CSE 421
Introduction to Algorithms

Lecture 4: BFS, DFS Properties/Applications, Topological Sort

HW1 Due today
Submit completed problems early
Don't wait until the end
HW2 out tonight

Undirected Graph Search Application: Connected Components

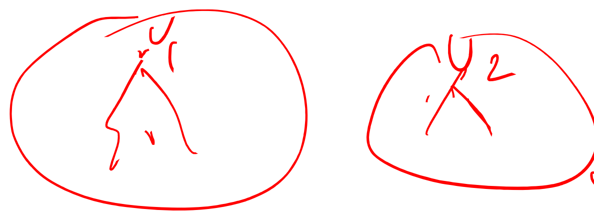
Want to answer questions of the form:

Given: vertices u and v in G

Is there a path from u to v?

Table

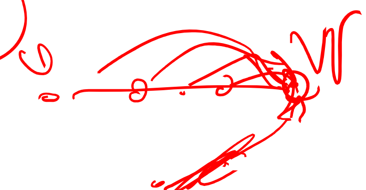
Path[u,v]



Idea: create array A s.t

$A[u]$ = smallest numbered vertex connected to u

smallest



Answer is yes iff $A[u] = A[v]$

$O(n)$

Q: Why is this better than an array $\mathbf{Path}[u, v]$?

$O(n^2)$

Undirected Graph Search Application: Connected Components

Initial state: all v unvisited

for $s \leftarrow 1$ to n do

  if state(s) $\neq$ fully-explored then

    BFS(s): setting $A[u] \leftarrow s$ for each u found

      (and marking u visited/fully-explored)

endfor

$\le n(n+m)$

Total cost: $O(n+m)$

- Each vertex is touched once in outer procedure and edges examined in different BFS runs are disjoint
- Works also with Depth First Search ...

DFS(u) – Recursive Procedure

Global Initialization: mark all vertices "unvisited"

DFS($\boldsymbol{u}$)

mark $\boldsymbol{u}$ “visited” and add $\boldsymbol{u}$ to $\boldsymbol{R}$

for each edge $(\boldsymbol{u}, \boldsymbol{v})$

if ($\boldsymbol{v}$ is “unvisited”)

DFS($\boldsymbol{v}$) ← parent(v) ← u

end for

mark $\boldsymbol{u}$ “fully-explored”

Properties of DFS(s)

Like BFS(s):

- DFS(s) visits x iff there is a path in G from s to x
- Edges into undiscovered vertices define depth-first spanning tree of G

Unlike the BFS tree:

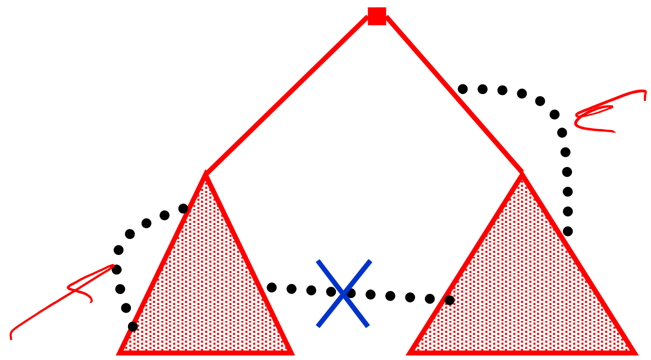
- the DFS spanning tree *isn't* minimum depth
- its levels *don't* reflect min distance from the root
- non-tree edges *never* join vertices on the same or adjacent levels

BUT…

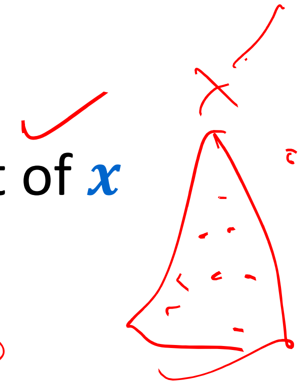
Non-tree edges in DFS tree of undirected graphs

Claim: All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree

- In other words ... No “cross edges”.



No cross edges in DFS on undirected graphs



Claim: During DFS($\boldsymbol{x}$) every vertex marked "visited" is a descendant of $\boldsymbol{x}$ in the DFS tree $\boldsymbol{T}$

Claim: For every $\boldsymbol{x}, \boldsymbol{y}$ in the DFS tree $\boldsymbol{T}$, if $(\boldsymbol{x}, \boldsymbol{y})$ is an edge *not* in $\boldsymbol{T}$ then one of $\boldsymbol{x}$ or $\boldsymbol{y}$ is an ancestor of the other in $\boldsymbol{T}$

Proof:

- One of DFS($\boldsymbol{x}$) or DFS($\boldsymbol{y}$) is called first, suppose WLOG that DFS($\boldsymbol{x}$) was called before DFS($\boldsymbol{y}$)
- During DFS($\boldsymbol{x}$), the edge $(\boldsymbol{x}, \boldsymbol{y})$ is examined
- Since $(\boldsymbol{x}, \boldsymbol{y})$ is a *not* an edge of $\boldsymbol{T}$, $\boldsymbol{y}$ was already visited when edge $(\boldsymbol{x}, \boldsymbol{y})$ was examined during DFS($\boldsymbol{x}$)
- Therefore $\boldsymbol{y}$ was visited during the call to DFS($\boldsymbol{x}$) so $\boldsymbol{y}$ is a descendant of $\boldsymbol{x}$. ■

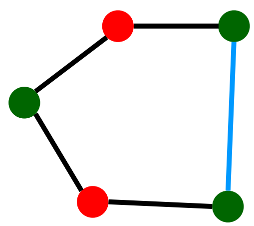
Applications of Graph Traversal: Bipartiteness Testing

Definition: An undirected graph G is bipartite iff we can color its vertices **red** and **green** so each edge has different color endpoints

Input: Undirected graph G
Goal: If G is bipartite, output a coloring;
otherwise, output "NOT Bipartite".

Fact: Graph G contains an odd-length cycle $\Rightarrow$ it is not bipartite

Just coloring the cycle part of G is impossible



On a cycle the two colors must alternate, so
- **green** every 2nd vertex
- **red** every 2nd vertex

Can't have either if length is not divisible by 2.

Applications of Graph Traversal: Bipartiteness Testing

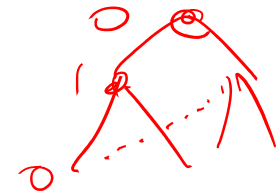
WLOG ("without loss of generality"): Can assume that G is connected

- Otherwise run on each component

Simple idea: start coloring nodes starting at a given node s

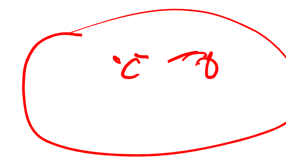
- Color s **red**
- Color all neighbors of s **green**
- Color all their neighbors **red**, etc.
- If you ever hit a node that was already colored
  - the **same** color as you want to color it, ignore it
  - the **opposite** color, output "NOT Bipartite" and halt

BFS gives Bipartiteness



Run BFS assigning all vertices from layer $\boldsymbol{L_i}$ the color $\boldsymbol{i} \bmod \mathbf{2}$

- i.e., **red** if they are in an even layer, **green** if in an odd layer



- if no edge joining two vertices of the same color
  - then it is a good coloring
- otherwise
  - there is a bad edge; output “Not Bipartite”

Why is that “Not Bipartite” output correct?

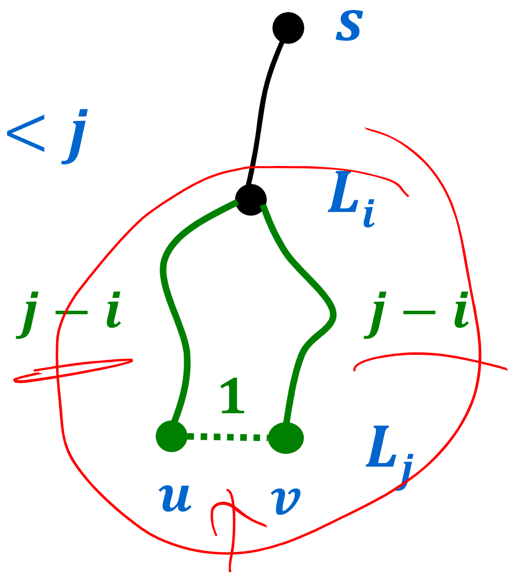
Why does BFS work for Bipartiteness?

Recall: All edges join vertices on the same or adjacent BFS layers

$\Rightarrow$ Any bad edge must join two vertices u and v in the same layer

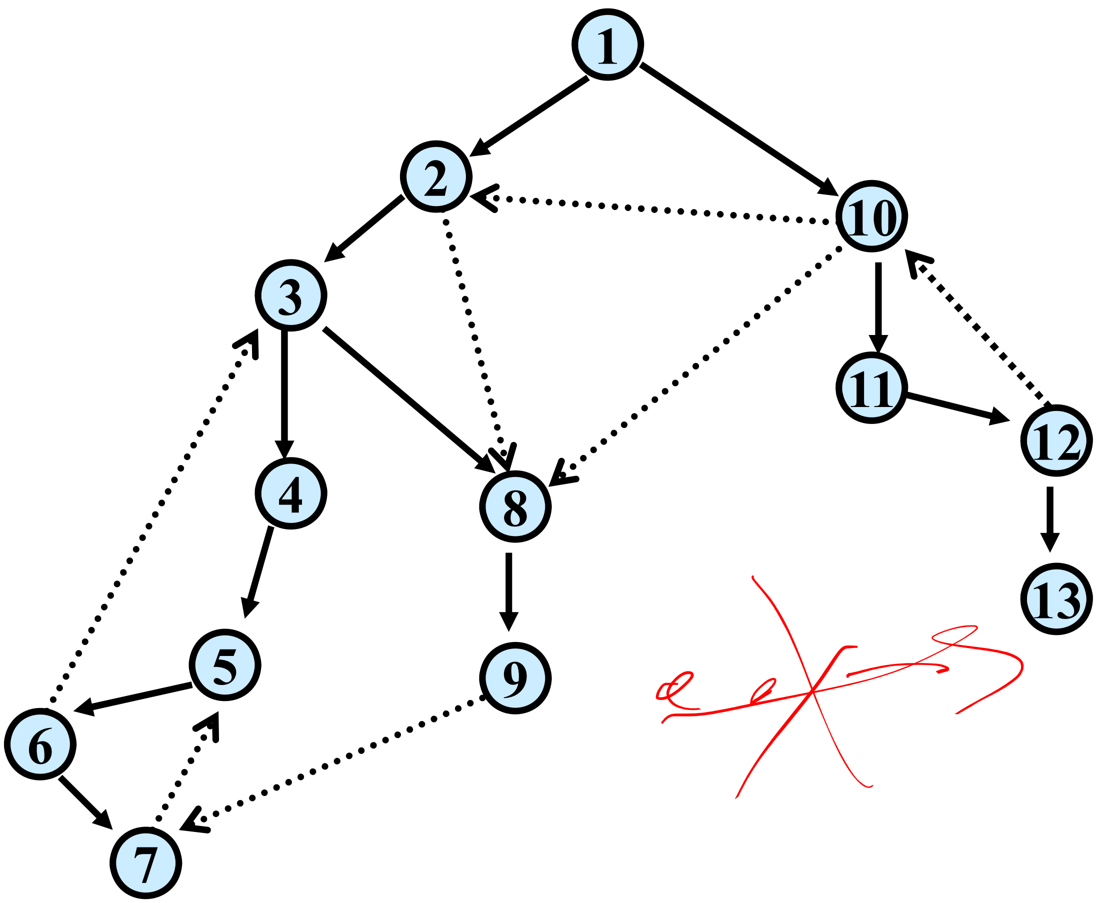
Say the layer with u and v is L_j

u and v have common ancestor at some level L_i for $i < j$

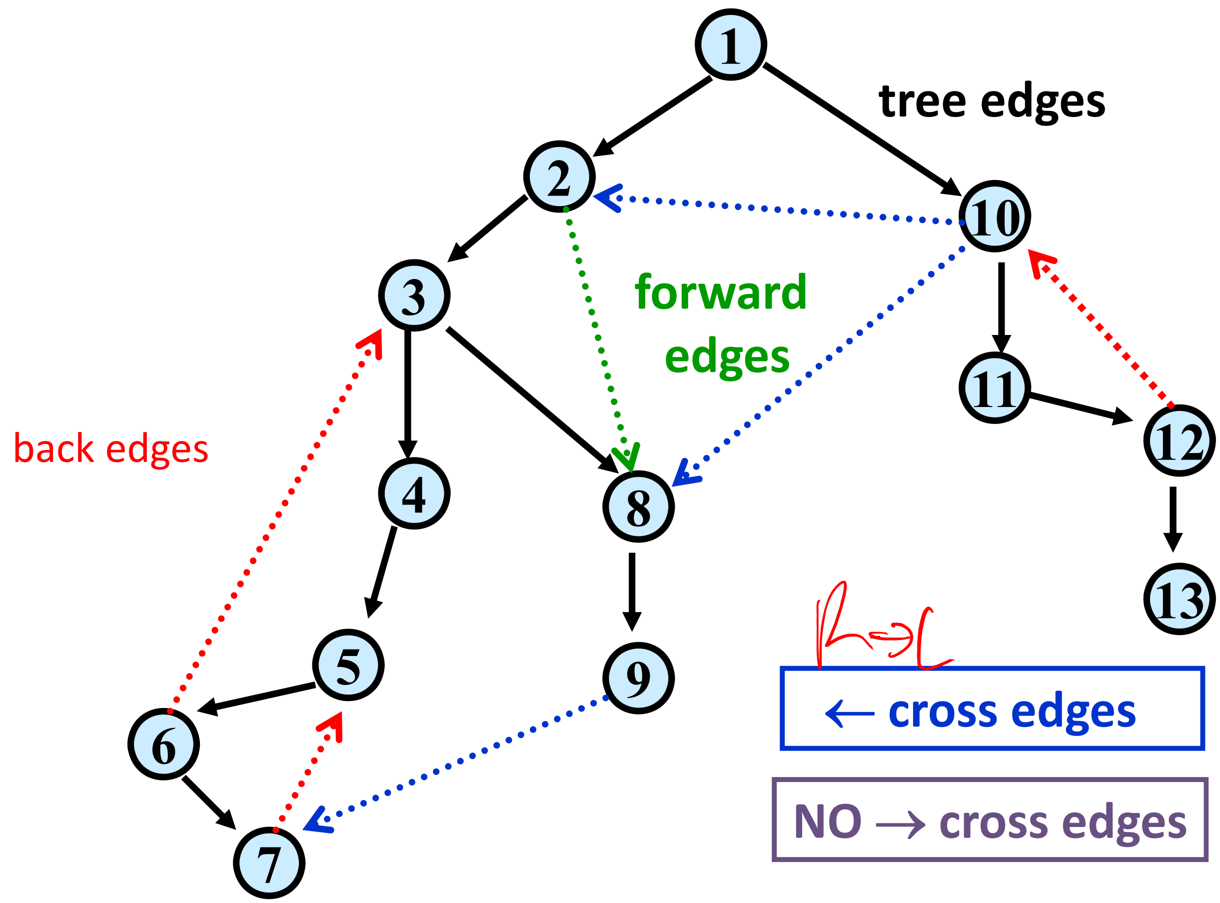


$\boxed{\text{Odd cycle of length } 2(j-i)+1 \Rightarrow \text{Not Bipartite}}$

DFS(v) for a directed graph



DFS(v)


1
2
3
4
5
6
7
8
9
10
11
12
13
tree edges
forward edges
back edges
R→L
← cross edges
NO → cross edges

Properties of Directed DFS

- Before DFS(s) returns, it visits all previously unvisited vertices reachable via directed paths from s

directed

- Every cycle contains a back edge in the DFS tree

Strongly Connected Components of Directed Graphs

Defn: Vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ are strongly connected iff they are on a directed cycle (there are paths from $\boldsymbol{u}$ to $\boldsymbol{v}$ and from $\boldsymbol{v}$ to $\boldsymbol{u}$).

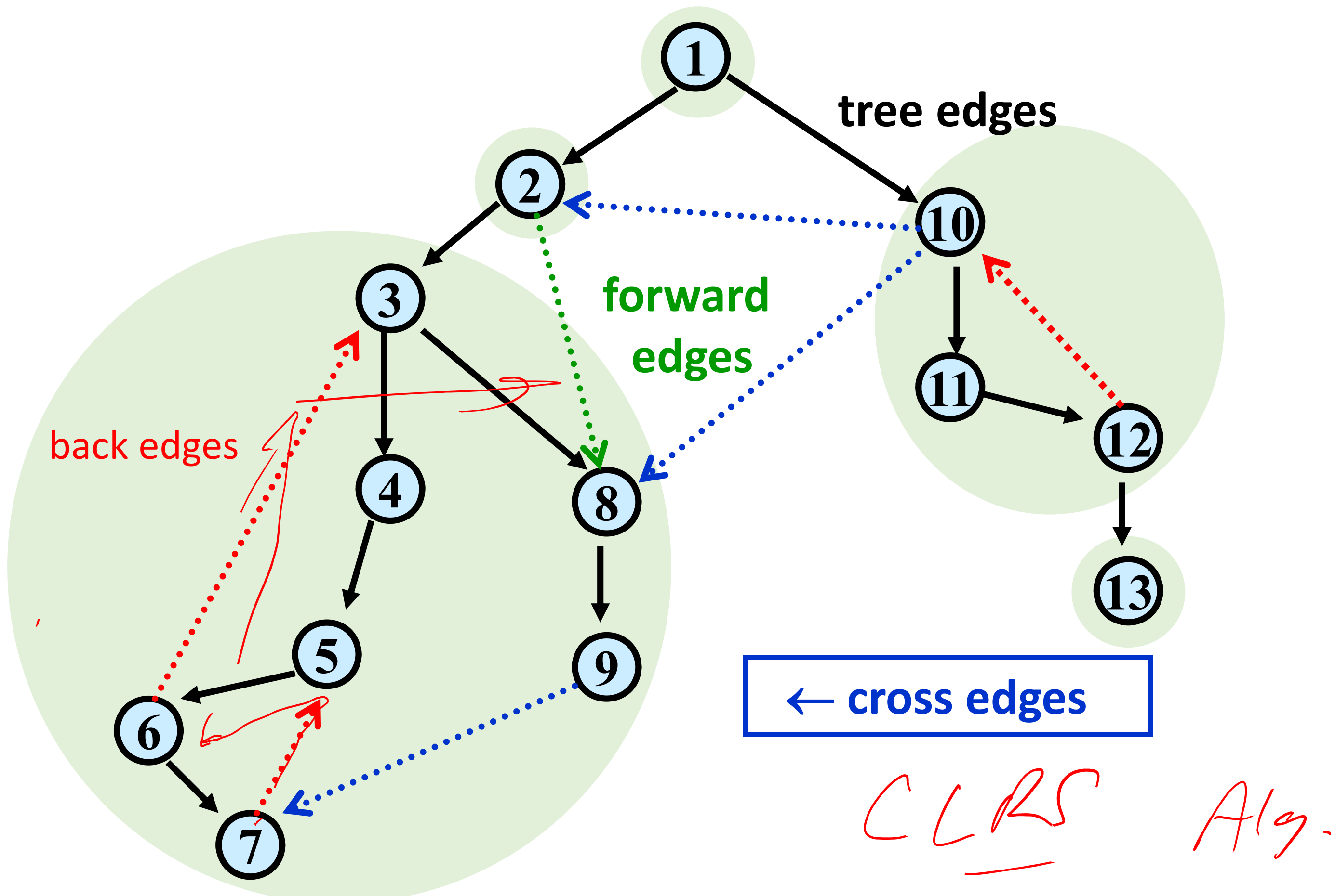
$u \rightleftarrows v$

Defn: Can partition vertices of any directed graph into strongly connected components:

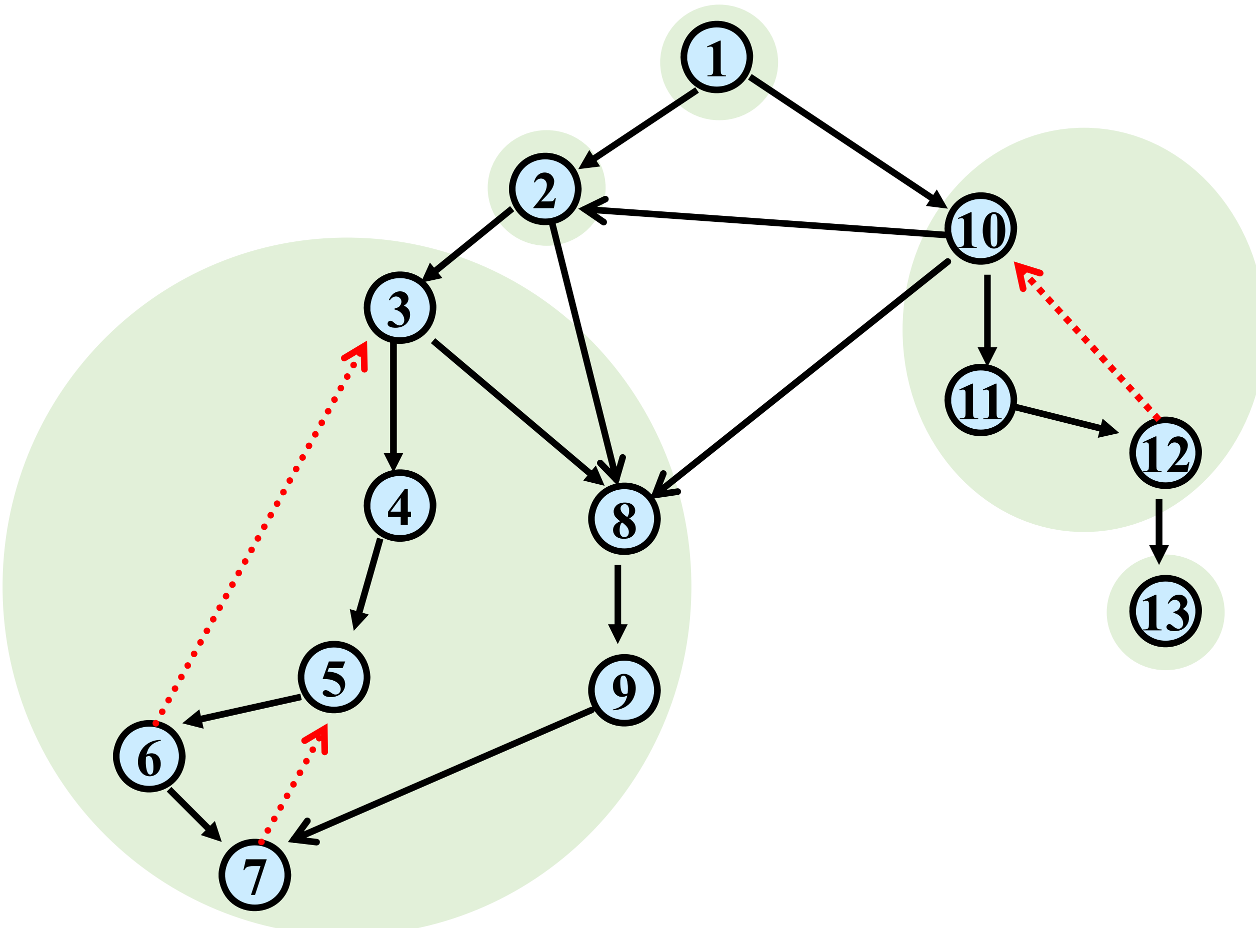
1. all pairs of vertices in the same component are strongly connected
2. can't merge components and keep property 1

- Strongly connected components can be stored efficiently just like connected components
- Can be found by extending DFS algorithm in $O(\boldsymbol{n} + \boldsymbol{m})$ time using extra bookkeeping
  - We won't cover the details

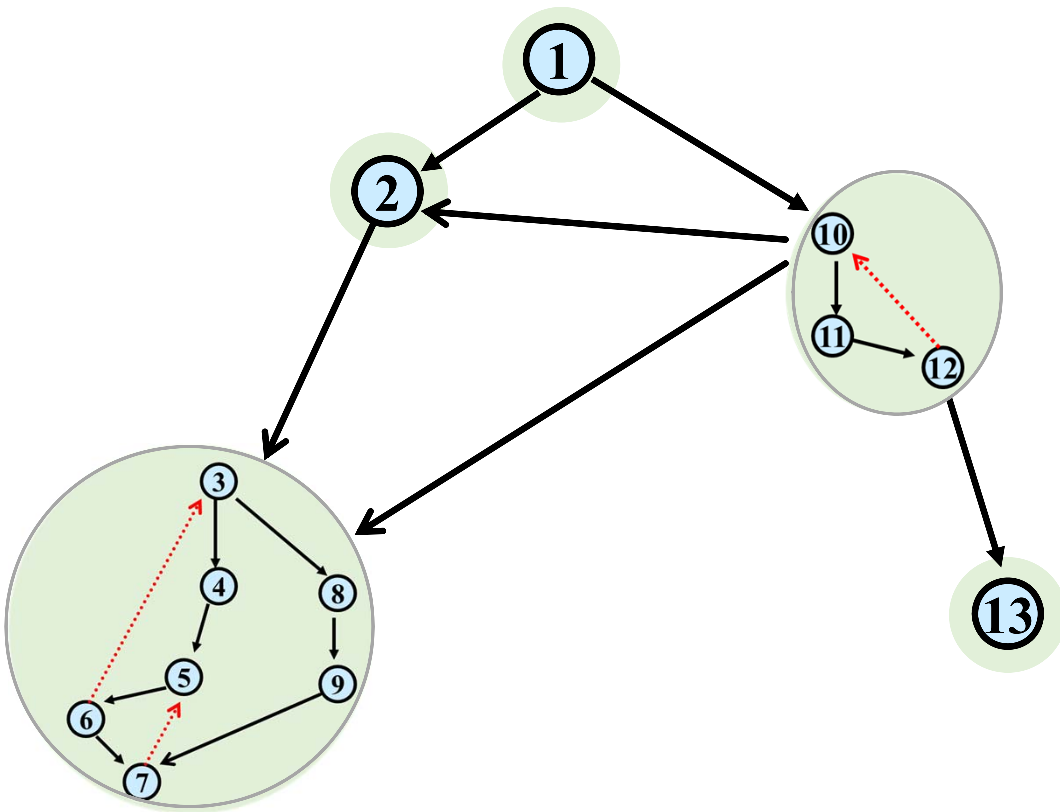
Strongly Connected Components



Strongly Connected Components



Strongly Connected Components



no cycles in reduced graph

Directed Acyclic Graphs

A directed graph $G = (V, E)$ is acyclic iff it has no directed cycles

Terminology: A directed acyclic graph is also called a DAG

After shrinking the strongly connected components of a directed graph to single vertices, the result is a DAG

Topological Sort

Given: a directed acyclic graph (DAG) $\boldsymbol{G = (V, E)}$

Output: numbering of the vertices of $\boldsymbol{G}$ with distinct numbers from $\mathbf{1}$ to $\boldsymbol{n}$ so that edges only go from lower numbered to higher numbered vertices

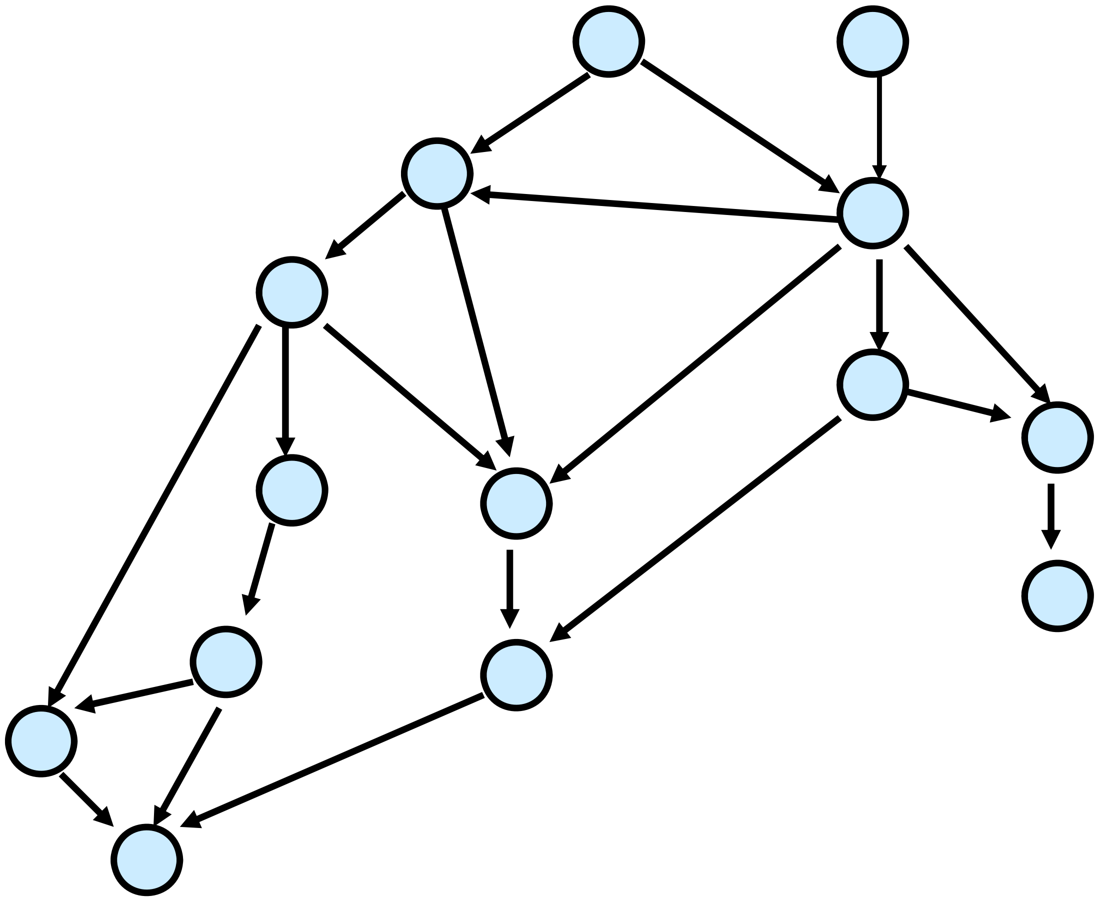
Applications:

- nodes represent tasks
- edges represent precedence between tasks
- topological sort gives a sequential schedule for solving them

Nice algorithmic paradigm for general directed graphs:

- Process strongly connected components one-by-one in the order given by topological sort of the DAG you get from shrinking them.

Directed Acyclic Graph



In-degree 0 vertices

Claim: Every DAG has a vertex of in-degree 0

Proof: By contradiction

Suppose every vertex has some incoming edge

Consider following procedure:

while (true) do

$\quad \boldsymbol{v} \leftarrow$ some predecessor of $\boldsymbol{v}$

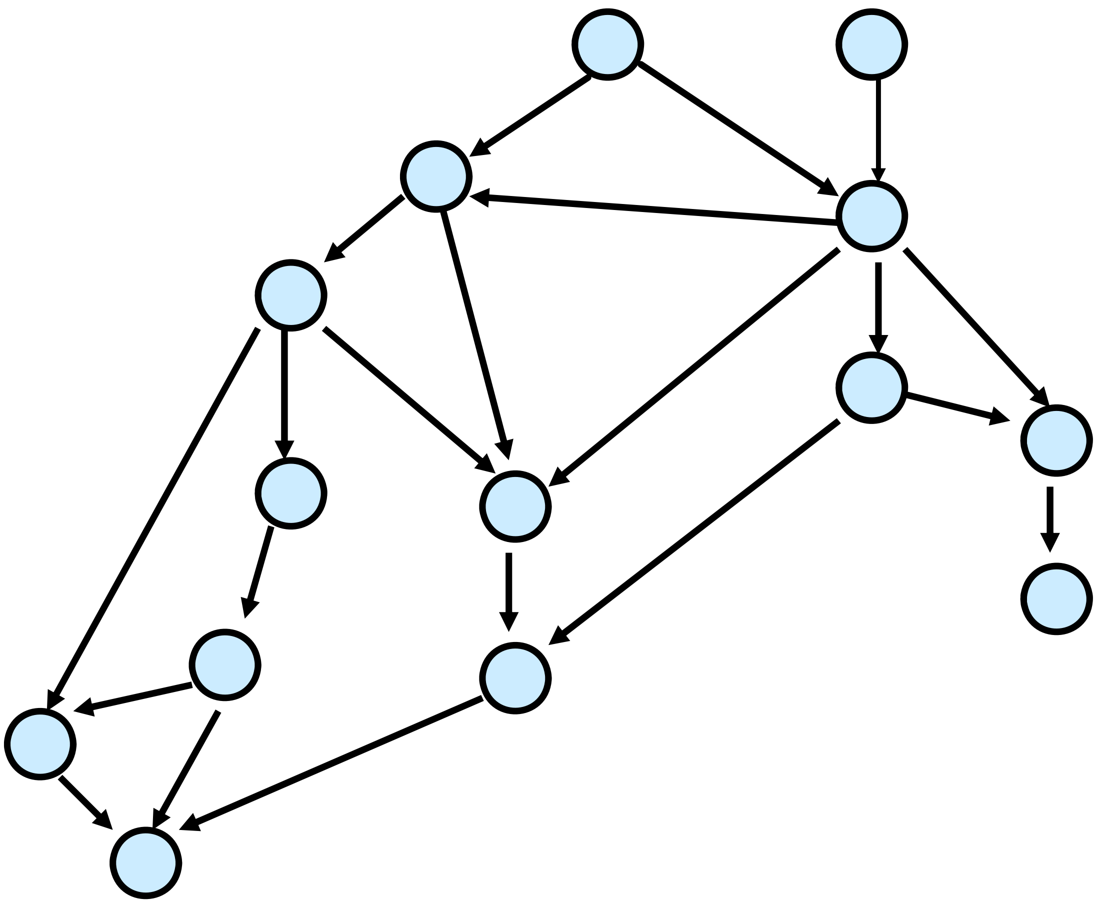
- After $\boldsymbol{n+1}$ steps where $\boldsymbol{n = |V|}$ there will be a repeated vertex
  - This yields a cycle, contradicting that it is a DAG. ■

Topological Sort

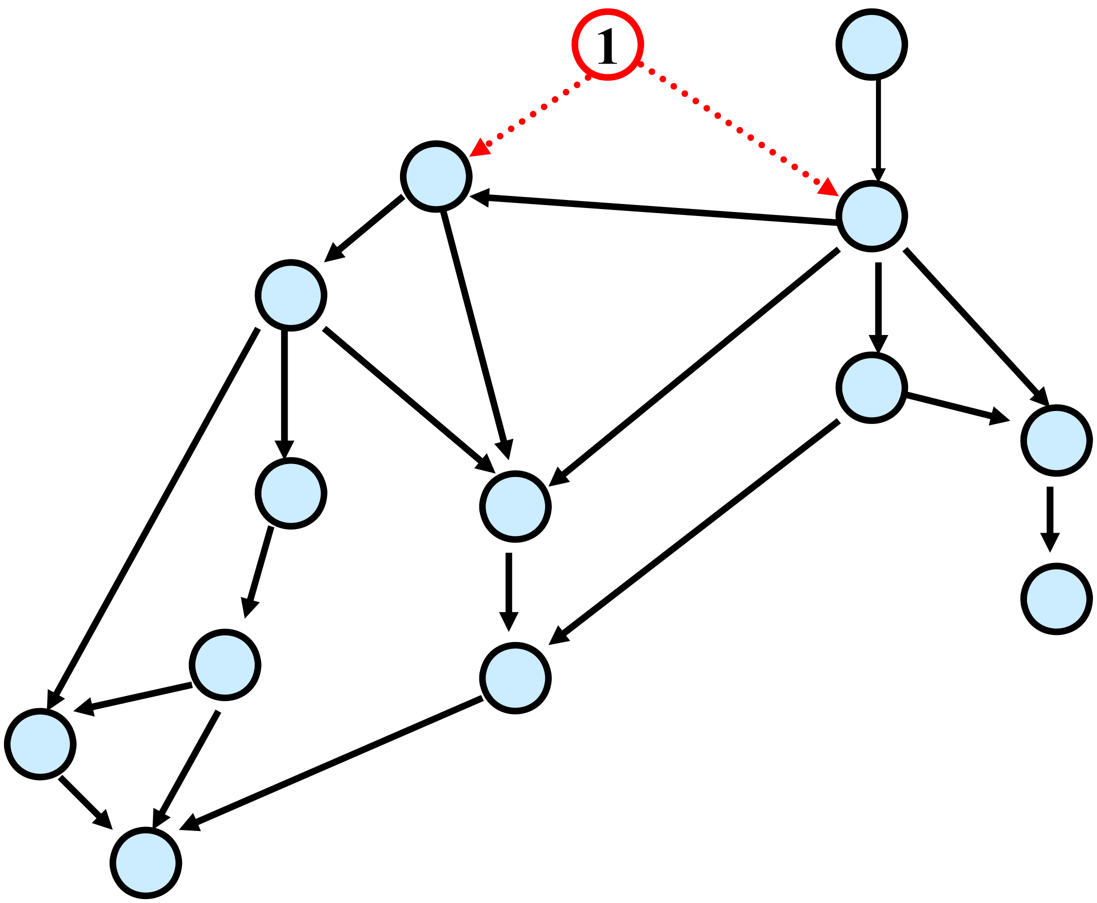
- Can do using DFS

- Alternative simpler idea:
  - Any vertex of in-degree 0 can be given number 1 to start
  - Remove it from the graph
  - Then give a vertex of in-degree 0 number 2
  - Etc.

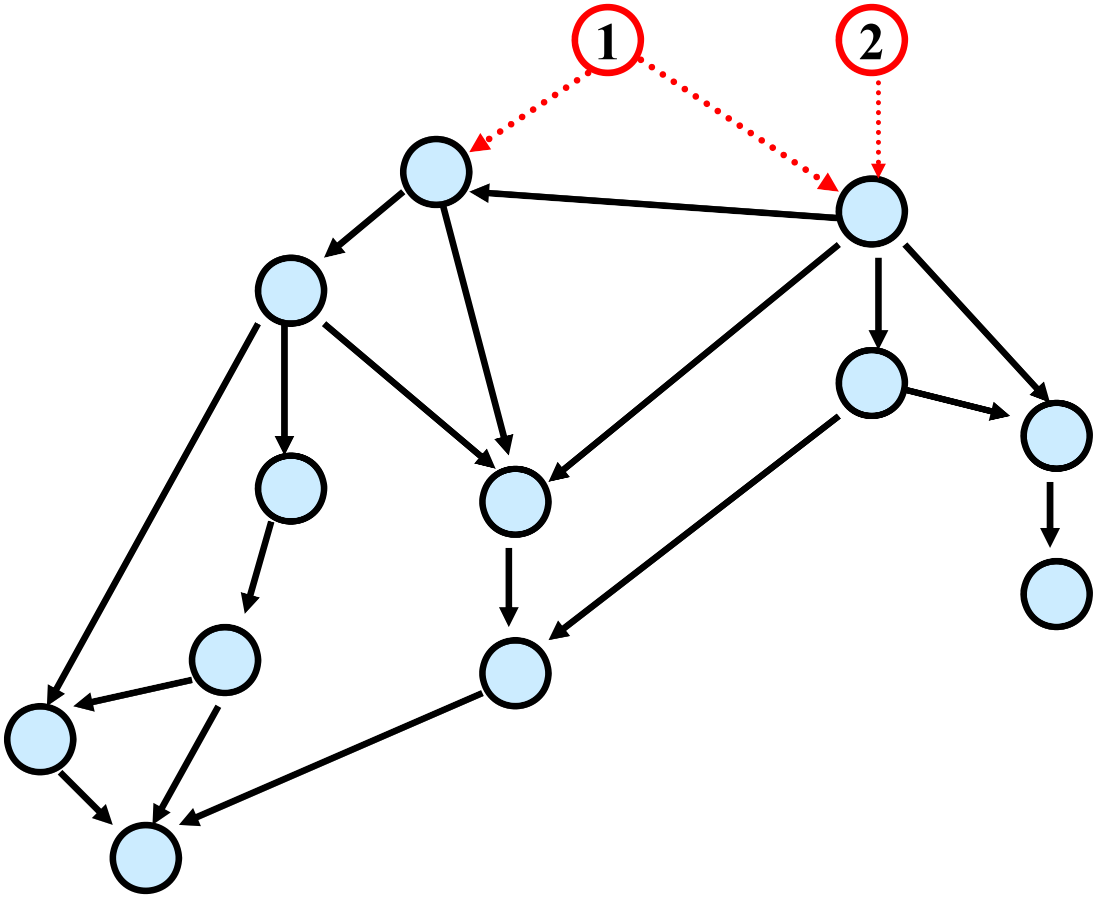
Topological Sort



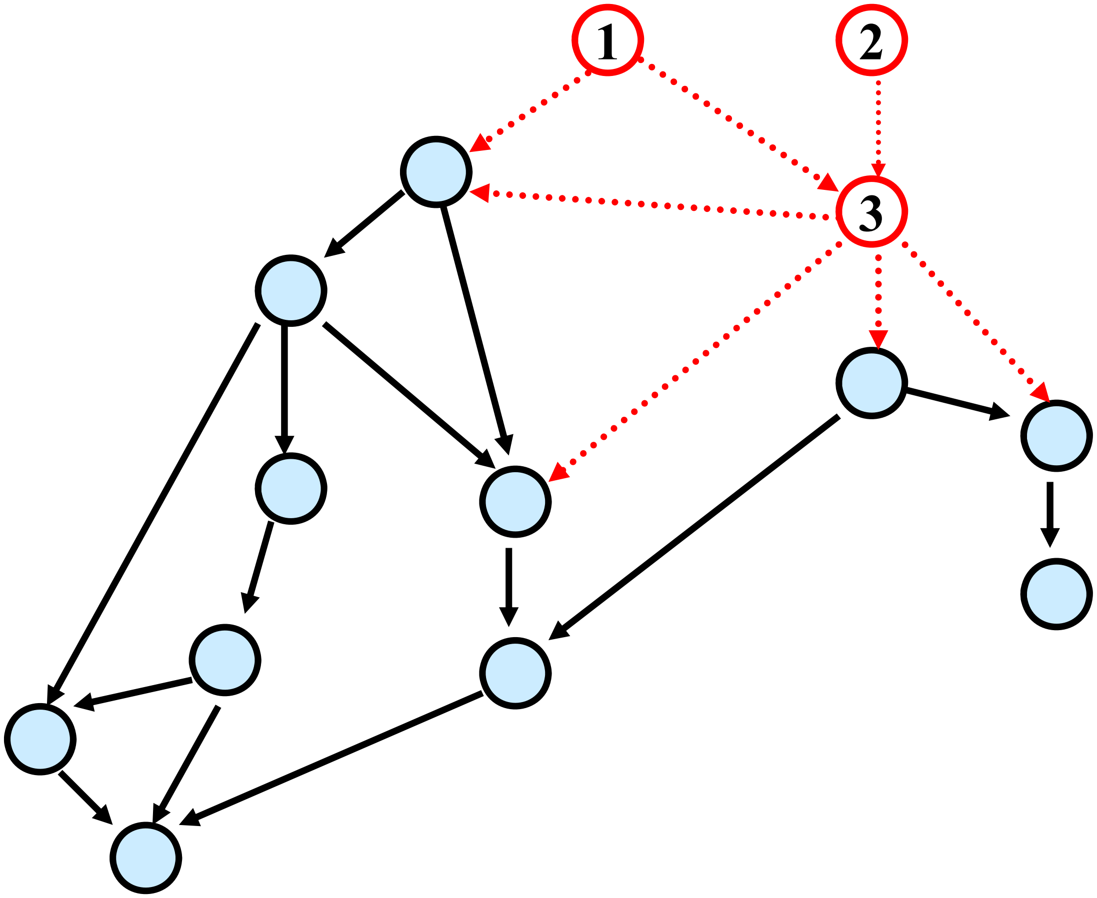
Topological Sort



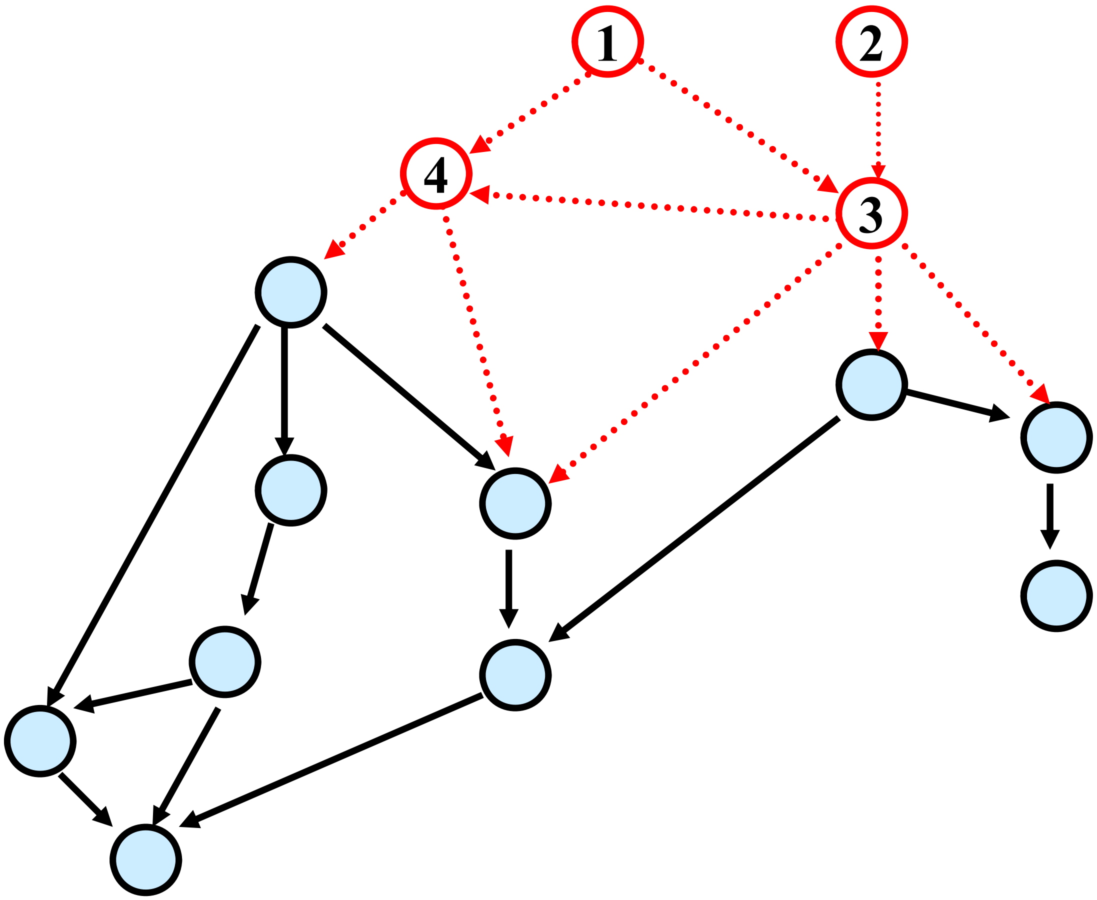
Topological Sort



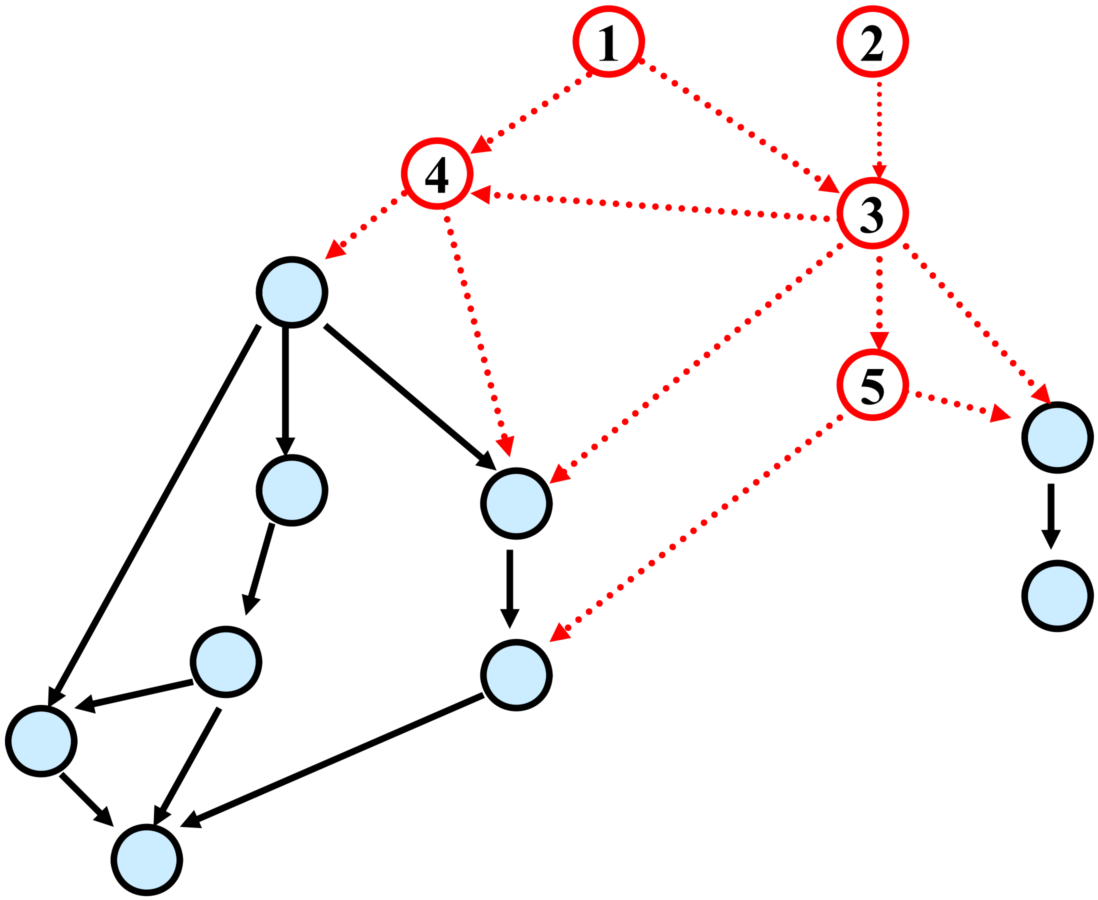
Topological Sort



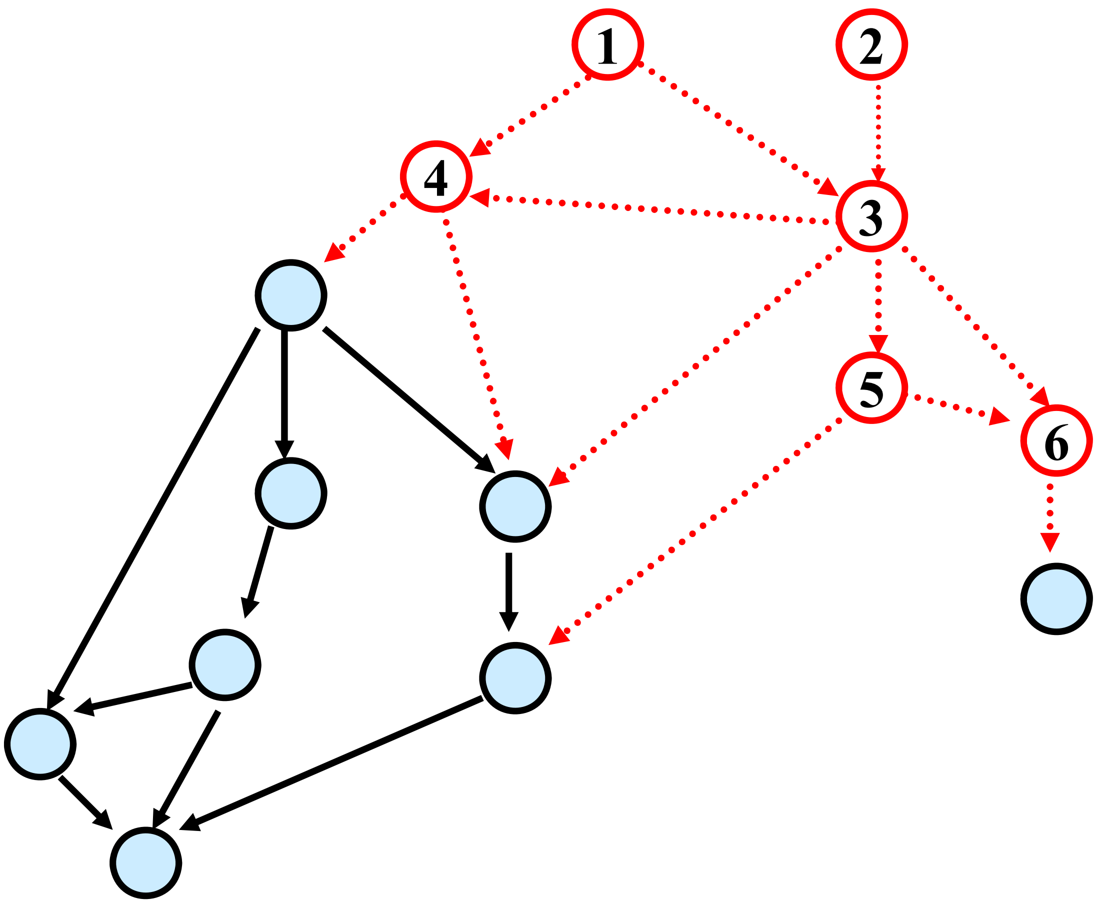
Topological Sort



Topological Sort



Topological Sort



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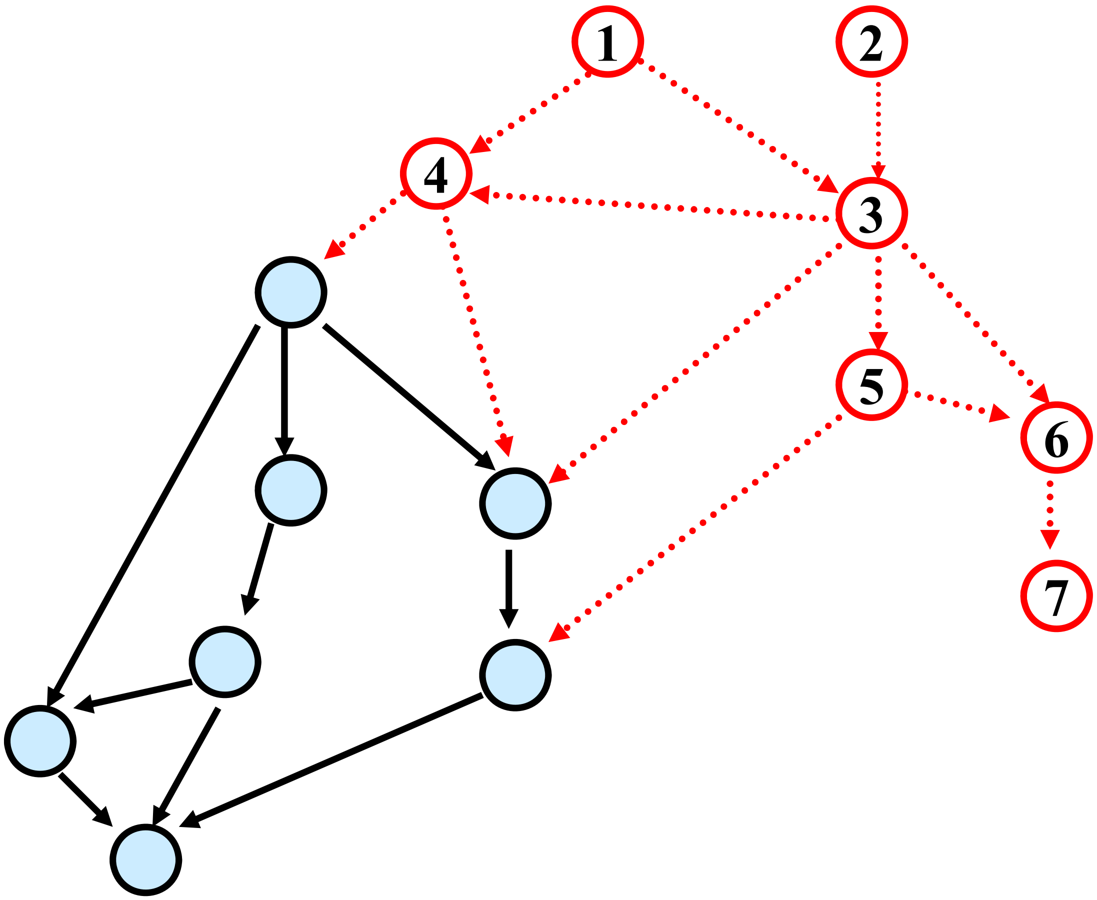
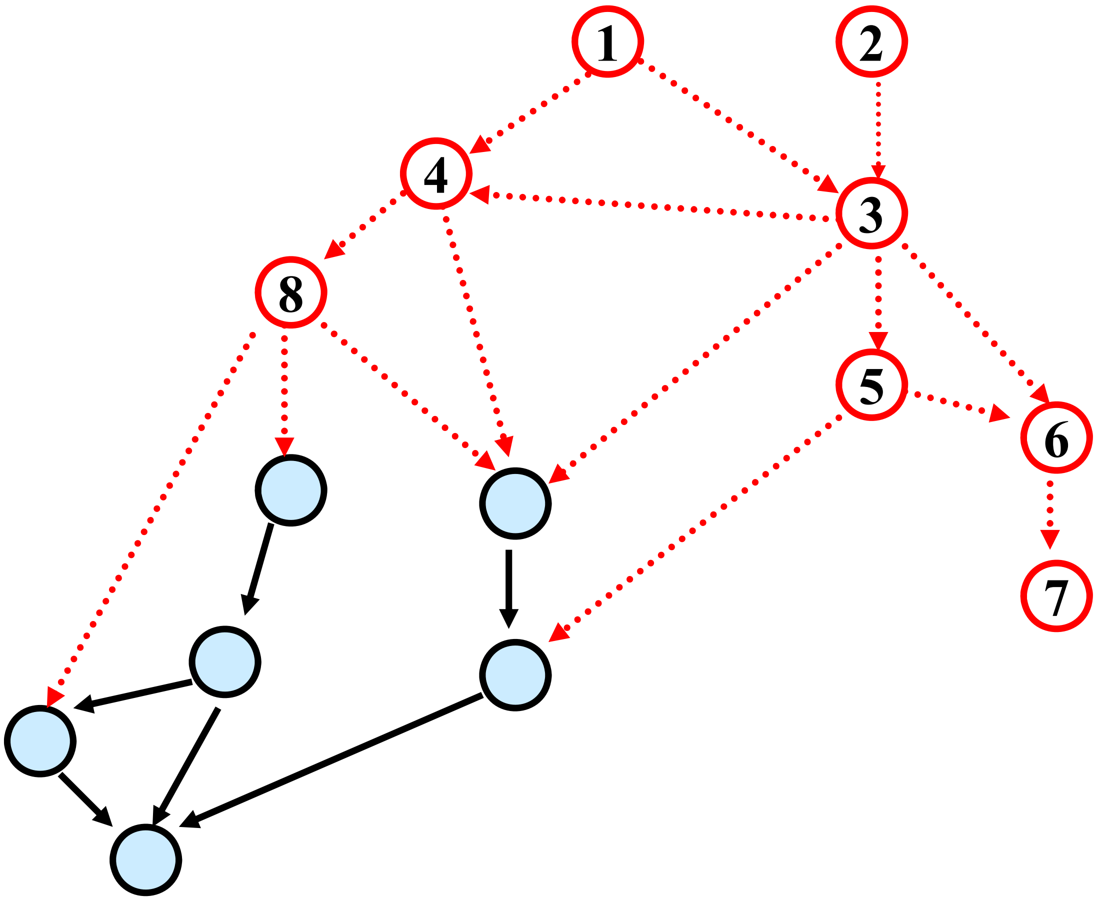
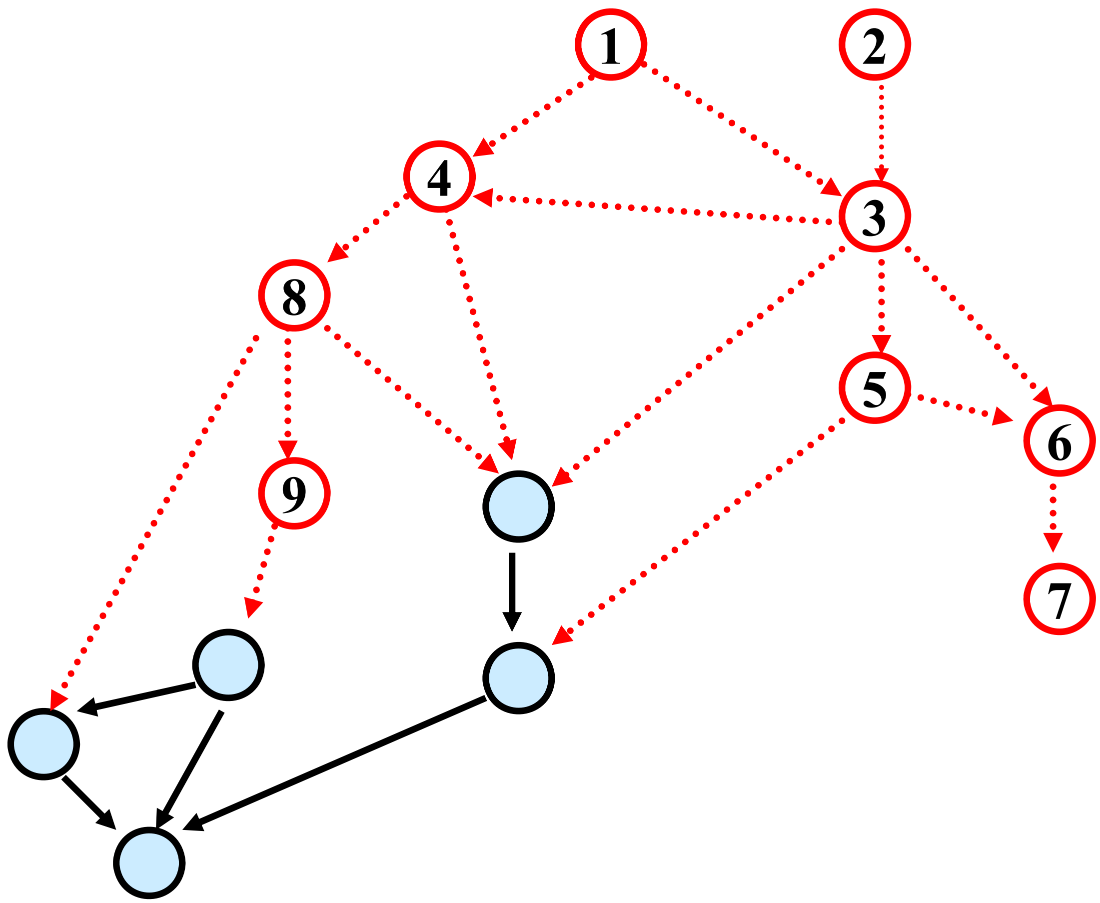


Topological Sort



Topological Sort



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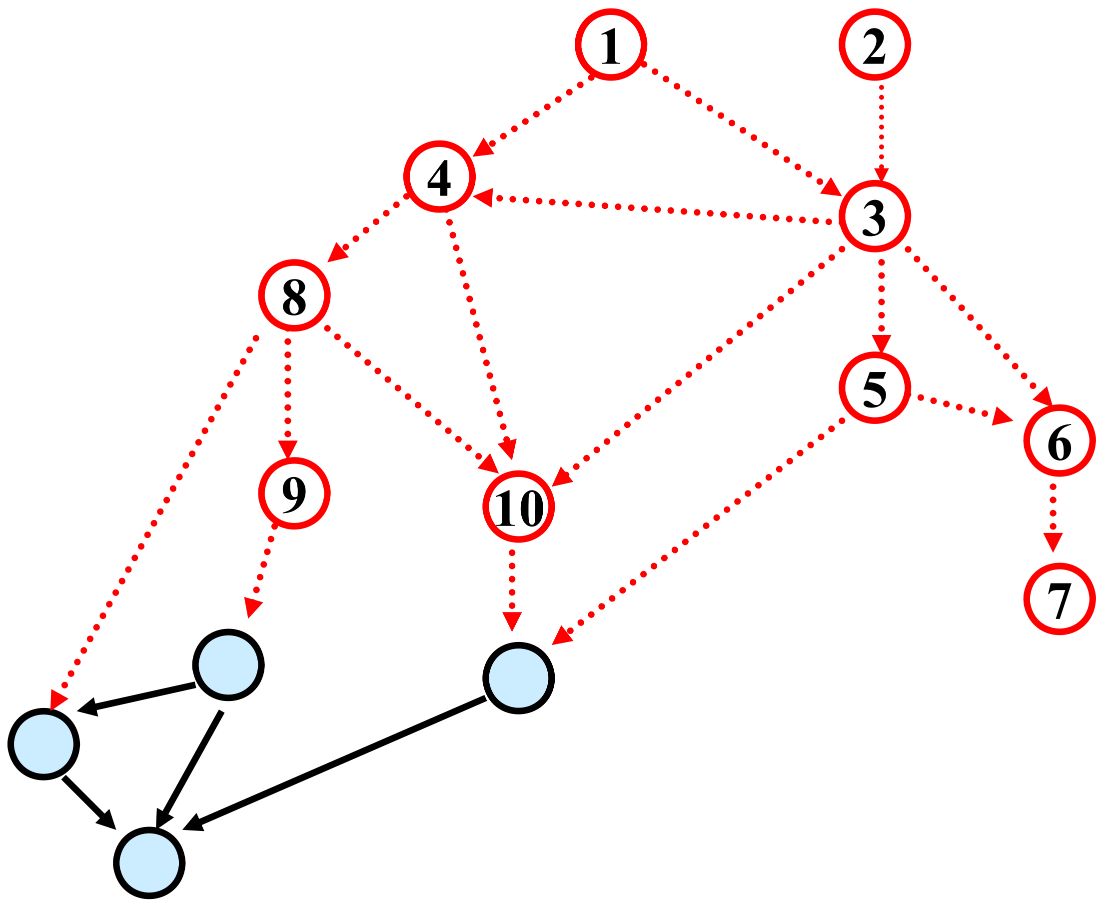

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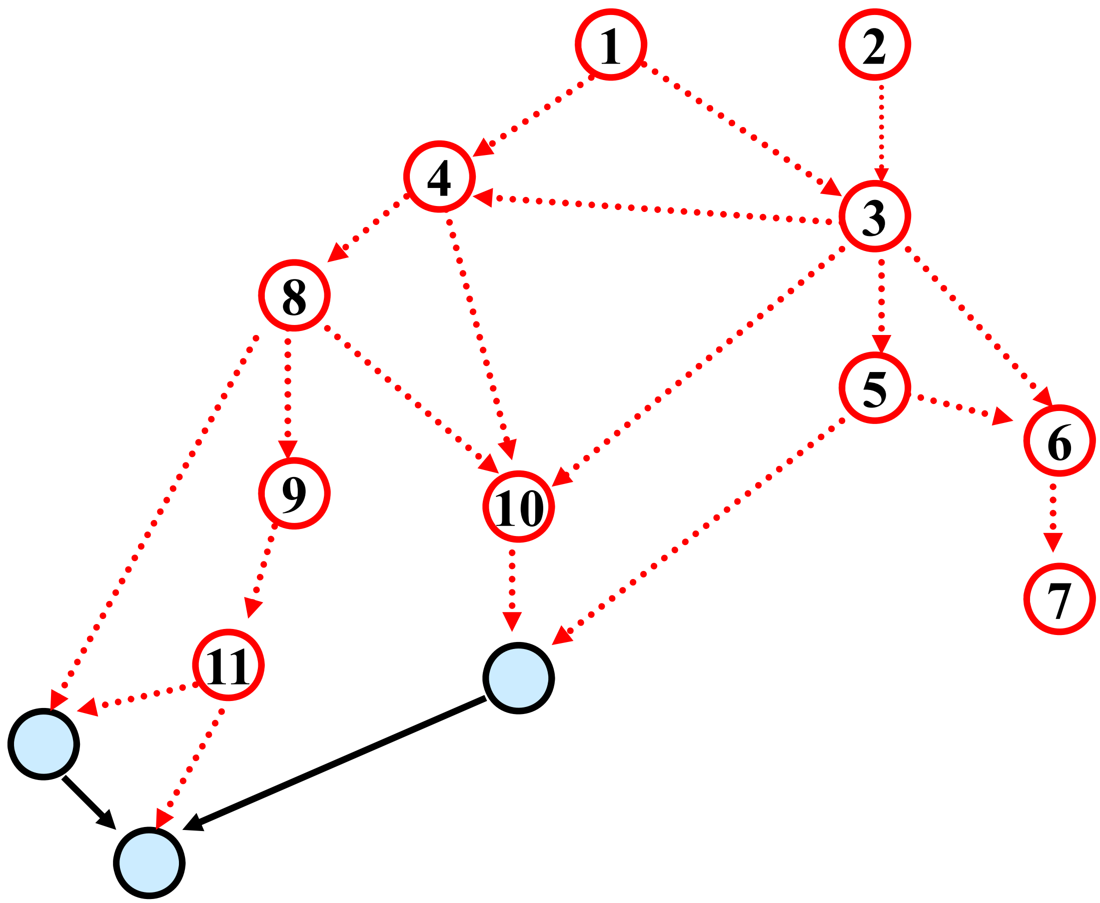

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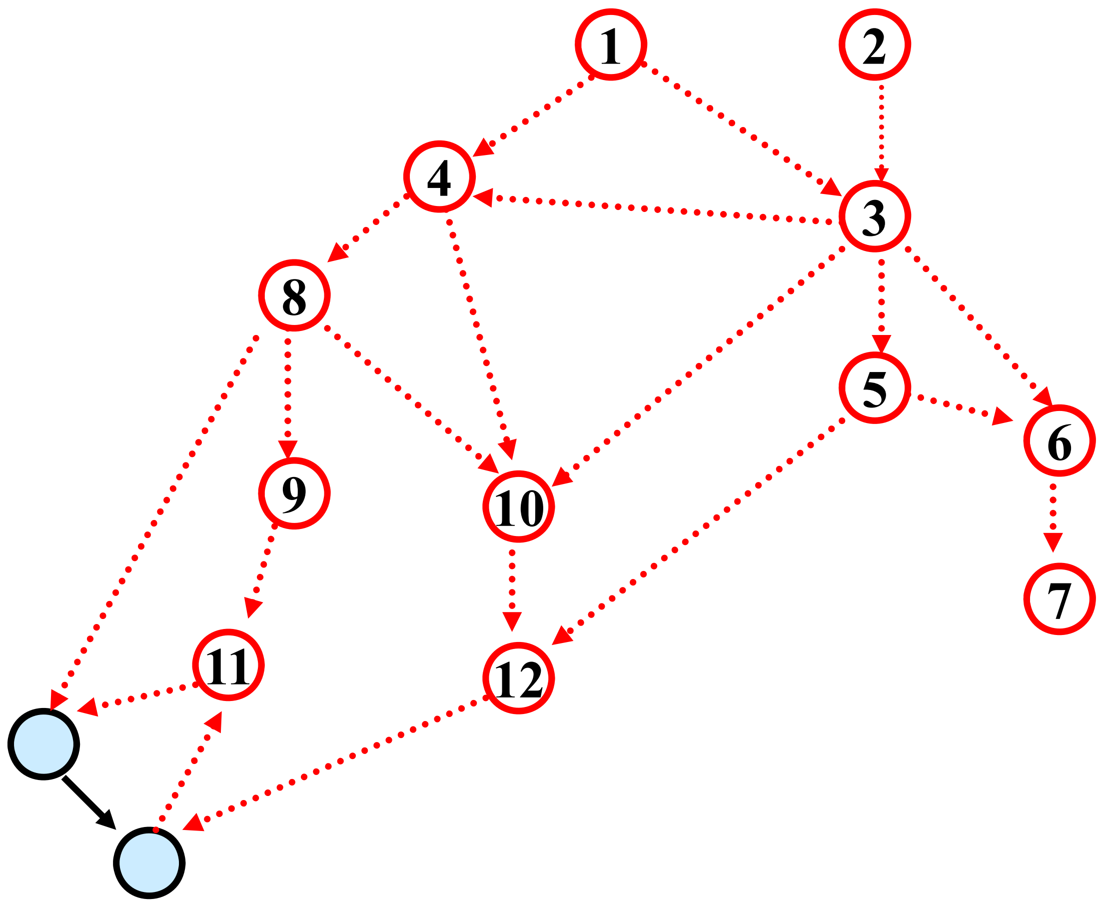
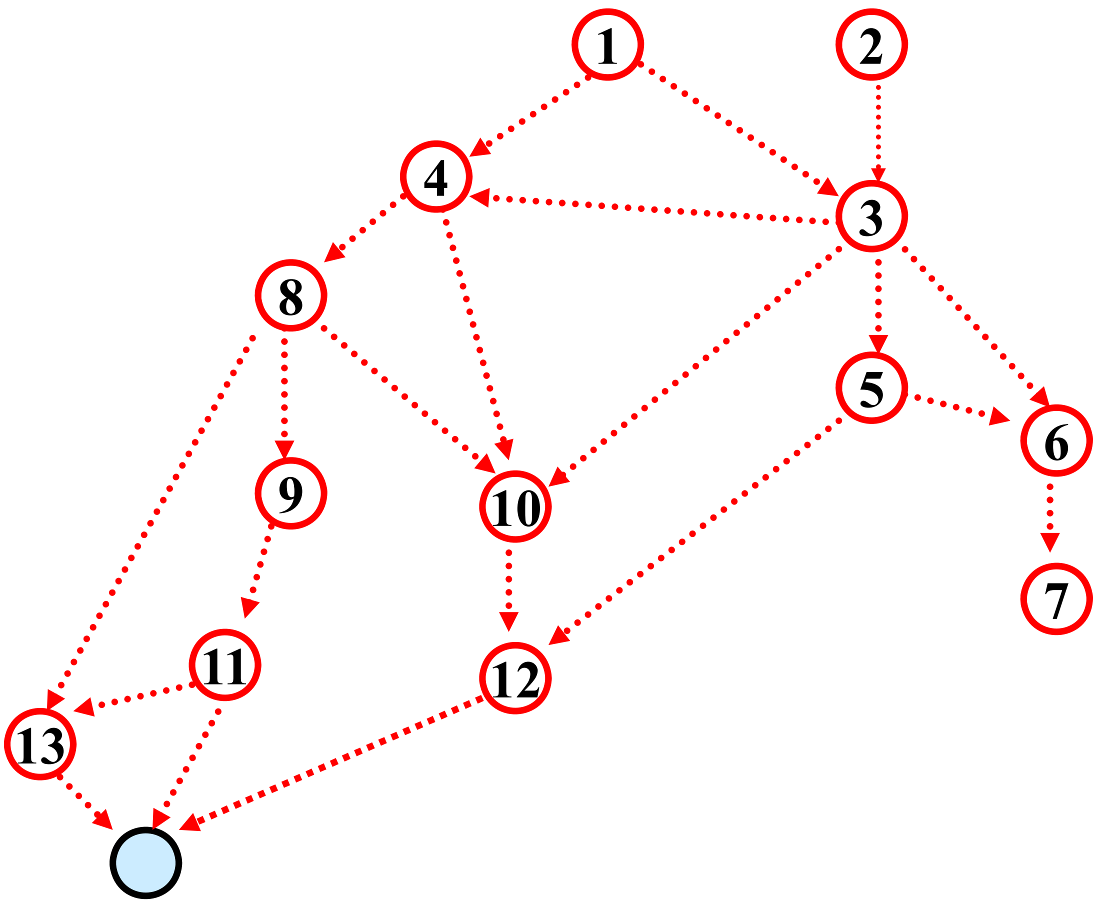


Topological Sort



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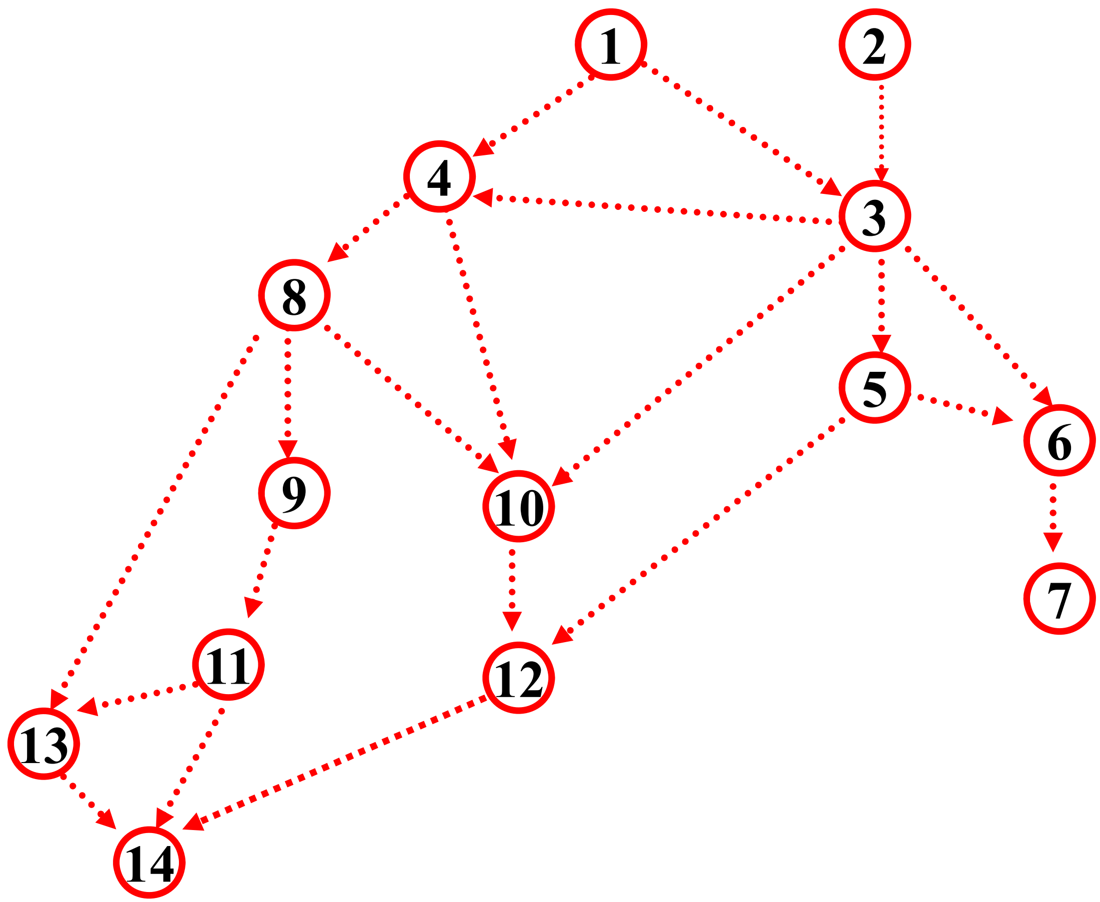

Implementing Topological Sort

- Go through all edges, computing array with in-degree for each vertex
  $O(\boldsymbol{m}+\boldsymbol{n})$

- Maintain a list of vertices of in-degree $\mathbf{0}$

- Remove any vertex in list and number it

- When a vertex is removed, decrease in-degree of each neighbor by $\mathbf{1}$ and add them to the list if their degree drops to $\mathbf{0}$

Total cost: $O(\boldsymbol{m}+\boldsymbol{n})$