels

Non-Tree Edges in DFS

Lemma: For every **undirected** edge $\{x, y\}$, then one of x or y is an ancestor of the other in the tree.

Proof:

Suppose that x is visited first.

Therefore DFS(x) was called before DFS(y)

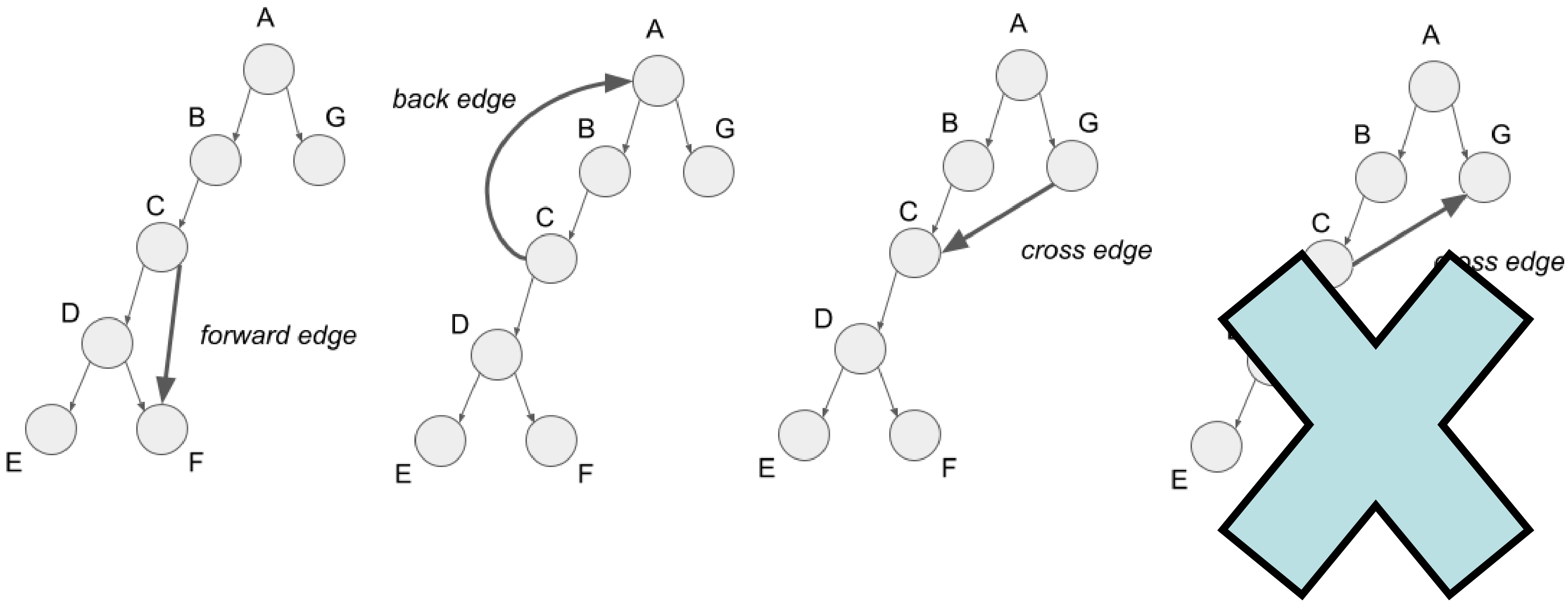
Since $\{x, y\}$ is not in DFS tree, y was visited when the edge $\{x, y\}$ was examined during DFS(x)

Therefore y was visited during the call to DFS(x) so y is a descendant of x.

Non-Tree Edges (Directed Graph)

Lemma: For every **directed** edge (x, y), then either

- y is visited first or
- y is a descendant of x



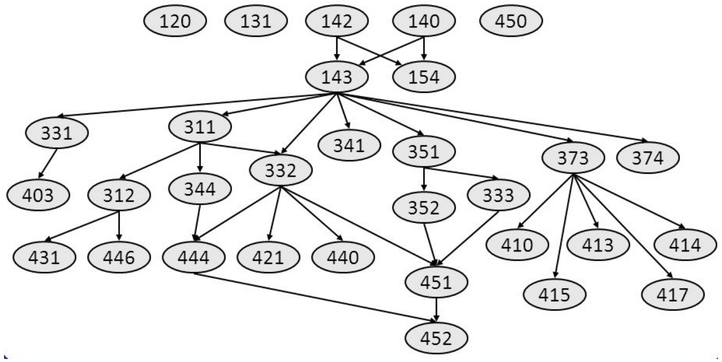
CSE 421

Applications of DFS
Topological sort

Yin Tat Lee

Precedence Constraints

In a directed graph, an edge (i, j) means task i must occur before task j.



Applications

- Course prerequisite:

  course i must be taken before j

- Compilation:

  must compile module i before j

- Computing overflow:

  output of job i is part of input to job j

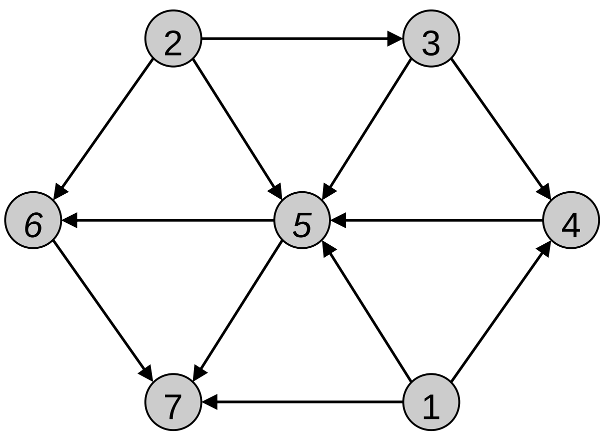
- Manufacturing or assembly:

  sand it before paint it

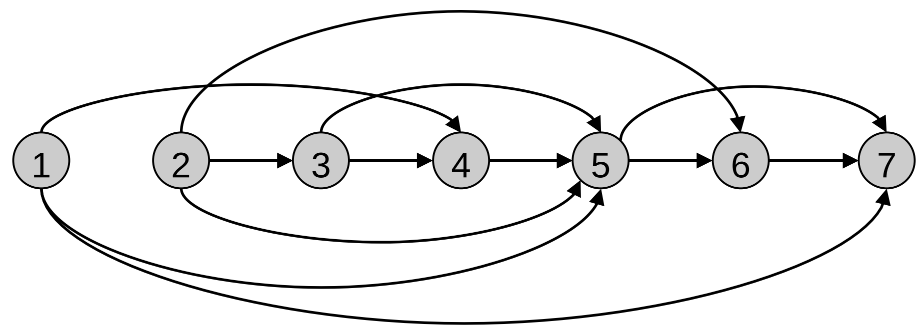
Directed Acyclic Graphs (DAG)

Def: A directed acyclic graph (DAG) is a graph that contains no directed cycles.

Def: A topological order of a directed graph $G = (V, E)$ is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge (v_i, v_j) we have $i < j$.



a DAG



a topological ordering of that DAG– all edges left-to-right

DAGs: A Sufficient Condition

Lemma: If G has a topological order, then G is a DAG.

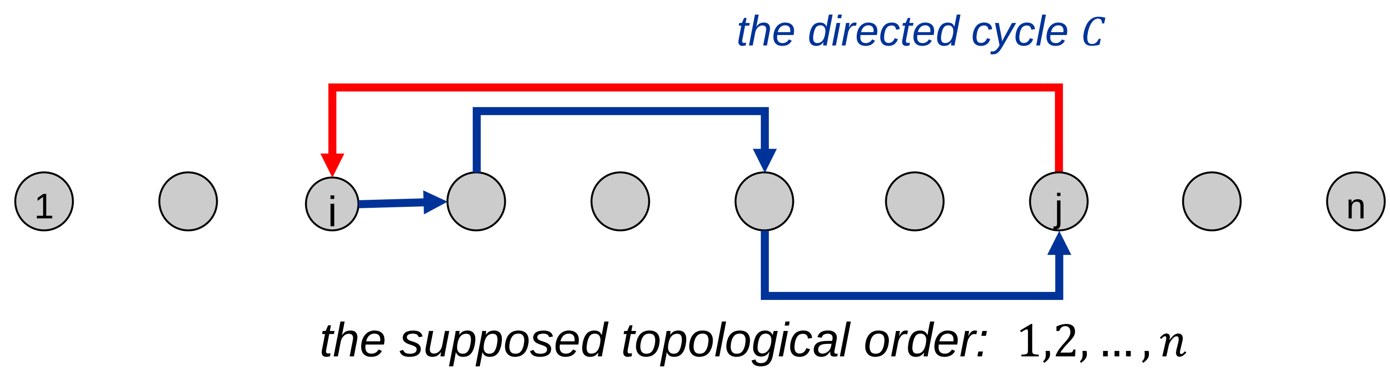
Proof. (by contradiction)

Suppose that G has a topological order $1,2,\ldots,n$ and that G also has a directed cycle C.

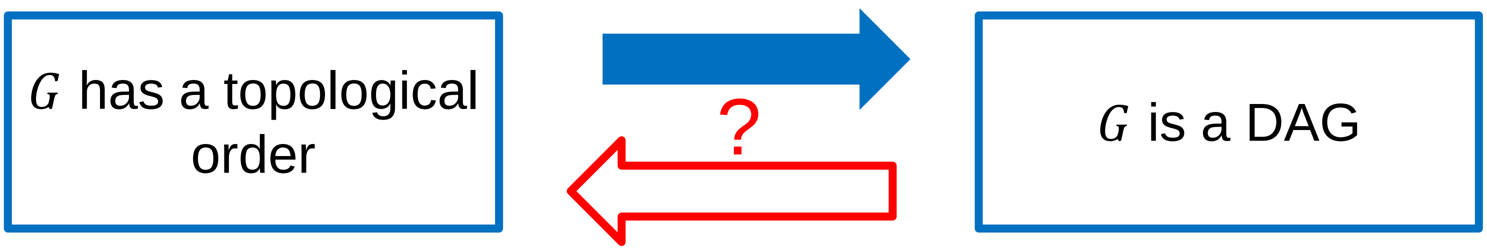
Let i be the lowest-indexed node in C, and let j be the node just before i; thus (j,i) is an (directed) edge.

By our choice of i, we have $i < j$.

On the other hand, since (j,i) is an edge and $1,2,\ldots,n$ is a topological order, we must have $j < i$, a contradiction



DAGs: A Sufficient Condition



Every DAG has a source node

Lemma: If G is a DAG, then G has a node with no incoming edges (i.e., a source).

The proof is similar to "tree has $n - 1$ edges".

Proof. (by contradiction)

Suppose that G is a DAG and it has no source

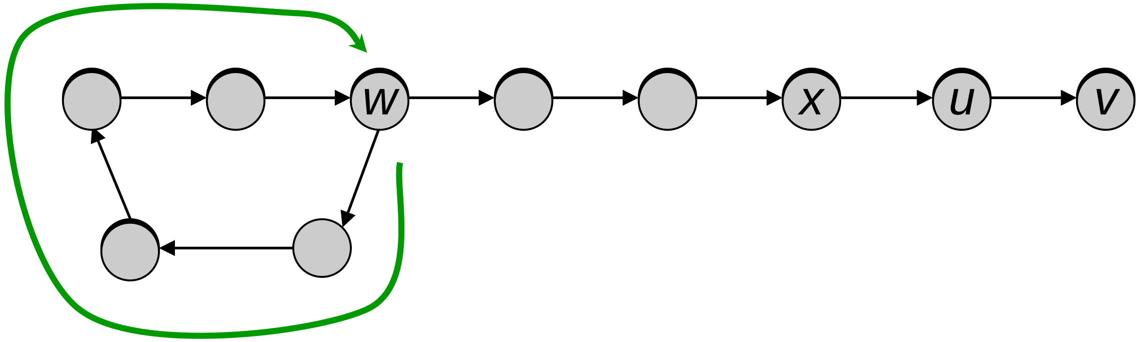
Pick any node v, and begin following edges backward from v. Since v has at least one incoming edge (u, v) we can walk backward to u.

Then, since u has at least one incoming edge (x, u), we can walk backward to x.

Repeat until we visit a node, say w, twice.

Let C be the sequence of nodes encountered between successive visits to w. C is a cycle.

C



DAG => Topological Order

Lemma: If G is a DAG, then G has a topological order

Proof. (by induction on n)

Base case: true if $n = 1$.

Hypothesis: Every DAG with $n - 1$ vertices has a topological ordering.

Inductive Step: Given DAG with $n > 1$ nodes, find a source node v.

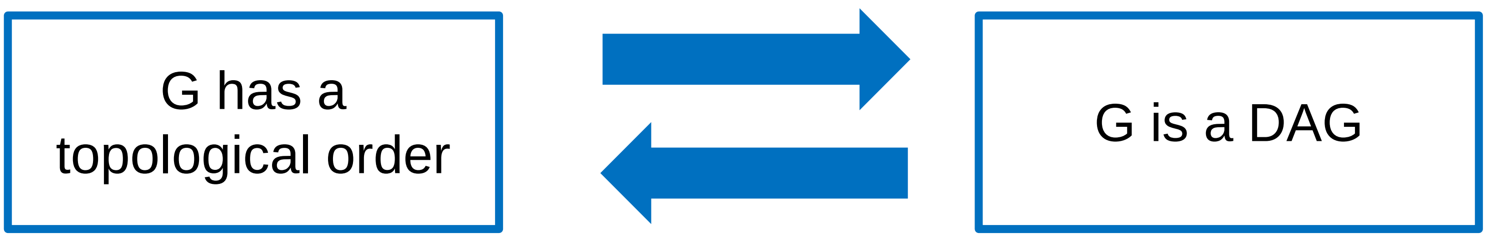
$G - \{v\}$ is a DAG, since deleting v cannot create cycles.

Reminder: Always remove vertices/edges to use IH

By hypothesis, $G - \{v\}$ has a topological ordering.

Place v first in topological ordering; then append nodes of $G - \{v\}$ in topological order. This is valid since v has no incoming edges.

A Characterization of DAGs



Quiz

How to find topological ordering in polynomial time?

Algorithm (n^2 time):

Function $\pi = Order(G)$

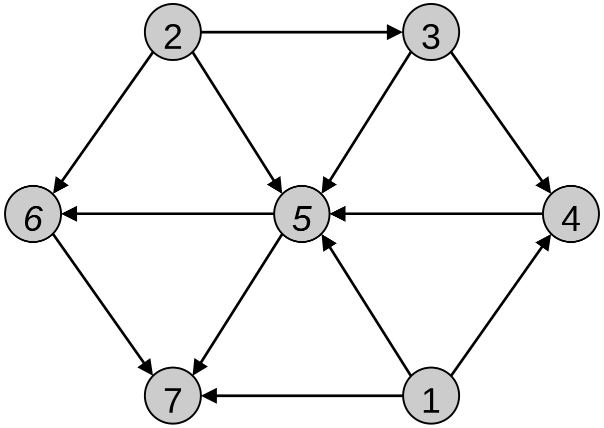
- Find a vertex v in G with no incoming edge (Time: n)
- Return $(v, Order(G - \{v\}))$. (Total Time: m)

How to improve the runtime?

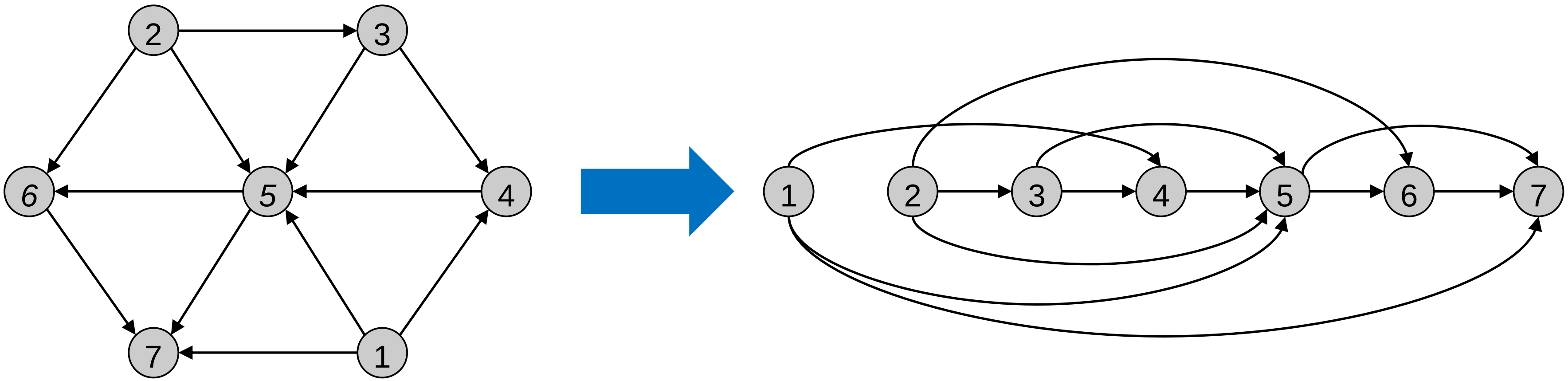
- Maintain the set of vertices with no incoming edge.

Alternatively, you can solve this problem by DFS.

Example



Example



Topological order: 1, 2, 3, 4, 5, 6, 7

Summary for last few classes

- Terminology: vertices, edges, paths, connected component, tree, bipartite…
- Vertices vs Edges: $m = O(n^2)$ in general, $m = n - 1$ for trees
- BFS: Layers, queue, shortest paths, all edges go to same or adjacent layer
- DFS: recursion/stack; all edges ancestor/descendant
- Algorithms: Connected Comp, bipartiteness, topological sort
- Techniques: Induction on vertices/layers