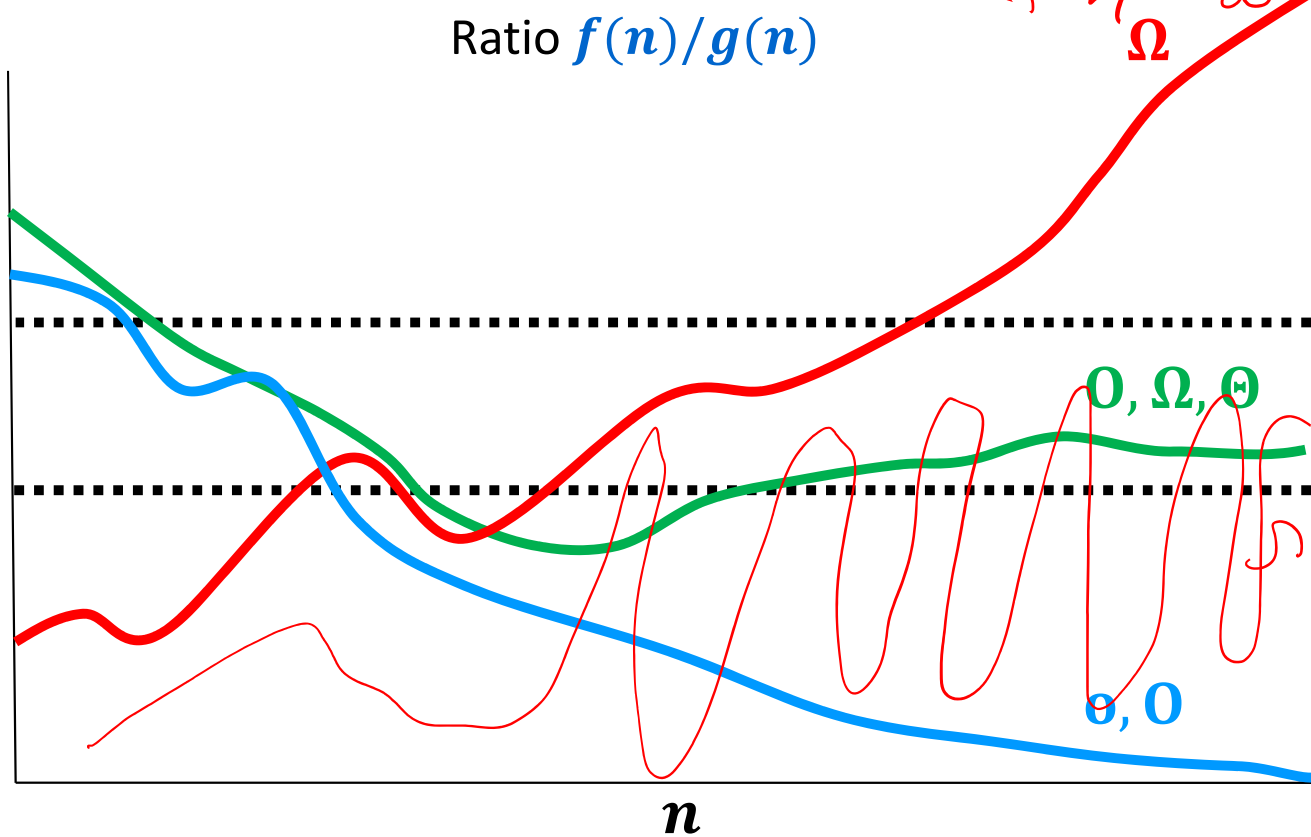
CSE 421
Introduction to Algorithms

Lecture 3: Overview, Graph Search

O, o, Ω, Θ-notation intuition

$f(n)$ is...

- $O(g(n))$: ratio eventually below a line forever
- $o(g(n))$: ratio goes to 0
- $\Omega(g(n))$: ratio eventually above a line forever
- $\Theta(g(n))$: both O and Ω



Introduction to Algorithms

- **Some representative problems**
  - Variety of techniques we'll cover
  - Seemingly small changes in a problem can require big changes in how we solve it

Some Representative Problems

Interval Scheduling:

- Single resource
- Reservation requests of form:

  "Can I reserve it from start time $\boldsymbol{s}$ to finish time $\boldsymbol{f}$?"

$$\boldsymbol{s} < \boldsymbol{f}$$

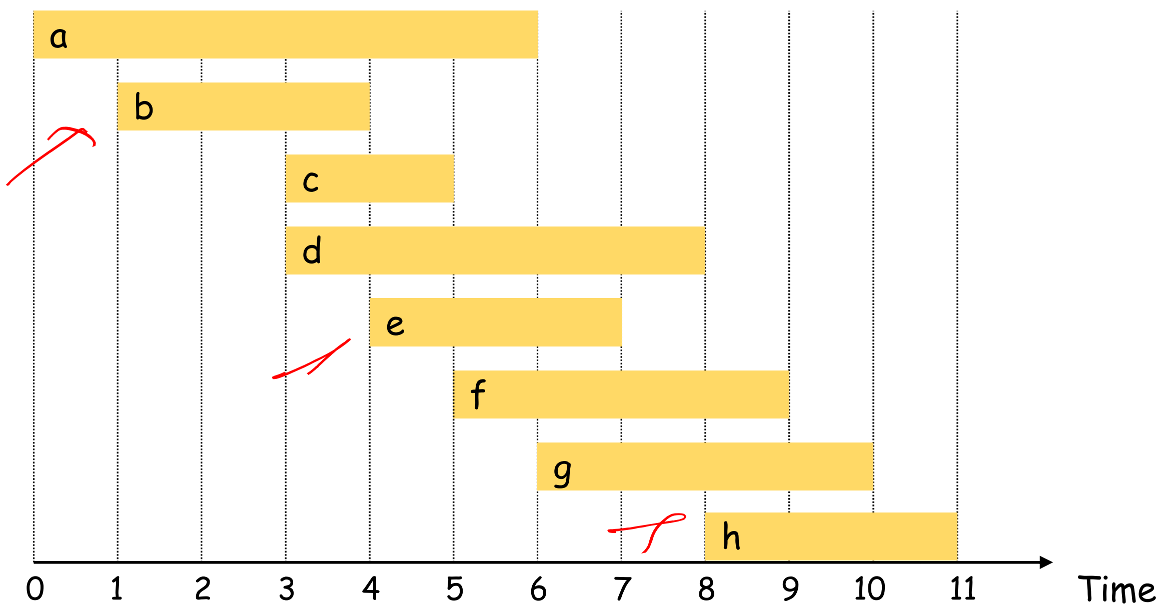
Interval Scheduling

Interval scheduling:

Input: set of jobs with start times and finish times

Goal: find maximum size subset of mutually compatible jobs.

jobs don't overlap



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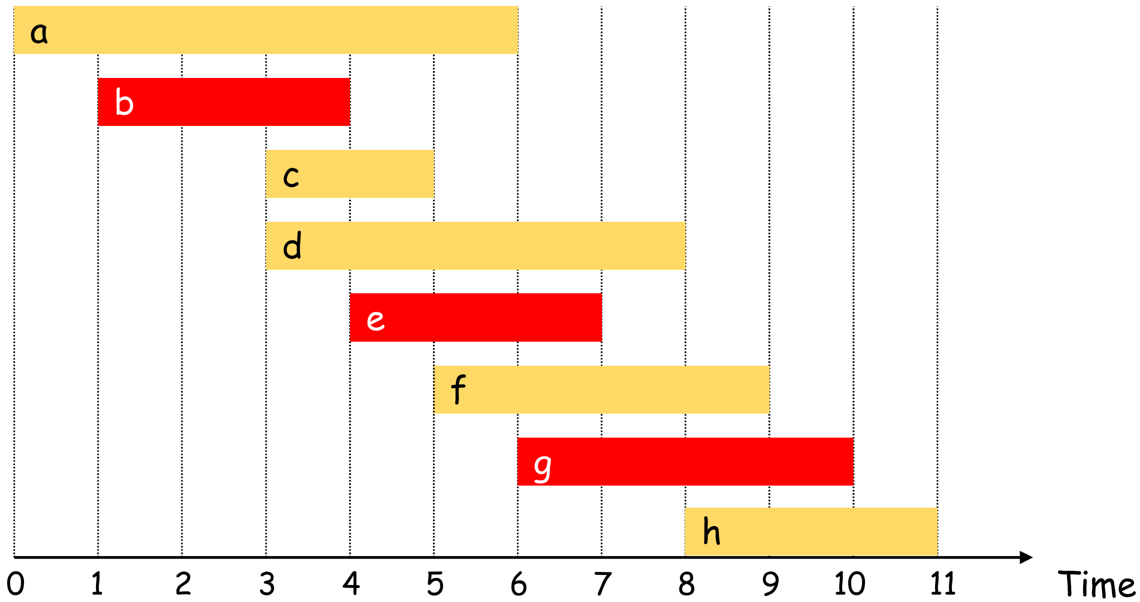


Interval Scheduling

- An optimal solution can be found using a “greedy algorithm”
  - Myopic kind of algorithm that seems to have no look-ahead
  - Greedy algorithms only work when the problem has a special kind of structure
  - When they do work they are typically very efficient

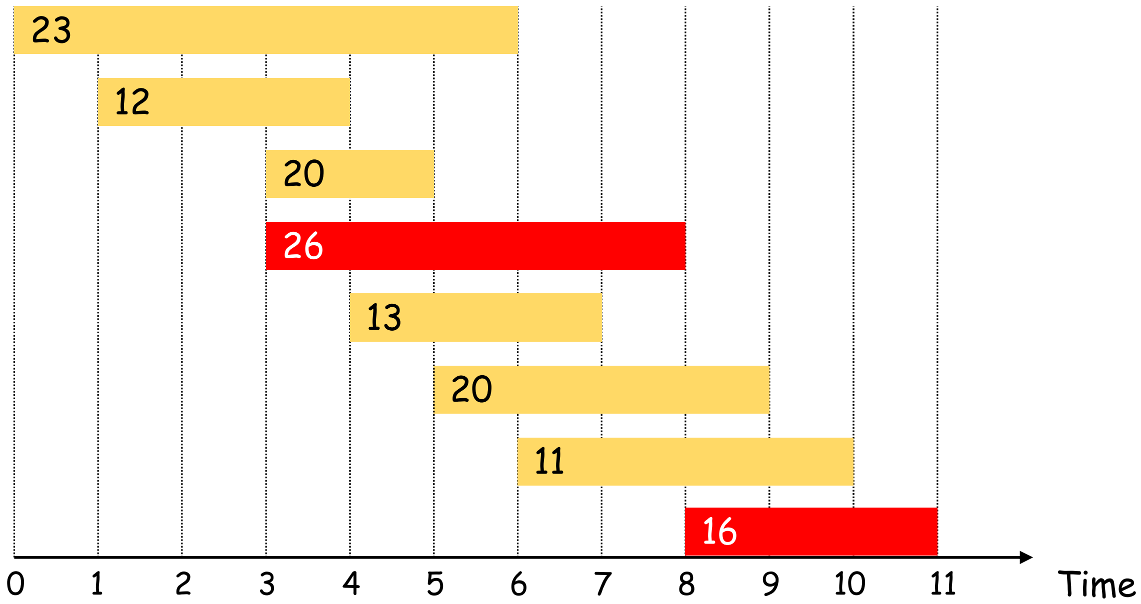
Weighted Interval Scheduling

- Same problem as interval scheduling except that each request i also has an associated value or weight w_i
  - w_i might be
    - amount of money we get from renting out the resource for that time period
    - amount of time the resource is being used

Weighted Interval Scheduling

Input: Set of jobs with start times, finish times, and weights

Goal: Find maximum weight subset of mutually compatible jobs.



Weighted Interval Scheduling

Ordinary interval scheduling is a special case of this problem

- Take all weights $\boldsymbol{w_i} = \mathbf{1}$

Problem is quite different though

- E.g. one weight might dwarf all others

“Greedy algorithms” don’t work

Solution: “Dynamic Programming”

- builds up optimal solutions from a table of solutions to smaller problems

Bipartite Matching

A graph $G = (V, E)$ is bipartite iff

- Set V of vertices has two disjoint parts X and Y
- Every edge in E joins a vertex from X and a vertex from Y

Set $M \subseteq E$ is a matching in G iff no two edges in M share a vertex

Goal: Find a matching M in G of maximum size.

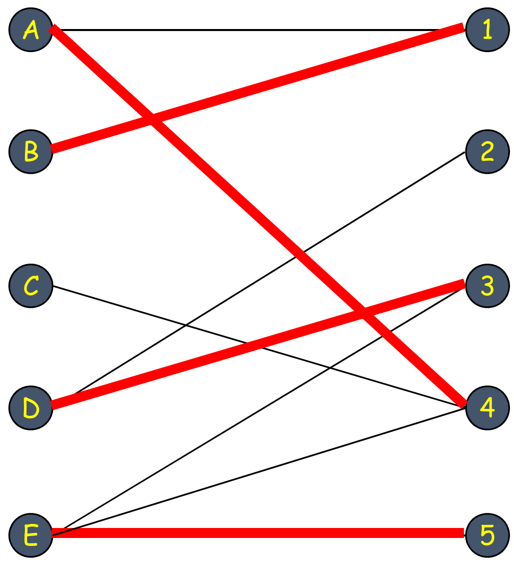
Differences from stable matching

- limited set of possible partners for each vertex
- sides may not be the same size
- no notion of stability; matching everything may be impossible.

Bipartite Matching

Input: Bipartite graph

Goal: Find maximum size matching.



Bipartite Matching

- Models assignment problems
  - X represents customers, Y represents salespeople
  - X represents professors, Y represents courses

- If $|X| = |Y| = n$
  - G has perfect matching iff maximum matching has size n

Solution: polynomial-time algorithm using "augmentation" technique

- Also used for solving more general class of network flow problems

Independent Set

Defn: For graph $G = (V, E)$ a set $I \subseteq V$ is independent iff no two nodes in I are joined by an edge

Input: Graph $G = (V, E)$

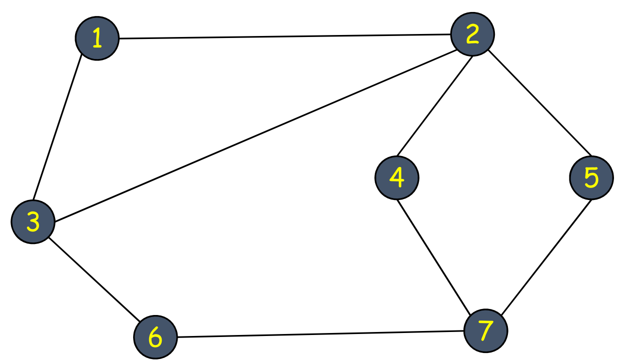
Goal: Find an independent set I in V of maximum possible size

- Models conflicts and mutual exclusion

Independent Set

Input: Graph.

Goal: Find a maximum size independent set.



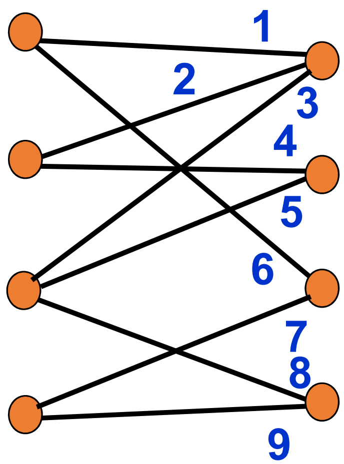
Independent Set

Generalizes

- **Interval Scheduling**
  - Vertices in the graph are the requests
  - Vertices are joined by an edge if they are **not** compatible

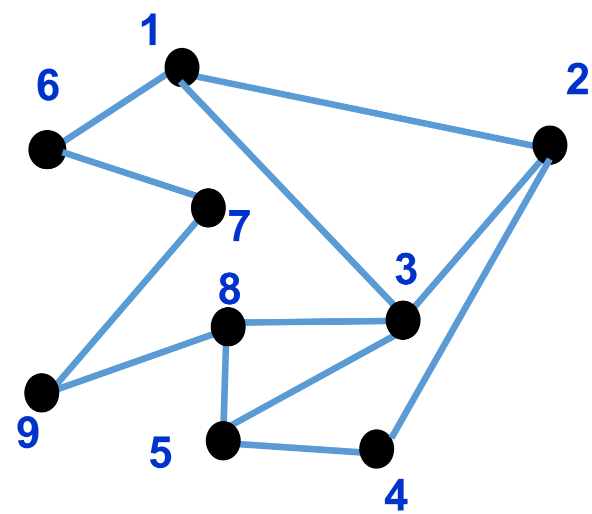
- **Bipartite Matching**
  - Given bipartite graph $G = (V, E)$ create new graph $G' = (V', E')$ (sometimes called the line-graph of G) where
    - $V' = E$
    - Two elements of V' (which are edges in G) are joined iff they touch
  - Independent set I in V' ⇒ no edges in I touch ⇒ I is matching in G

Bipartite Matching



$G = (V, E)$

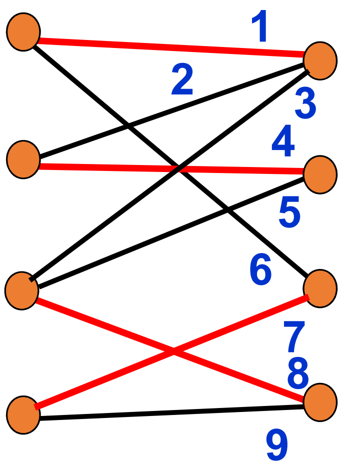
Independent Set



$G' = (V', E')$

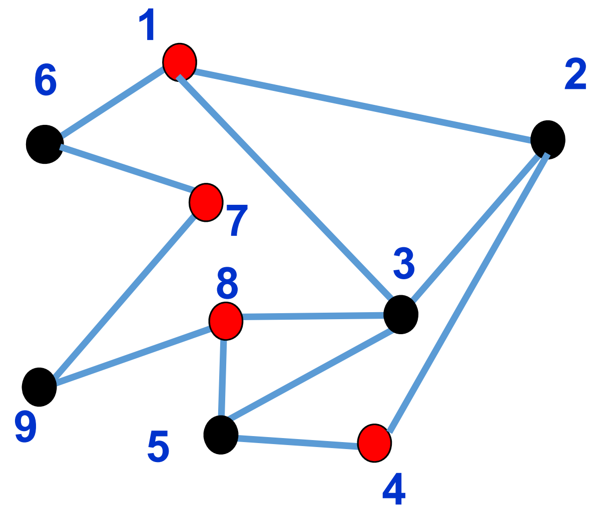
Line graph of G

Bipartite Matching



$G = (V, E)$

Independent Set



$G' = (V', E')$

Independent Set

No polynomial-time algorithm is known ← if we could do this poly-time Algo for 10,000s of

- But to convince someone that there is a large independent set all you'd only need to tell them what the set is
  - they can easily convince themselves that the set is large enough and independent

- Convincing someone that there isn't such a set seems much harder

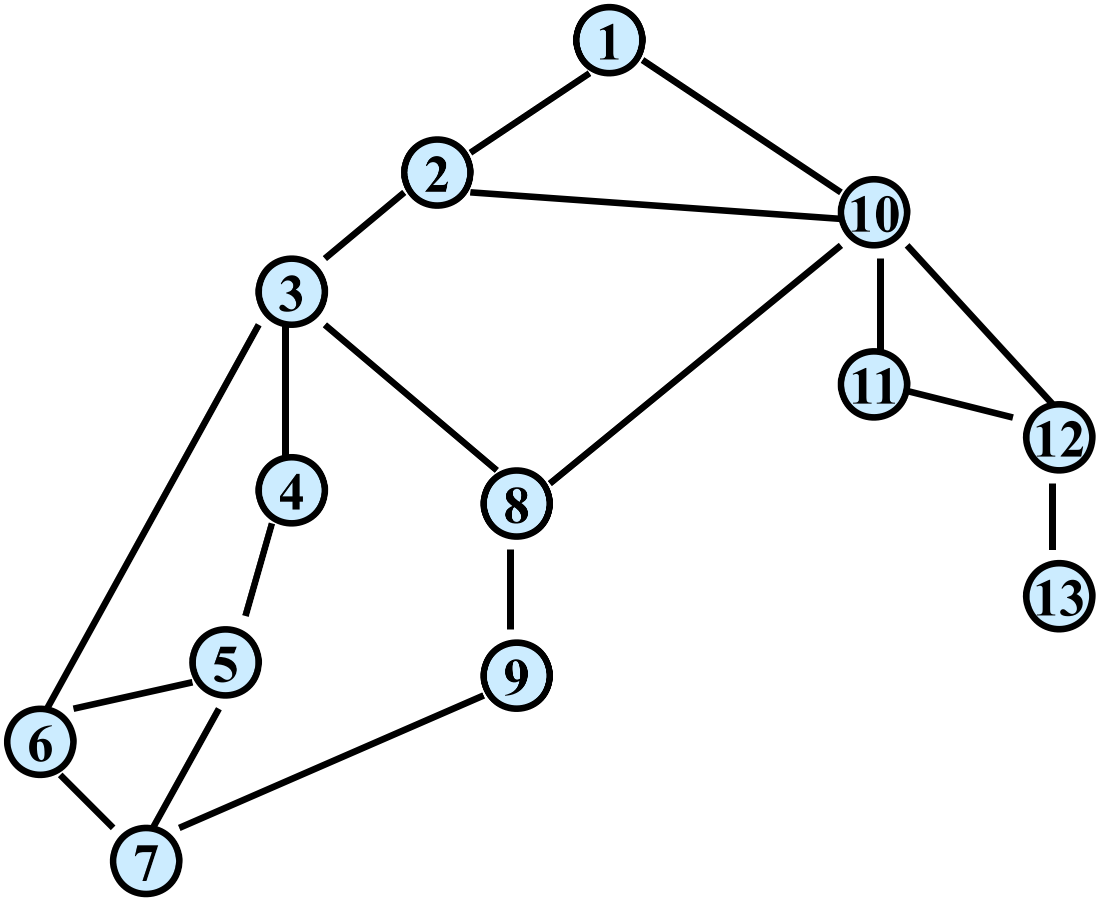
We will show that **Independent Set** is NP-complete

- Class of all the hardest problems that have the property above

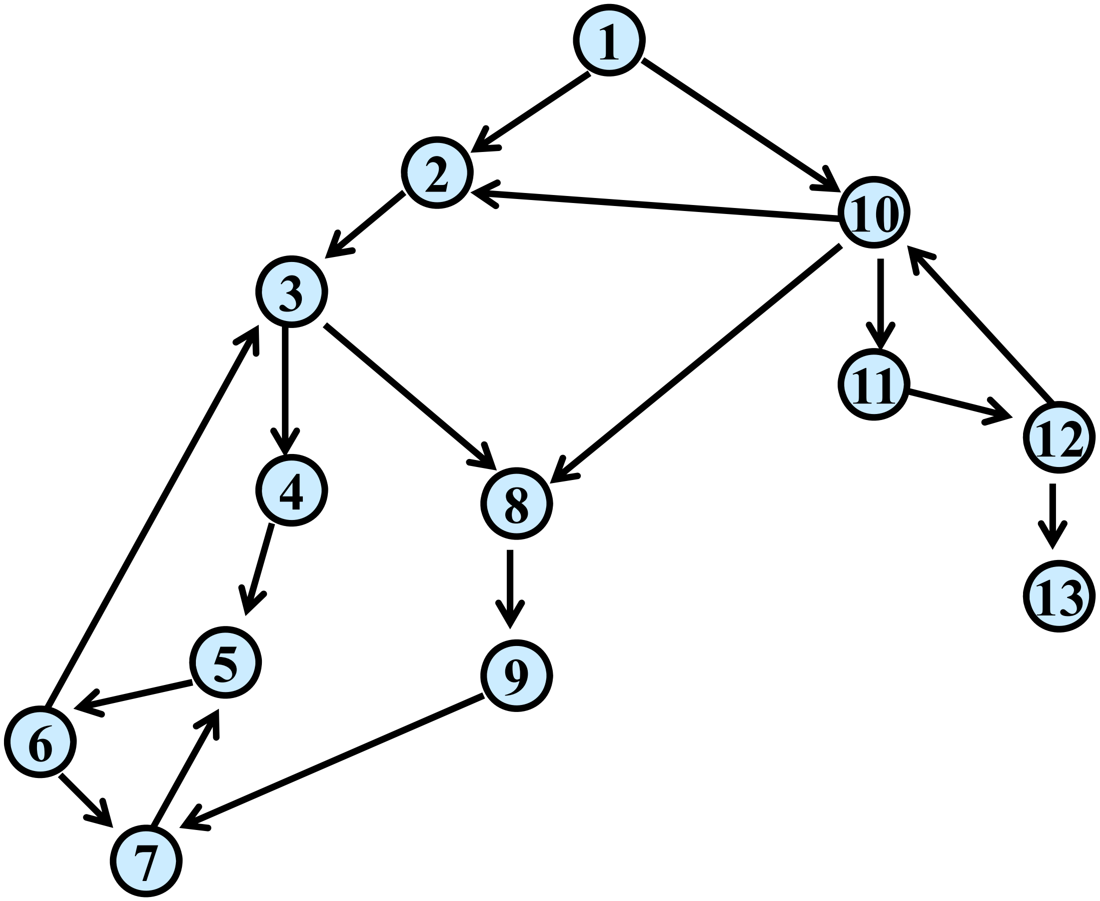
Introduction to Algorithms

- Graph Search/Traversal

Undirected Graph G = (V,E)



Directed Graph G = (V,E)



Graph Traversal

Learn the basic structure of a graph

Walk from a fixed starting vertex s to find all vertices reachable from s

Generic Graph Traversal Algorithm

Given: Graph graph $G = (V, E)$ vertex $s \in V$

Find: set R of vertices reachable from $s \in V$

Reachable(s):

$R \leftarrow \{s\}$

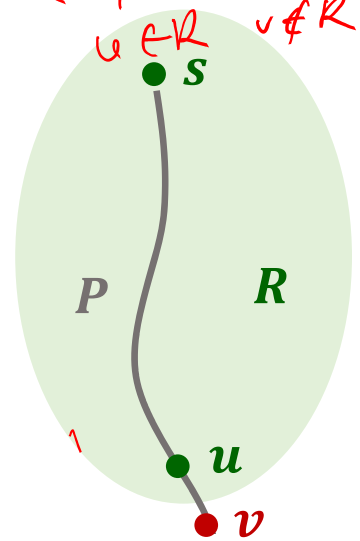
while there is a $(u, v) \in E$ where $u \in R$ and $v \notin R$

Add v to R

return R

Generic Traversal Always Works

Claim: At termination, $\boldsymbol{R}$ is the set of nodes reachable from $\boldsymbol{s}$



Proof

$\subseteq$: For every node $\boldsymbol{v} \in \boldsymbol{R}$ there is a path from $\boldsymbol{s}$ to $\boldsymbol{v}$

- Easy induction based on edges found.

$\supseteq$: Suppose there is a node $\boldsymbol{w} \notin \boldsymbol{R}$ reachable from $\boldsymbol{s}$ via a path $\boldsymbol{P}$

- Take first node $\boldsymbol{v}$ on $\boldsymbol{P}$ such that $\boldsymbol{v} \notin \boldsymbol{R}$
- Predecessor $\boldsymbol{u}$ of $\boldsymbol{v}$ in $\boldsymbol{P}$ satisfies
  - $\boldsymbol{u} \in \boldsymbol{R}$
  - $(\boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{E}$
- But this contradicts the fact that the algorithm exited the while loop. ■

Graph Traversal

Learn the basic structure of a graph

Walk from a fixed starting vertex s to find all vertices reachable from s

Three states of vertices

- **unvisited**
- **visited/discovered** (in R)
- **fully-explored** (in R and all neighbors in R)

Breadth-First Search

Completely explore the vertices in order of their distance from s

Naturally implemented using a queue

BFS(s)

Global initialization: mark all vertices "unvisited"

BFS(s)

mark s "visited"; $R \leftarrow \{s\}$; layer $L_0 \leftarrow \{s\}$; $i \leftarrow 0$

while L_i not empty

$L_{i+1} \leftarrow \varnothing$

for each $u \in L_i$

for each edge (u, v)

if (v is "unvisited")

mark v "visited"

Add v to set R and to layer L_{i+1}

mark u "fully-explored"

$i \leftarrow i + 1$

L_i distance i

BFSparent[v]

BFSparent[v] ← u

Properties of BFS

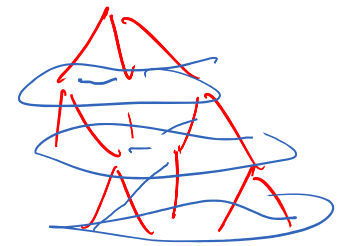
BFS(s) visits x iff there is a path in G from s to x.

Edges followed to undiscovered vertices define a breadth first spanning tree of G

Layer i in this tree:

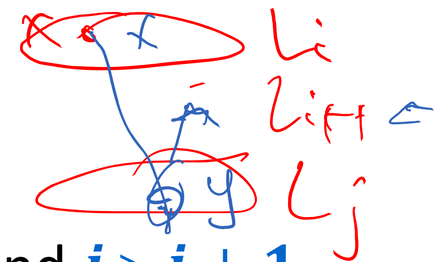
L_i = set of vertices u with shortest path in G from root s of length i.

Properties of BFS



Claim: For undirected graphs:

All edges join vertices on the same or adjacent layers of BFS tree



Proof: Suppose not...

Then there would be vertices (x, y) s.t. $x \in L_i$ and $y \in L_j$ and $j > i + 1$.

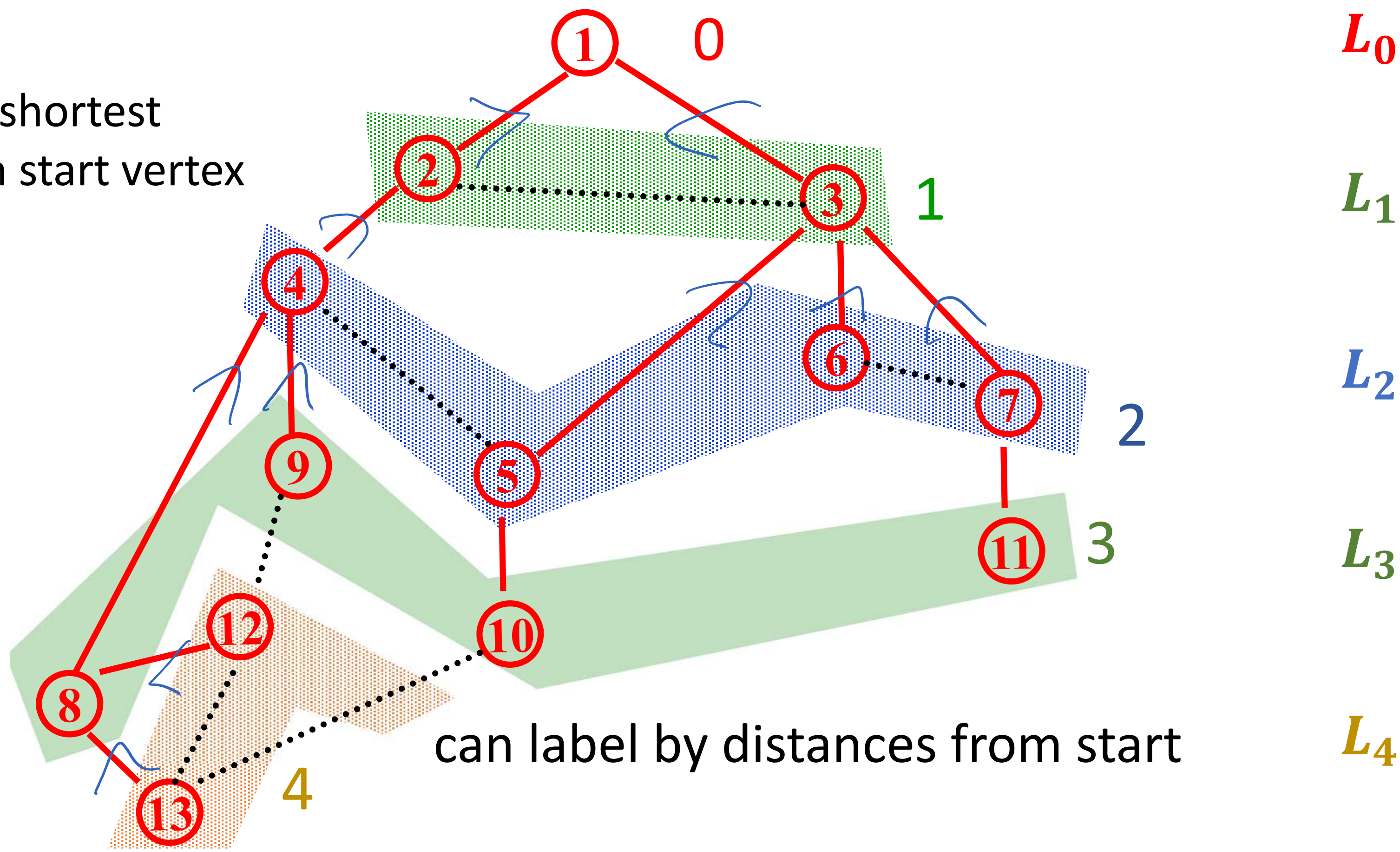
Then, when vertices adjacent to x are considered in BFS,

y would be added to L_{i+1} and not to L_j.

Contradiction. ■

BFS Application: Shortest Paths

Tree gives shortest paths from start vertex



can label by distances from start

Undirected Graph Search Application: Connected Components

Want to answer questions of the form:

Given: vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\boldsymbol{G}$

Is there a path from $\boldsymbol{u}$ to $\boldsymbol{v}$?