CSE 421
Introduction to Algorithms

Lecture 28: Dealing with NP-completeness:

Fixed Parameter Tractability

SAT Solving

Reminder/Announcement

- The Final Exam is Monday December 11, 2:30-4:20 pm here
  - If nobody has a conflict that would prevent them staying longer, I will extend the time available to 4:45 pm.
  - Email me (enter a Private post on Edstem) by the end of day today if you can't stay that long.

- See the pinned Edstem posts on Final Exam Information.

- I will run a Zoom review session on Sunday. Fill out the Edstem poll about the time for this session by end-of-day today.

Fixed Parameter Algorithms

The theory of **fixed parameter tractability** looks at $\mathbf{NP}$ problems using a second parameter $\boldsymbol{k}$ in addition to input size $\boldsymbol{n}$ and seeks algorithms with running times $\boldsymbol{f(k) \cdot n^{O(1)}}$ where $\boldsymbol{f}$ might be exponential.

Clique: Extra parameter $\boldsymbol{k}$ for clique size target:

Brute force algorithm: try all subsets of size $\boldsymbol{k}$ and check: $\boldsymbol{\Theta(k^2 n^k)}$ time.

Vertex-Cover: Extra parameter $\boldsymbol{k}$ for clique size target:

Brute force algorithm: try all subsets of size $\boldsymbol{k}$ and check: $\boldsymbol{\Theta(m n^k)}$ time.

- Neither is a good fixed parameter algorithm

Vertex-Cover Fixed Parameter Algorithm

FPT

Vertex-Cover(C, b) **{**

  if there is an edge (u, v) not covered by C **{**

    if $b > 0$ **{**

      Vertex-Cover$(C \cup \{u\}, b-1)$

      Vertex-Cover$(C \cup \{v\}, b-1)$

    }

  }

  else

    Output **YES** (and set C) and halt

  }

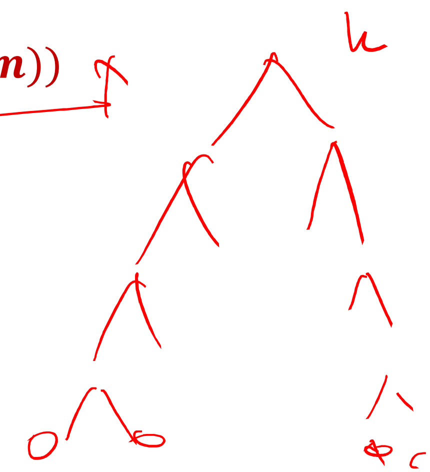
}

Call **Vertex-Cover**$(\emptyset, k)$

if no answer, output **NO**

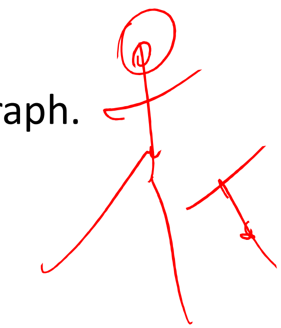
Analysis:

- Time to identify possible edge (u, v) not covered (and modify C) is $O(m+n)$
- # of recursive calls $\leq 2^{k+1}$
- Total runtime $O(2^k(m+n))$

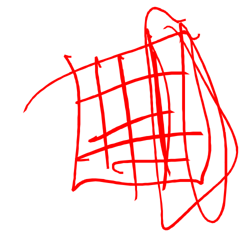


More on Fixed Parameter Algorithms

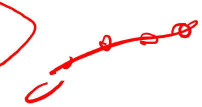
Many graph problems can be given a second parameter k called the treewidth of the input graph.



- Treewidth 1 graphs are trees (technically forests).
- Multiple natural definitions of treewidth (here's one):
  - Graph $G = (V, E)$ is treewidth at most k iff there is a tree T such that
    - each node u of T is labelled by a subset V_u of $\leq k$ vertices in V
    - for every edge $(v, w) \in E$ there is a node u of T such that both $v, w \in V_u$.
    - for every $v \in V$ the set of nodes u in T with $v \in V_u$ is connected in T
  - The tree with the sets are called the tree decomposition of G.
    The minimum k and tree decomposition can be found in linear time.
    The tree defines a natural elimination ordering for recursive algorithms on the graph.



- **Fact:** Obstacle to treewidth $k - 1$: the $k \times k$ grid graph.



Many NP-hard problems are efficiently solvable on graphs of bounded treewidth.

Treewidth also comes up in route-finding in Google Maps: Can't run full-blown Dijkstra on the whole graph every time a user requests a route.

What to do if the problem you want to solve is NP-hard

Try to make an exponential-time solution as efficient as possible.

e.g. Try to search the space of possible hints/certificates in a more efficient way and hope that it is quick enough.

Backtracking search

e.g., for **SAT**, search through the 2^n possible truth assignments...

...but set the truth values one-by-one so we can able to figure out whole parts of the space to avoid,

e.g. Given $F = (\neg x_1 \vee x_2) \wedge (\neg x_2 \vee x_3) \wedge (x_4 \vee \neg x_3) \wedge (x_1 \vee x_4)$

after setting $x_1 = 1$ and $x_2 = 0$ we don't even need to set x_3 or x_4 to know that it won't satisfy F.

Now: More clever backtracking search for **SAT** solutions

SAT Solving

SAT is an extremely flexible problem:

- The fact that **SAT** is an **NP**-complete problem says that we can re-express a huge range of problems as **SAT** problems

This means that good algorithms for **SAT** solving would be useful for a huge range of tasks.

Since roughly 2001, there has been a massive improvement in our ability to solve **SAT** on a wide range of practical instances

- These algorithms aren't perfect. They fail on many worst-case instances.

Satisfiability Algorithms

Local search: Solve **SAT** as a special case of **MaxSAT**

(incomplete, may fail to find satisfying assignment)

GSAT – random local search [Selman,Levesque,Mitchell 92]

Walksat – Metropolis [Kautz,Selman 96]

Backtracking search (complete)

- DPLL [Davis,Putnam 60], [Davis,Logeman,Loveland 62]
- CDCL: Adds clause learning and restarts

  GRASP, SATO, zchaff, MiniSAT, Glucose, etc.

CNF Satisfiability

SAT: satisfiability problem for CNF formulas with any clause size

Write CNFs with the $\land$ between clauses implicit:

$$F = (x_1 \lor \overline{x_2} \lor x_4)(\overline{x_1} \lor x_3)(\overline{x_3} \lor x_2)(\overline{x_4} \lor \overline{x_3})$$

Write assignment as literals assigned true: $x_1, x_2, x_3, \overline{x_4}$

Defn: Given partial assignment x_3 where

$$F = (x_1 \lor \overline{x_2} \lor x_4)(\overline{x_1} \lor x_3)(\overline{x_3} \lor x_2)(\overline{x_4} \lor \overline{x_3})$$

define **simplify**(F, x_3) by

simplify(F, x_3)$= (x_1 \lor \overline{x_2} \lor x_4) \qquad x_2 \quad \overline{x_4}$

That is: remove satisfied clauses and remove unsatisfied literals from clauses.

Note: F is satisfiable iff all clauses disappear under some assignment.

Backtracking search/DPLL

$\mathrm{t} \leftarrow \varepsilon$

repeat

select a literal ℓ (some $\boldsymbol{x}$ or $\overline{\boldsymbol{x}}$)

$\boldsymbol{F} \leftarrow$ **simplify**$(\boldsymbol{F}, \ell)$; $\mathrm{t} \leftarrow \mathrm{append}(\mathrm{t}, \ell)$

free step

while $\boldsymbol{F}$ contains a $\mathbf{1}$-clause ℓ'

$\boldsymbol{F} \leftarrow$ **simplify**$(\boldsymbol{F}, \ell')$; $\mathrm{t} \leftarrow \mathrm{append}(\mathrm{t}, \ell')$

unit propagation

if $\boldsymbol{F}$ has no clauses **return** t as satisfying assignment

if $\boldsymbol{F}$ has an empty clause

backtrack to last free step and flip assignment (step no longer free)

Recursive view of DPLL (without unit propagation)

DPLL(F):

  if F is empty **report** satisfiable and **halt**

  if F contains the empty clause

    return

  else choose a literal x

    DPLL(**simplify**(F, x))

    DPLL(**simplify**$(F, \overline{x})$)

with unit propagation choose x to be the literal of a 1-clause if possible

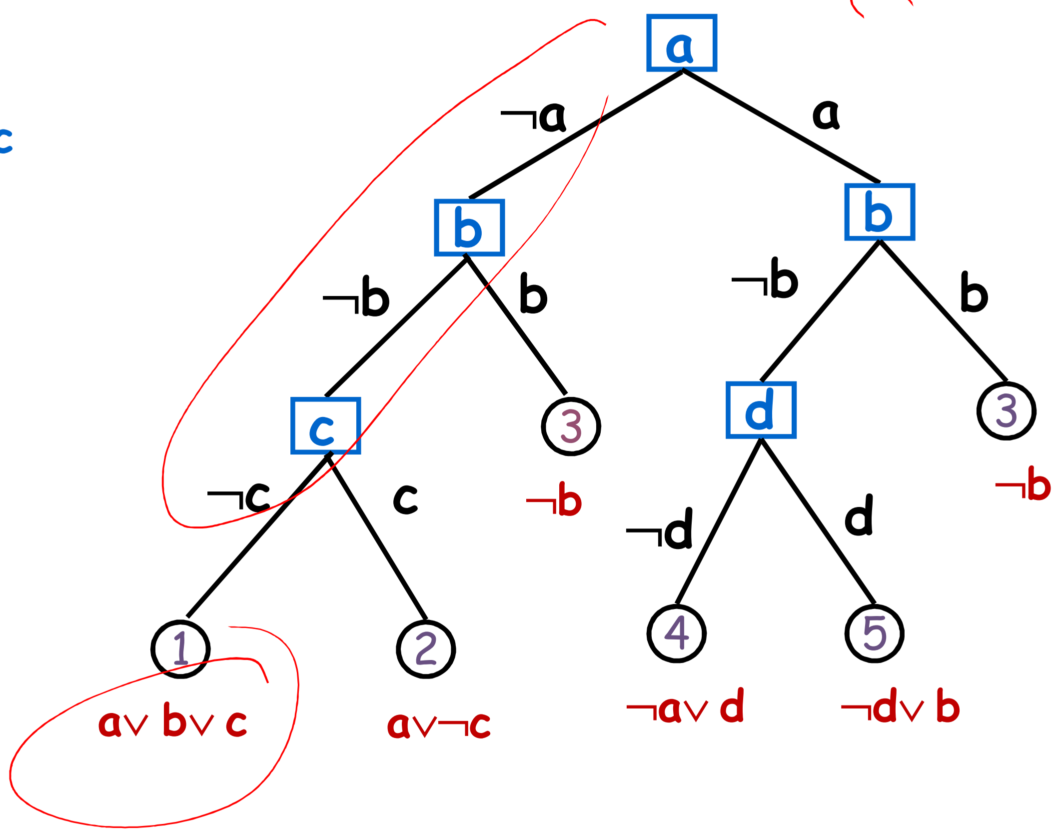
DPLL on UNSAT formula

$\neg(\neg a \wedge \neg b \wedge \neg c)$

Clauses

1. a∨ b∨ c
2. a∨¬c
3. ¬b
4. ¬a∨ d
5. ¬d∨ b

Residual Formula



Extending DPLL: Clause Learning

- When backtracking in DPLL, add new clauses corresponding to causes of failure of the search

- Added conflict clauses
  - Capture *reasons* of conflicts
  - Obtained via *unit propagations* from known ones
  - Reduce future search by producing conflicts sooner

Conflict Graph: Graph of Unit Propagations

At each conflict (derivation of them empty clause) the negations of the predecessor node labels across any cut form an implied clause.

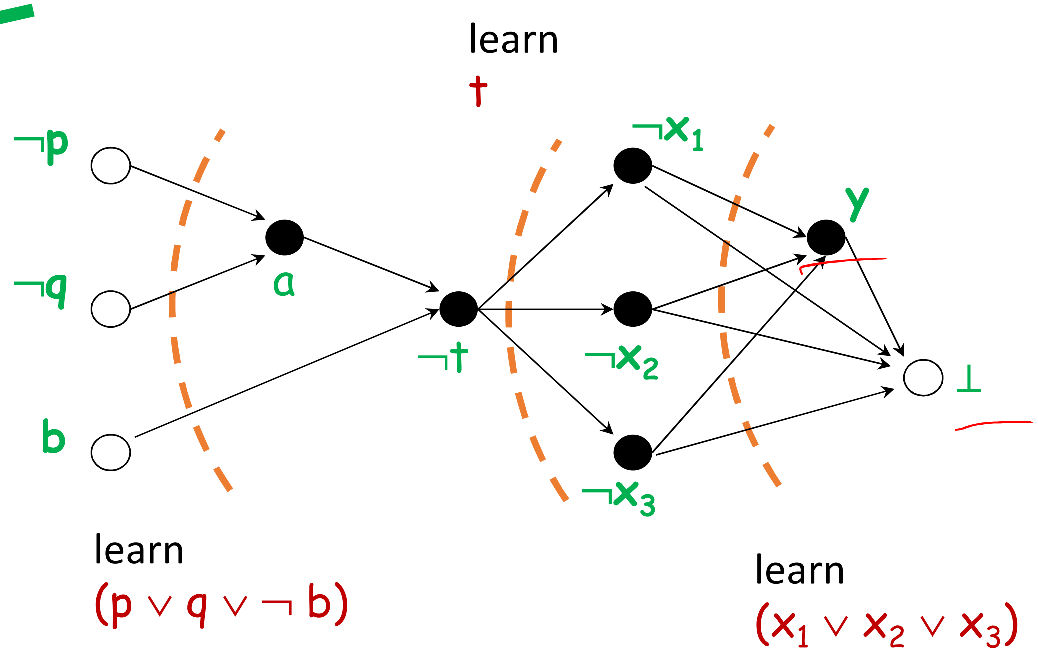
- if clause is false then could derive ⊥

Known Clauses

$(p \lor q \lor a)$
$(\neg a \lor \neg b \lor \neg t)$
$(t \lor \neg x_1)$
$(t \lor \neg x_2)$
$(t \lor \neg x_3)$
$(x_1 \lor x_2 \lor x_3 \lor y)$
$(x_1 \lor x_2 \lor x_3 \lor \neg y)$

Decisions

p = false
q = false
b = true



Best Current SAT Solvers

Conflict-Directed Clause-Learning (CDCL) Algorithms
Minisat, Glucose, MapleSAT, CaDiCaL

They rely on many optimizations:

- No explicit computation of residual formulas, just fast calculation of the unit propagations that will happen. "watched literals"
- No explicit backtracking: New clauses always chosen to generate unit propagations higher in the tree. "asserting clauses"
- Heuristics based on learned clauses to decide what free choices to make. "VSIDS"
- Pruning of cache of learned clauses so only recently used ones are kept.
- Periodic restarting search with original formula plus learned clauses.
- etc...

Best Current SAT Solvers

Conflict-Directed Clause-Learning (CDCL) Algorithms

Minisat, Glucose, MapleSAT, CaDiCaL

They work well on many practical formulas even with hundreds of thousands of variables or more.

- Often used in proving properties of human-produced designs.
- They are incorporated in software verification tools and a variety of automated reasoning (SMT Solvers)
- We really don't know why they work so well.
- Definitely worth a try!

However, they provably perform very badly even on some small formulas of a few hundred or thousand variables. We have a pretty good idea why.