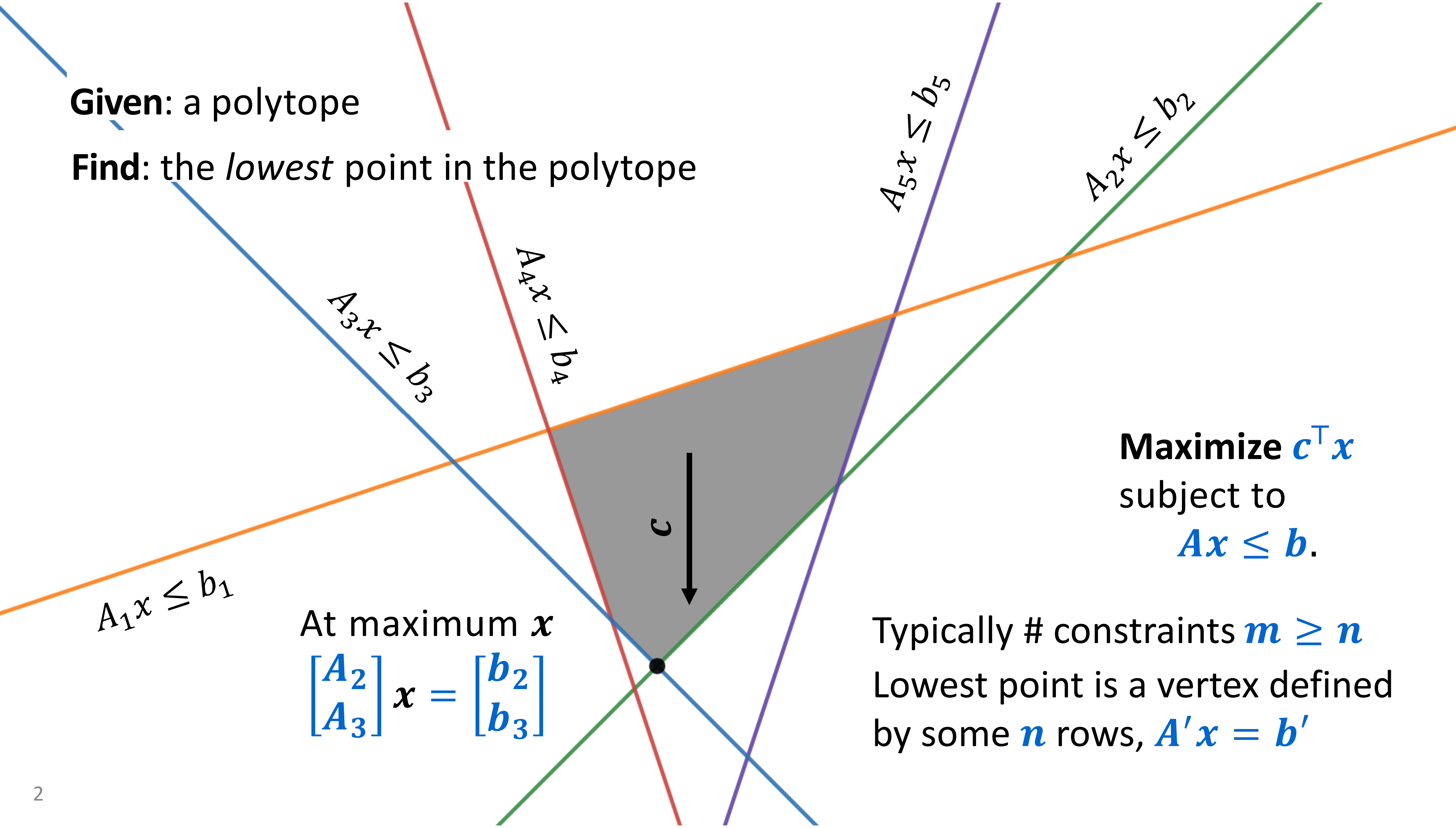
CSE 421
Introduction to Algorithms

Lecture 21: Linear Programming Duality

Given: a polytope

Find: the *lowest* point in the polytope



At maximum $\boldsymbol{x}$

$$\begin{bmatrix} \boldsymbol{A_2} \\ \boldsymbol{A_3} \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} \boldsymbol{b_2} \\ \boldsymbol{b_3} \end{bmatrix}$$

Maximize $\boldsymbol{c}^\top \boldsymbol{x}$
subject to
$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$.

Typically # constraints $\boldsymbol{m} \geq \boldsymbol{n}$

Lowest point is a vertex defined by some $\boldsymbol{n}$ rows, $\boldsymbol{A}'\boldsymbol{x} = \boldsymbol{b}'$

Max Flow in Standard Form LP

Maximize $\sum_{e \text{ out of } s} x_e$

subject to

$0 \le x_e \le c(e)$ for every $e \in E$

$$\sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e$$

for every node $v \in V - \{s, t\}$

Replace equality constraints by a pair of inequalities

Maximize $c^{\top} x$

subject to

$$Ax \le b$$
$$x \ge 0$$

This is for the c above. Nothing to do with capacities!

1. $c_e = \begin{cases} 1 & \text{if } e \text{ out of } s \\ 0 & \text{otherwise} \end{cases}$
2. $x_e \le c(e)$
3. $\sum_{e \text{ out of } v} x_e - \sum_{e \text{ into } v} x_e \le 0$
4. $\sum_{e \text{ into } v} x_e - \sum_{e \text{ out of } v} x_e \le 0$
5. $x \ge 0$

Minimization or Maximization

Minimize $\boldsymbol{c}^{\top}\boldsymbol{x}$
subject to
$$\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

⟷

Maximize $(-\boldsymbol{c})^{\top}\boldsymbol{x}$
subject to
$$(-\boldsymbol{A})\boldsymbol{x} \leq (-\boldsymbol{b})$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

Shortest Paths

Given: Directed graph $G = (V, E)$
vertices s, t in V

Find: shortest path from s to t

Claim: *Length* ℓ of the shortest path is the solution to this program.

Proof sketch: A shortest path yields a solution of cost ℓ. Optimal solution must be a combination of flows on shortest paths also cost ℓ; otherwise there is a part of the 1 unit of flow that gets counted on more than ℓ edges.

Minimize $\sum_{e} x_e$ Total flow

subject to

$$x \geq 0$$

$$\sum_{e \text{ out of } s} x_e = 1$$ Flow out of s is 1

$$\sum_{e \text{ into } t} x_e = 1$$ Flow into t is 1

$$\sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e$$

for every node $v \in V - \{s, t\}$

Flow conservation

Duality

Maximize $x_1 + 2x_3$
subject to

$$
\begin{array}{ll}
a & 2x_1 - x_2 + 3x_3 \leq 1 \\
b & -x_1 + x_2 - \quad x_3 \leq 5 \\
 & x \geq 0
\end{array}
$$

Claim: Optimum ≤ 6
Proof: Add the two LHS:

$$
\begin{array}{l}
2x_1 - x_2 + 3x_3 \\
+ (-x_1 + x_2 - \quad x_3) \\
= \quad x_1 \qquad\qquad + 2x_3.
\end{array}
$$

Must be $\leq$ sum of RHS $= 6$.

We multiplied the 1st inequality by $a = 1$, the 2nd by $b = 1$ and added.

Claim: For all $a, b \geq 0$ if

$$
\begin{array}{c}
2a - b \geq 1 \\
-a + b \geq 0 \\
3a - b \geq 2
\end{array}
$$

then Optimum $\leq a + 5b$

Proof:

$$
\begin{array}{l}
x_1 \qquad + \; 2x_3 \\
\leq a(2x_1 - x_2 + 3x_3) \\
+ b\,(-x_1 + x_2 - \quad x_3) \\
\leq 1a + 5b.
\end{array}
$$

Duality

Maximize $x_1 + 2x_3$
subject to

$$\begin{array}{ll} a & 2x_1 - x_2 + 3x_3 \le 1 \\ b & -x_1 + x_2 - \quad x_3 \le 5 \end{array}$$

$$x \ge 0$$

primal

Minimize $a + 5b$
subject to

$$\begin{array}{l} 2a - b \ge 1 \\ -a + b \ge 0 \\ 3a - b \ge 2 \end{array}$$

$$a, b \ge 0$$

dual

We multiplied the 1st inequality by $a = 1$, the 2nd by $b = 1$ and added.

Claim: For all $a, b \ge 0$ if

$$2a - b \ge 1$$
$$-a + b \ge 0$$
$$3a - b \ge 2$$

then Optimum $\le a + 5b$

Proof: $x_1 \quad + \; 2x_3$

$$\le a(2x_1 - x_2 + 3x_3)$$
$$+b\,(-x_1 + x_2 - \quad x_3)$$
$$\le 1a + 5b.$$

Duality

Maximize $x_1 + 2x_3$
subject to

$a \quad 2x_1 - x_2 + 3x_3 \leq 1$
$b \quad -x_1 + x_2 - \quad x_3 \leq 5$ primal
$x \geq 0$

Minimize $a + 5b$
subject to

$2a - b \geq 1$
$-a + b \geq 0$ dual
$3a - b \geq 2$
$a, b \geq 0$

We multiplied the 1st inequality by $a = 1$, the 2nd by $b = 1$ and added.

Claim: For all $a, b \geq 0$ if

$2a - b \geq 1$
$-a + b \geq 0$
$3a - b \geq 2$

then Optimum $\leq a + 5b$

Proof: $x_1 \quad + \ 2x_3$
$\leq a(2x_1 - x_2 + 3x_3)$
$+b\,(-x_1 + x_2 - \quad x_3)$
$\leq 1a + 5b$.

Duality

Maximize $x_1 + 2x_3$
subject to

$$a \quad 2x_1 - x_2 + 3x_3 \le 1$$
$$b \quad -x_1 + x_2 - \quad x_3 \le 5$$
$$x \ge 0$$

primal

Maximize $-a - 5b$
subject to

$$-2a + b \le -1$$
$$a - b \le \quad 0$$
$$-3a + b \le -2$$
$$a, b \ge 0$$

dual

We multiplied the 1st inequality by $a = 1$, the 2nd by $b = 1$ and added.

Claim: For all $a, b \ge 0$ if

$$2a - b \ge 1$$
$$-a + b \ge 0$$
$$3a - b \ge 2$$

then Optimum $\le a + 5b$

Proof: $x_1 \quad + \; 2x_3$
$$\le a(2x_1 - x_2 + 3x_3)$$
$$+b\,(-x_1 + x_2 - \quad x_3)$$
$$\le 1a + 5b.$$

Duality

Maximize $x_1 + 2x_3$
subject to

$$\begin{array}{ll} a & 2x_1 - x_2 + 3x_3 \leq 1 \\ b & -x_1 + x_2 - \quad x_3 \leq 5 \\ & x \geq 0 \end{array}$$

primal

Maximize $-a - 5b$
subject to

$$\begin{array}{ll} y_1 & -2a + b \leq -1 \\ y_2 & a - b \leq \ \ 0 \\ y_3 & -3a + b \leq -2 \\ & a, b \geq 0 \end{array}$$

dual

What is the dual of the dual?

Minimize $-1y_1 \ \ - 2y_3$
subject to

$$-2y_1 + y_2 - 3y_3 \geq -1$$
$$y_1 - y_2 + \ \ y_3 \geq -5$$
$$y \geq 0$$

equivalent to

Maximize $y_1 + 2y_3$
subject to

$$2y_1 - y_2 + 3y_3 \leq 1$$
$$-y_1 + y_2 \ - y_3 \leq 5$$
$$y \geq 0$$

Duality

primal

Maximize $\boldsymbol{c}^\top \boldsymbol{x}$
subject to
$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

dual

Minimize $\boldsymbol{b}^\top \boldsymbol{y}$
subject to
$$\boldsymbol{A}^\top \boldsymbol{y} \geq \boldsymbol{c}$$
$$\boldsymbol{y} \geq \boldsymbol{0}$$

dual

Maximize $(-\boldsymbol{b})^\top \boldsymbol{y}$
subject to
$$(-\boldsymbol{A})^\top \boldsymbol{y} \leq -\boldsymbol{c}$$
$$\boldsymbol{y} \geq \boldsymbol{0}$$

Theorem: The dual of the dual is the primal.

Proof:

dual of dual

Minimize $(-\boldsymbol{c})^\top \boldsymbol{x}$
subject to
$$((-\boldsymbol{A})^\top)^\top \boldsymbol{x} \geq (-\boldsymbol{b})^\top$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

$\equiv$

dual of dual

Minimize $-\boldsymbol{c}^\top \boldsymbol{x}$
subject to
$$-\boldsymbol{A}\boldsymbol{x} \geq -\boldsymbol{b}^\top$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

dual of dual

Maximize $\boldsymbol{c}^\top \boldsymbol{x}$
subject to
$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}^\top$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

Duality

primal

Maximize $\boldsymbol{c}^\top \boldsymbol{x}$
subject to
$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

dual

Minimize $\boldsymbol{b}^\top \boldsymbol{y}$
subject to
$$\boldsymbol{A}^\top \boldsymbol{y} \geq \boldsymbol{c}$$
$$\boldsymbol{y} \geq \boldsymbol{0}$$

Theorem: The dual of the dual is the primal.

Theorem (Weak Duality): Every solution to primal has a value that is at most that of every solution to dual.

Proof: We constructed the dual to give upper bounds on the primal.

Duality

primal

Maximize $\boldsymbol{c}^\top \boldsymbol{x}$
subject to
$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

dual

Minimize $\boldsymbol{b}^\top \boldsymbol{y}$
subject to
$$\boldsymbol{A}^\top \boldsymbol{y} \geq \boldsymbol{c}$$
$$\boldsymbol{y} \geq \boldsymbol{0}$$

Theorem: The dual of the dual is the primal.

Theorem (Weak Duality): Every solution to primal has a value that is at most that of every solution to dual.

Theorem (Strong Duality): If primal has a solution of finite value, then that value is equal to optimal solution of dual.

Duality

primal

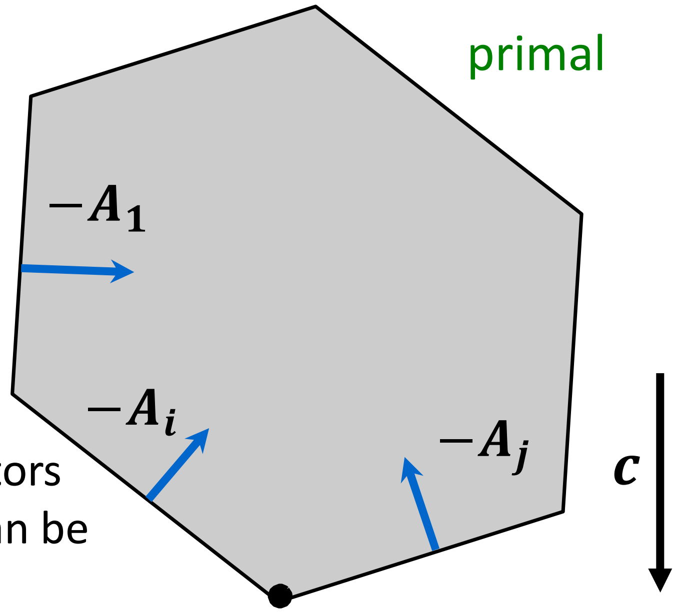
Maximize $\boldsymbol{c}^\top \boldsymbol{x}$
subject to
$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

dual

Minimize $\boldsymbol{b}^\top \boldsymbol{y}$
subject to
$$\boldsymbol{A}^\top \boldsymbol{y} \geq \boldsymbol{c}$$
$$\boldsymbol{y} \geq \boldsymbol{0}$$

Theorem (Strong Duality): If primal has a solution of finite value, then that value is equal to optimal solution of dual.

Fact: At vertex, $\boldsymbol{n}$ inequalities are tight
$$\boldsymbol{A}'\boldsymbol{x} = \boldsymbol{b}'.$$



Physics: Coefficient vectors $\boldsymbol{y}' \geq \boldsymbol{0}$ for tight rows can be combined to get $\boldsymbol{c}^\top$.

E.g. there are $\boldsymbol{y}_i, \boldsymbol{y}_j \geq \boldsymbol{0}$ s.t. $\boldsymbol{y}_i\boldsymbol{A}_i + \boldsymbol{y}_j\boldsymbol{A}_j = \boldsymbol{c}^\top$.

Set $\boldsymbol{y}_k$ for all other rows to $\boldsymbol{0}$, get $\boldsymbol{y}\,\boldsymbol{A} = \boldsymbol{y}'\boldsymbol{A}' = \boldsymbol{c}^\top$

so $\boldsymbol{A}^\top\boldsymbol{y} = \boldsymbol{c}$.

Then
$$\boldsymbol{b}^\top\boldsymbol{y} = (\boldsymbol{b}')^\top\boldsymbol{y}' = (\boldsymbol{A}'\boldsymbol{x})^\top\boldsymbol{y}' = \boldsymbol{x}^\top(\boldsymbol{A}')^\top\boldsymbol{y}' = \boldsymbol{x}^\top\boldsymbol{A}^\top\boldsymbol{y}$$
$$= \boldsymbol{x}^\top\boldsymbol{c} = \boldsymbol{c}^\top\boldsymbol{x}$$

since $\boldsymbol{x}^\top\boldsymbol{c}$ and $\boldsymbol{c}^\top\boldsymbol{x}$ are just numbers.

Saving dual variables for equalities

Maximize $x_1 + 4x_2$
subject to

a' $\quad 3x_1 - 2x_2 \leq 5$

a'' $\quad -3x_1 + 2x_2 \leq -5$

$\dots$

$x \geq 0$

Dual →

Minimize $5(a' - a'') + \dots$
subject to

$3(a' - a'') + \dots \geq 1$

$-2(a' - a'') + \dots \geq 4$

$a', a'' \dots \geq 0$

$a' - a''$ can take on any real value

Standard form conversion for equality ↑

↓

Maximize $x_1 + 4x_2$
subject to

a $\quad 3x_1 - 2x_2 = 5$

$\dots$

$x \geq 0$

Dual →

use direct conversion!

Minimize $5a + \dots$
subject to

$3a + \dots \geq 1$

$-2a + \dots \geq 4$

$\dots \geq 0$

No requirement that $a \geq 0$

Dual of Max Flow

Use a different names to avoid confusion with capacity vector

Maximize $g^\top x$
subject to
$$Ax \le h$$
$$x \ge 0$$

1. $g_e = \begin{cases} 1 & \text{if } e \text{ out of } s \\ 0 & \text{otherwise} \end{cases}$

a_e 2. $x_e \le c(e)$

b_v 3. $\sum_{e \text{ into } v} x_e - \sum_{e \text{ out of } v} x_e = 0$

4. $x \ge 0$

$v \in S - \{s, t\}$

Minimize $\sum_e c(e) a_e \equiv c^\top a$
subject to

$a_e + b_v \ge 1$ if $e = (s, v)$

$a_e - b_u \ge 0$ if $e = (u, t)$

$a_e - b_u + b_v \ge 0$ if $e = (u, v)$

$a \ge 0$ $\quad u, v \in S - \{s, t\}$

More uniform way to write Max Flow Dual

Minimize $\sum_e c(e)a_e \equiv c^\top a$

subject to

$a_e + b_v \geq 1$ if $e = (s, v)$

$a_e - b_u \geq 0$ if $e = (u, t)$

$a_e - b_u + b_v \geq 0$ if $e = (u, v)$

$u, v \in S - \{s, t\}$

$a \geq 0$

Define

$b_s = 1$

$b_t = 0$

Minimize $\sum_e c(e)a_e \equiv c^\top a$

subject to

$b_s = 1$

$b_t = 0$

$a_e - b_u + b_v \geq 0$

for $e = (u, v)$

$a \geq 0$

Simpler to read Max Flow Dual

Minimize $\sum_e c(e)a_e \equiv c^\top a$
subject to

$$b_s = 1$$
$$b_t = 0$$

$$a_e - b_u + b_v \geq 0$$
for $e = (u, v)$

$$a \geq 0$$

All the $c(e) \geq 0$, so we want the a_e as small as possible.

Minimize $\sum_e c(e)a_e \equiv c^\top a$
subject to

$$b_s = 1$$
$$b_t = 0$$

$$a_e = \max(b_u - b_v, 0)$$
for $e = (u, v)$

Minimize

$\sum_e c(e)a_e \equiv c^\top a$

subject to

$$b_s = 1$$
$$b_t = 0$$

$$a_e = \max(b_u - b_v, 0)$$
$$\text{for } e = (u, v)$$

Claim: Optimum is achieved with $0 \leq b_v \leq 1$ for every vertex v.

Proof:

Move b_v values between $\mathbf{0}$ and $\mathbf{1}$

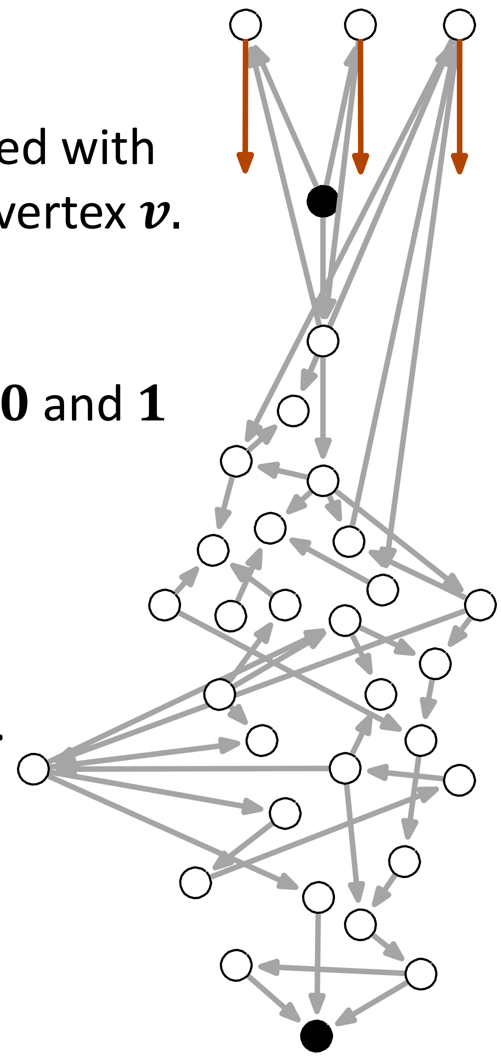
Reduces:

a_e = length if e is down

Doesn't change:

$a_e = \mathbf{0}$ if e is up

Overall solution improves.



$b_s = 1$

$b_t = 0$

Minimize

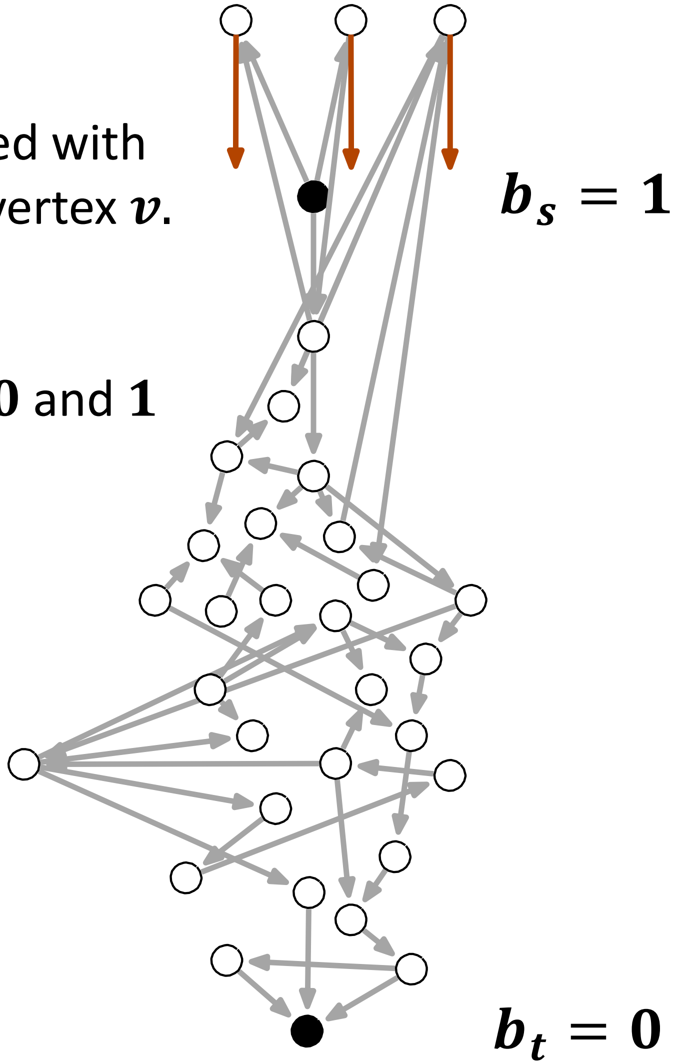
$\sum_e c(e)a_e \equiv c^\top a$

subject to

$$b_s = 1$$
$$b_t = 0$$
$$0 \le b_v \le 1$$
$$a_e = \max(b_u - b_v, 0)$$
for $e = (u, v)$

Claim: Optimum is achieved with
$0 \le b_v \le 1$ for every vertex v.

Proof:
Move b_v values between $\mathbf{0}$ and $\mathbf{1}$
Reduces:
a_e = length if e is down
Doesn't change:
$a_e = 0$ if e is up
Overall solution improves.



Minimize

$\sum_e c(e) a_e \equiv c^\top a$

subject to

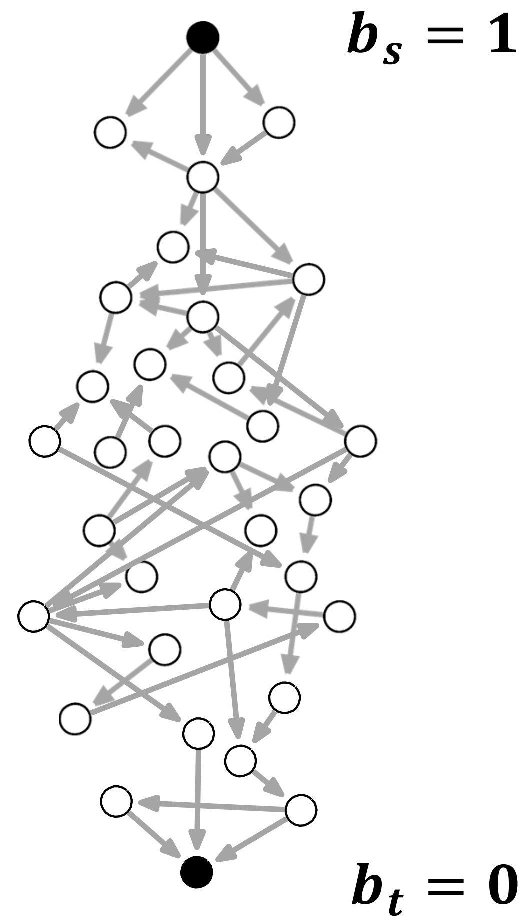
$$b_s = 1$$

$$b_t = 0$$

$$0 \le b_v \le 1$$

$$a_e = \max(b_u - b_v, 0)$$

$$\text{for } e = (u, v)$$



Minimize

$\sum_e c(e)a_e \equiv c^\top a$

subject to

$$b_s = 1$$
$$b_t = 0$$
$$0 \le b_v \le 1$$
$$a_e = \max(b_u - b_v, 0)$$
for $e = (u, v)$

Claim: Optimum is achieved with $b_v = 0$ or $b_v = 1$ for every vertex v.

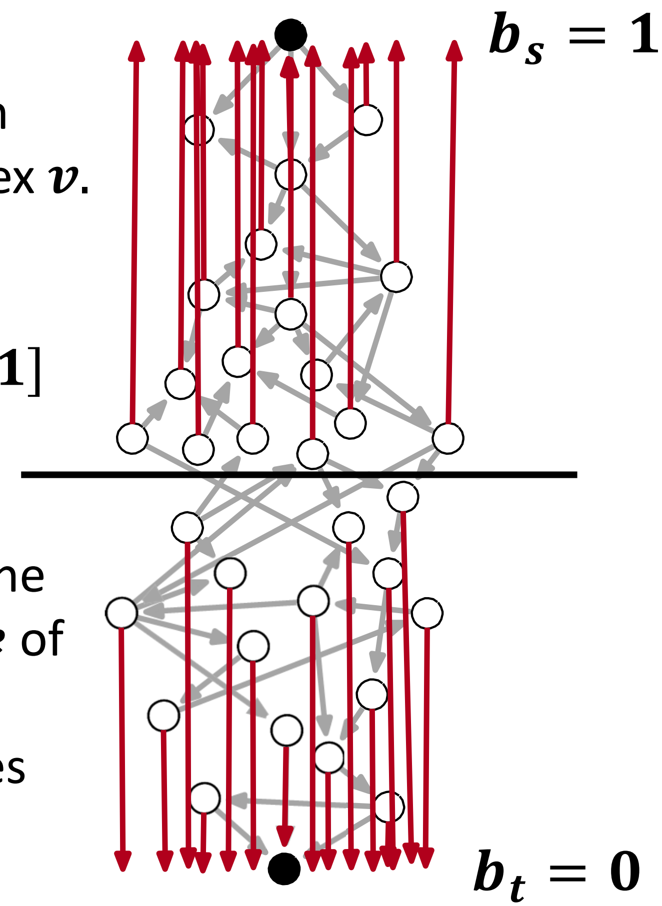
Proof:

Choose uniform random $r \in [0, 1]$

Set $b_v = \begin{cases} 1 & \text{if } b_v \ge r \\ 0 & \text{if } b_v < r \end{cases}$

Expected value for random r is the same as the original since edge e of length a_e is cut w.p. a_e.

So... one of those random choices must be at least as good.



Minimize

$\sum_e \boldsymbol{c}(\boldsymbol{e})\boldsymbol{a}_{\boldsymbol{e}} \equiv \boldsymbol{c}^\top \boldsymbol{a}$

subject to

$$\boldsymbol{b}_{\boldsymbol{s}} = \mathbf{1}$$

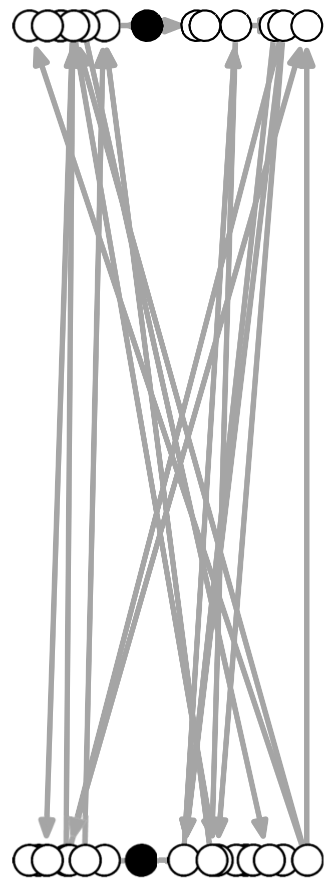
$$\boldsymbol{b}_{\boldsymbol{t}} = \mathbf{0}$$

$$\boldsymbol{b}_{\boldsymbol{v}} \in \{\mathbf{0}, \mathbf{1}\}$$

$$\boldsymbol{a}_{\boldsymbol{e}} = \max(\boldsymbol{b}_{\boldsymbol{u}} - \boldsymbol{b}_{\boldsymbol{v}}, \mathbf{0})$$

for $\boldsymbol{e} = (\boldsymbol{u}, \boldsymbol{v})$

MinCut!



$\boldsymbol{b}_{\boldsymbol{s}} = \mathbf{1}$

$\boldsymbol{b}_{\boldsymbol{t}} = \mathbf{0}$

Duality of Shortest Paths

Minimize $\sum_e x_e$

subject to

$\sum_{e \text{ out of } s} x_e = 1$

$\sum_{e \text{ into } t} x_e = 1$

$\sum_{e \text{ into } v} x_e - \sum_{e \text{ out of } v} x_e = 0$
for all $v \in V - \{s, t\}$

$x \geq 0$

Duality of Shortest Paths

Minimize $\sum_e x_e$

subject to

a_s $\quad \sum_{e \text{ into } s} x_e - \sum_{e \text{ out of } s} x_e = -1$

a_t $\quad \sum_{e \text{ into } t} x_e - \sum_{e \text{ out of } t} x_e = 1$

a_v $\quad \sum_{e \text{ into } v} x_e - \sum_{e \text{ out of } v} x_e = 0$

for all $v \in V - \{s, t\}$

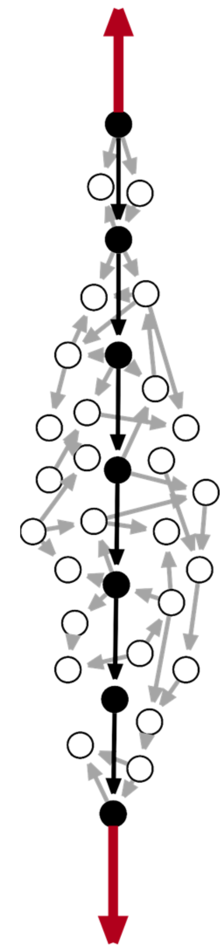
$x \geq 0$

Maximize $a_s - a_t$

subject to

$a_u - a_v \leq 1$

if $e = (u, v)$



Duality and Zero-Sum Games

Two player zero-sum game:

An $m \times n$ matrix G

$G_{i,j}$ = payoff to row player assuming:
row player uses strategy i, and
column player uses strategy j.

Column player's payoff for game $= -G_{i,j}$

Example: Chess (idealized)

i specifies how white would move in every possible board configuration.

j specifies how black would move.

$$G_{i,j} = \begin{cases} +1 & \text{White checkmates} \\ -1 & \text{Black checkmates} \\ 0 & \text{Draw on board} \end{cases}$$

Randomized Strategy:

Probability distribution on row strategies:

- A column vector x with each $x_i \geq 0$

$$\sum_i x_i = 1$$

Probability distribution on column strategies:

- A column vector y with each $y_j \geq 0$

$$\sum_j y_j = 1$$

Expected payoff to row player:

$$x^\top G\, y$$

Who decides on their strategy first

If row player commits to $\boldsymbol{x}$:

Row player will get payoff

$$\min_{\boldsymbol{y}} \boldsymbol{x}^\top \boldsymbol{G} \boldsymbol{y} = \min_{j} (\boldsymbol{x}^\top \boldsymbol{G})_j$$

So if row player plays first they can get payoff

$$\max_{\boldsymbol{x}} \min_{\boldsymbol{y}} \boldsymbol{x}^\top \boldsymbol{G} \boldsymbol{y}$$

If column player commits to $\boldsymbol{y}$:

Row player will get payoff

$$\max_{\boldsymbol{x}} \boldsymbol{x}^\top \boldsymbol{G} \boldsymbol{y} = \max_{i} (\boldsymbol{G}\ \boldsymbol{y})_i$$

So if column player plays first, row player can get payoff

$$\min_{\boldsymbol{y}} \max_{\boldsymbol{x}} \boldsymbol{x}^\top \boldsymbol{G} \boldsymbol{y}$$

Randomized Strategy:

Probability distribution on row strategies:

- A column vector $\boldsymbol{x}$ with each $\boldsymbol{x}_i \geq \boldsymbol{0}$

$$\sum_{i} \boldsymbol{x}_i = \boldsymbol{1}$$

Probability distribution on column strategies:

- A column vector $\boldsymbol{y}$ with each $\boldsymbol{y}_j \geq \boldsymbol{0}$

$$\sum_{j} \boldsymbol{y}_j = \boldsymbol{1}$$

Expected payoff to row player:

$$\boldsymbol{x}^\top \boldsymbol{G}\ \boldsymbol{y}$$

Von Neumann's MiniMax Theorem

If row player commits to $\boldsymbol{x}$:

Row player will get payoff

$$\min_{\boldsymbol{y}} \boldsymbol{x}^\top \boldsymbol{G} \boldsymbol{y} = \min_{\boldsymbol{j}} (\boldsymbol{x}^\top \boldsymbol{G})_{\boldsymbol{j}}$$

So if row player plays first they can get payoff

$$\max_{\boldsymbol{x}} \min_{\boldsymbol{y}} \boldsymbol{x}^\top \boldsymbol{G} \boldsymbol{y}$$

If column player commits to $\boldsymbol{y}$:

Row player will get payoff

$$\max_{\boldsymbol{x}} \boldsymbol{x}^\top \boldsymbol{G} \boldsymbol{y} = \max_{\boldsymbol{i}} (\boldsymbol{G}\, \boldsymbol{y})_{\boldsymbol{i}}$$

So if column player plays first, row player can get payoff

$$\min_{\boldsymbol{y}} \max_{\boldsymbol{x}} \boldsymbol{x}^\top \boldsymbol{G} \boldsymbol{y}$$

It doesn't matter who plays first!

Theorem:

$$\max_{\boldsymbol{x}} \min_{\boldsymbol{y}} \boldsymbol{x}^\top \boldsymbol{G} \boldsymbol{y} = \min_{\boldsymbol{y}} \max_{\boldsymbol{x}} \boldsymbol{x}^\top \boldsymbol{G} \boldsymbol{y}$$

Use Strong Duality to prove MiniMax Theorem

Theorem: $\max_{\boldsymbol{x}} \min_{\boldsymbol{y}} \boldsymbol{x}^\top \boldsymbol{G} \boldsymbol{y} = \min_{\boldsymbol{y}} \max_{\boldsymbol{x}} \boldsymbol{x}^\top \boldsymbol{G} \boldsymbol{y}$

i.e., $\max_{\boldsymbol{x}} \min_{\boldsymbol{j}} (\boldsymbol{x}^\top \boldsymbol{G})_{\boldsymbol{j}} = \min_{\boldsymbol{y}} \max_{\boldsymbol{i}} (\boldsymbol{G}\, \boldsymbol{y})_{\boldsymbol{i}}$

Primal

Maximize $\boldsymbol{z}$

subject to

$\boldsymbol{w}$ $\quad \sum_i \boldsymbol{x_i} = \boldsymbol{1}$

$\boldsymbol{y_j}$ $\quad \boldsymbol{z} - (\boldsymbol{x}^\top \boldsymbol{G})_{\boldsymbol{j}} \leq \boldsymbol{0}^*$

for all $\boldsymbol{j}$

$\boldsymbol{x} \geq \boldsymbol{0}$

*equivalent to $\boldsymbol{z} \leq \min_{\boldsymbol{j}} (\boldsymbol{x}^\top \boldsymbol{G})_{\boldsymbol{j}}$

Dual

Minimize $\boldsymbol{w}$

subject to

$\sum_j \boldsymbol{y_j} = \boldsymbol{1}$ Coefficient of $\boldsymbol{z}$ must be $\boldsymbol{1}$

$\boldsymbol{w} - (\boldsymbol{G}\, \boldsymbol{y})_{\boldsymbol{i}} \geq \boldsymbol{0}^*$ Coefficient of $\boldsymbol{x_i}$ must be $\geq \boldsymbol{0}$

for all $\boldsymbol{i}$

$\boldsymbol{y} \geq \boldsymbol{0}$

*equivalent to $\boldsymbol{w} \geq \max_{\boldsymbol{i}} (\boldsymbol{G}\, \boldsymbol{y})_{\boldsymbol{i}}$