CSE 421 Lecture 7

1 Dijkstra's Algorithm

Theorem 1. *Let T be the spanning tree found by* Dijkstra(s). *Then, $d_G(s,u) = d_T(s,u)$ for all u.*

Proof:
Let $S_k$be the set S in the algorithm before step k.
Let induction statement $P(k)$ be "$d_T(s,u) = d_G(s,u)$ for all $u \in S_k$"
Base case $k = 1$**:**
$S_1 = \{s\}$. $d_T(s,s) = 0 = d_G(s,s)$.
Induction step:
Let v be the new vertex in S_k.
Let P be the path from s to v using the tree and the addition edge.

// The idea: Consider a shortest path P^* from s to v. By the choice of the algorithm, P is the shortest path exiting the set S_{k-1}. So, $c(P^*) \geq c(P)$.
Let P^* be some shortest path from s to v.
Let (u,v) be the edge that P exit S_{k-1}
Let (x,y) be the first edge that P^* exit S_{k-1}
Note that

$$\begin{aligned} c(P^*) &\geq d_G(s,x) + c_{(x,y)} \text{ (it is a subpath of } P^*) \\ &= d_T(s,x) + c_{(x,y)} \ (x \in S_{k-1}) \\ &\geq d_T(s,u) + c_{(u,v)} \text{ (by the choice of algo)} \\ &= c(P). \end{aligned}$$

2 Quiz

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Algorithm:
Run dijkstra to find a shortest path from s to t with the new length

$$\tilde{l}_e = l_e + \frac{1}{n}.$$

Output the shortest path dijkstra gives.
Runtime:
$O(m + n \log n)$ due to dijkstra.
Correctness:
Claim: Any shortest path for the distance $\tilde{l}$ is a shortest path for the distance l.
Proof: Any shortest path has length at most $n-1$. So, we at most add $\frac{n-1}{n} < 1$. Since all the costs are integer, any shortest path for the distance $\tilde{l}$ is a shortest path for the distance l.

Now, note that any shortest path in l with less length is shorter in $\tilde{l}$. So, the algo correctly outputs a shortest path with minimum length.