CSE 421
Introduction to Algorithms

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Lecture 9, Winter 2024

Recurrences

Announcements

- Divide and Conquer and Recurrences
  - Recurrence Techniques
  - Fast Matrix Multiplication
  - Counting Inversions (5.3)
  - Closest Pair (5.4)
  - Multiplication (5.5)
  - Quicksort and Median Finding
- Dynamic Programming
- Midterm, Friday, February 9

Divide and Conquer

```
Array Mergesort(Array a){
        n = a.Length;
        if (n <= 1)
                return a;
        b = Mergesort(a[0 .. n/2]);
        c = Mergesort(a[n/2+1 .. n-1]);
        return Merge(b, c);
}
```

Algorithm Analysis

- Cost of Merge
- Cost of Mergesort

$$T(n) = 2T(n/2) + cn; \; T(1) = c;$$

Recurrence Analysis

- Solution methods
  - Unrolling recurrence
  - Guess and verify
  - Plugging in to a “Master Theorem”

Useful Math Facts

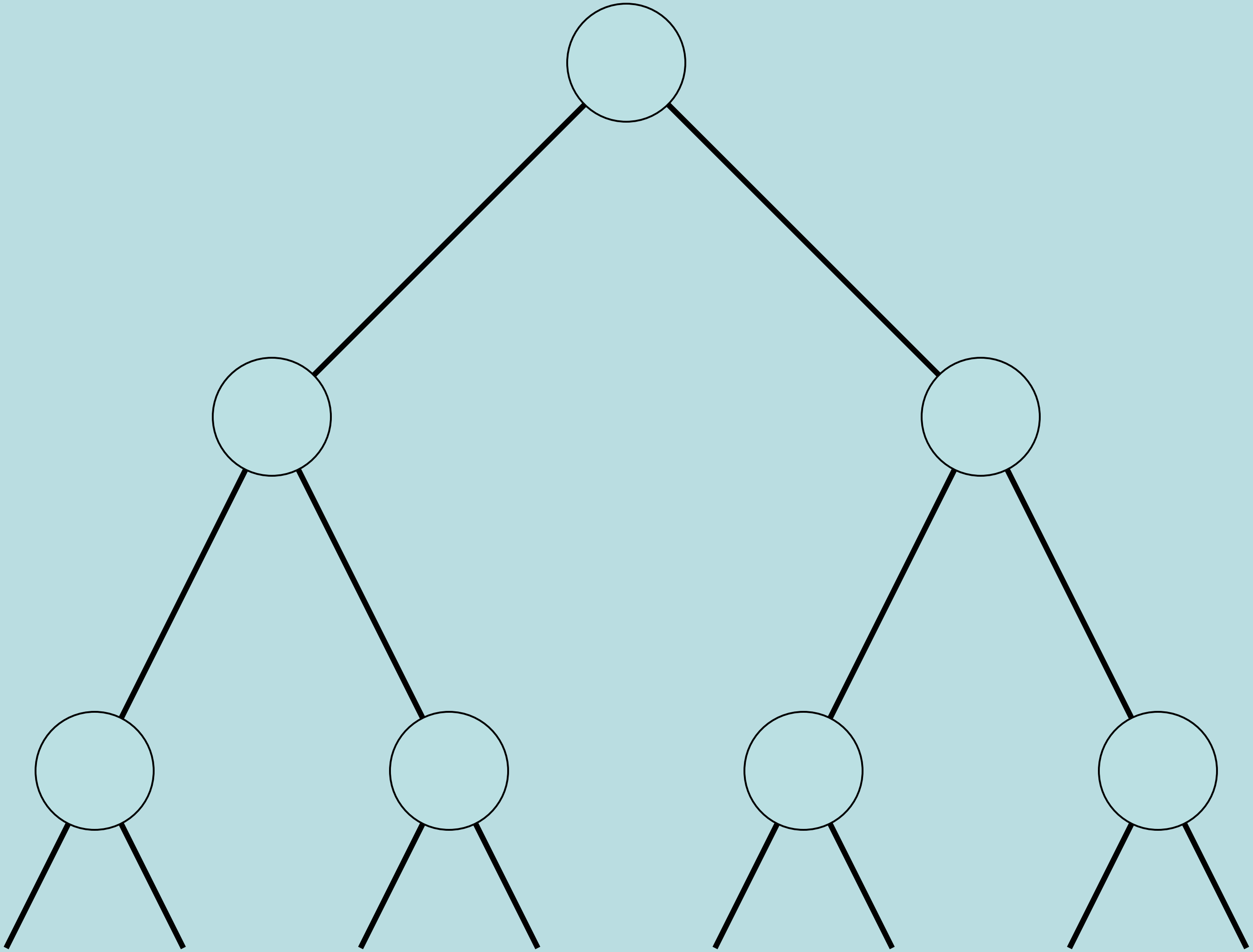
$$k^{\log_k n} = n$$

$$\log_k n = \frac{\log_2 n}{\log_2 k}$$

$$k^{\log_2 n} = n^{\log_2 k}$$

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}$$

Unrolling the recurrence



$T(n) = 2T(n/2) + n; T(1) = 1;$

Substitution

Prove $T(n) \leq n (\log_2 n + 1)$ for $n \geq 1$

Induction:

Base Case:

Induction Hypothesis:

Master Theorem

- $T(n) = a\,T(n/b) + O(n^d)$

- $T(n) = O(n^d)$ if $d > \log_b a$
- $T(n) = O(n^d \log n)$ if $d = \log_b a$
- $T(n) = O(n^{\log_b a})$ if $d < \log_b a$

A better mergesort (?)

- Divide into 3 subarrays and recursively sort
- Apply 3-way merge

What is the recurrence?

Unroll recurrence for $T(n) = 3T(n/3) + dn$

$$T(n) = aT(n/b) + f(n)$$

$T(n) = T(n/2) + cn$

Where does this recurrence arise?

Recursive Matrix Multiplication

Multiply 2 x 2 Matrices:

$$\begin{vmatrix} r & s \\ t & u \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & g \\ f & h \end{vmatrix}$$

$r = ae + bf$

$s = ag + bh$

$t = ce + df$

$u = cg + dh$

A N x N matrix can be viewed as a 2 x 2 matrix with entries that are (N/2) x (N/2) matrices.

The recursive matrix multiplication algorithm recursively multiplies the (N/2) x (N/2) matrices and combines them using the equations for multiplying 2 x 2 matrices

Recursive Matrix Multiplication

- How many recursive calls are made at each level?

- How much work in combining the results?

- What is the recurrence?

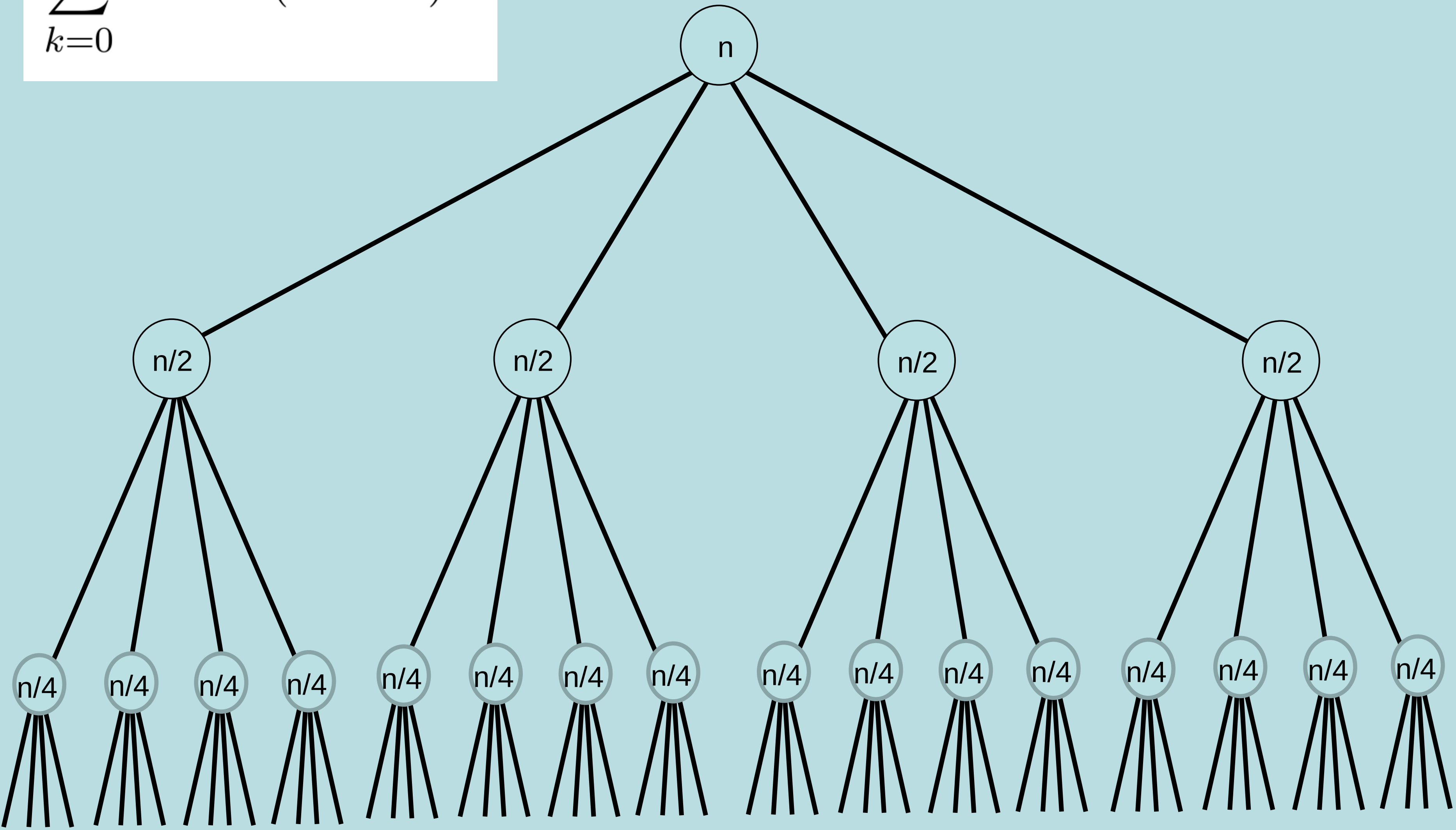
What is the run time for the recursive Matrix Multiplication Algorithm?

- Recurrence:

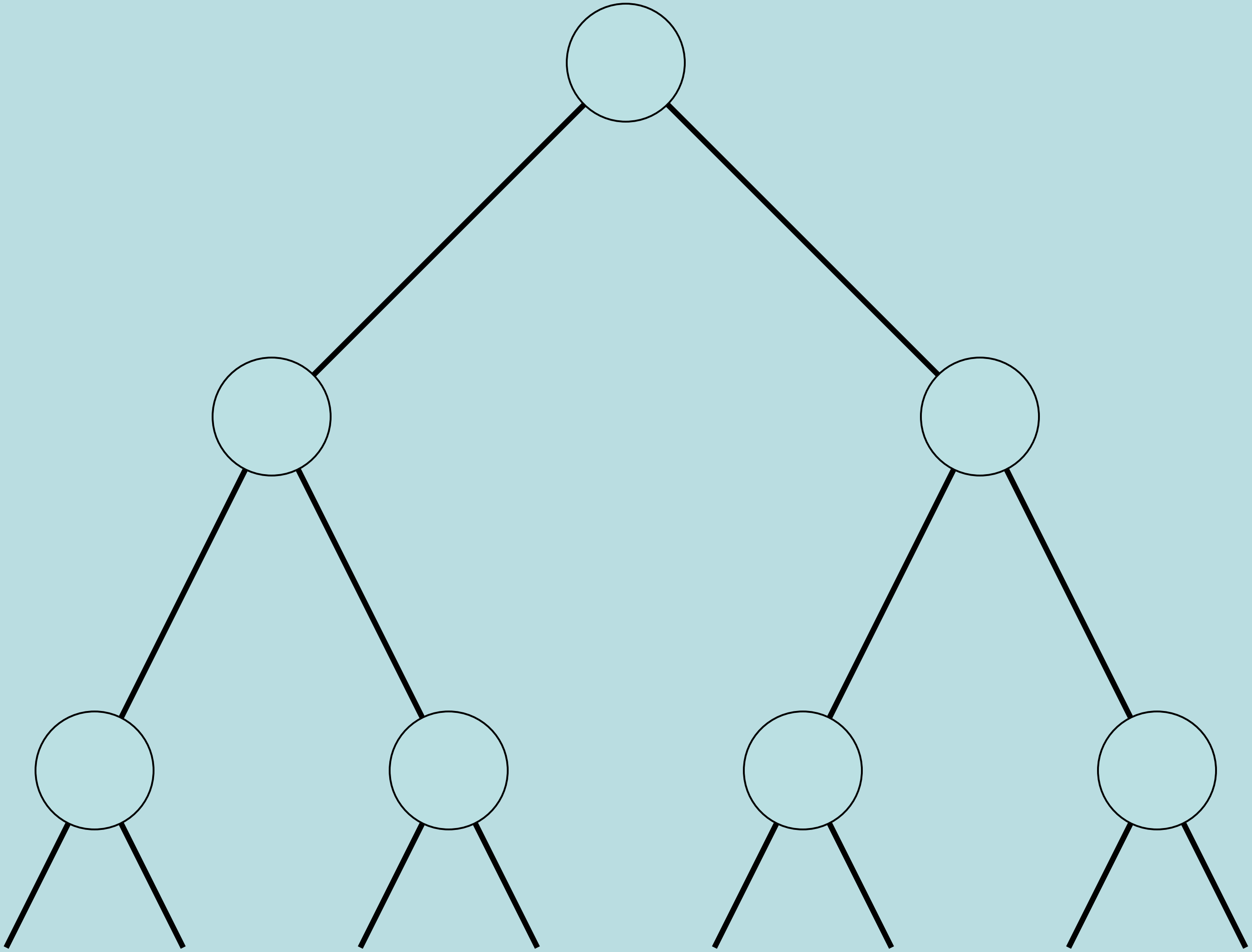
Total Work

$$\sum_{k=0}^{\log n} 2^k n = (2n-1)n$$

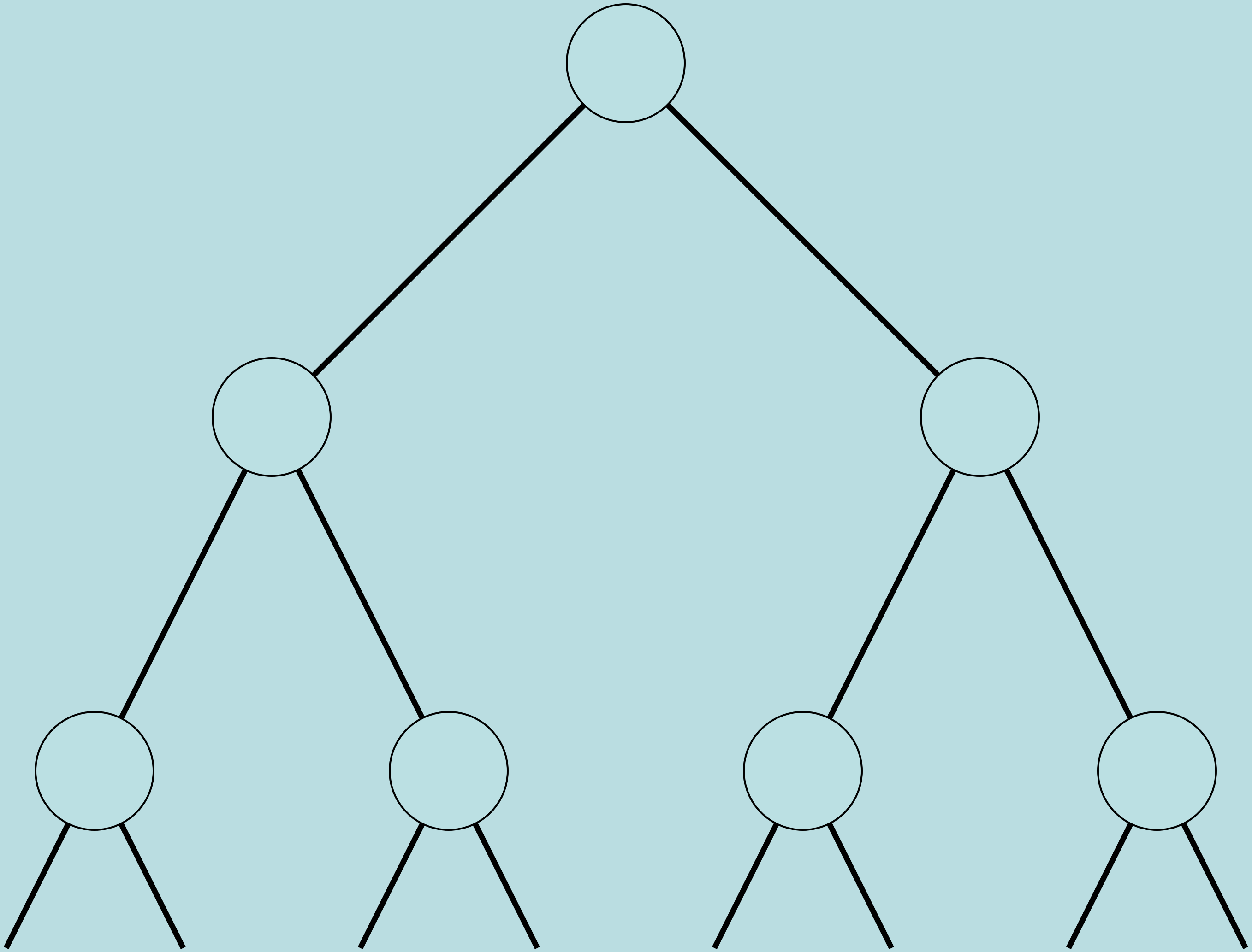
T(n) = 4T(n/2) + n



$$T(n) = 2T(n/2) + n^2$$



$$T(n) = 2T(n/2) + n^{1/2}$$



Recurrences

- Three basic behaviors
  - Dominated by initial case
  - Dominated by base case
  - All cases equal – we care about the depth

What you really need to know about recurrences

- Work per level changes geometrically with the level
- Geometrically increasing ($x > 1$)
  - The bottom level wins
- Geometrically decreasing ($x < 1$)
  - The top level wins
- Balanced ($x = 1$)
  - Equal contribution

Classify the following recurrences (Increasing, Decreasing, Balanced)

- $T(n) = n + 5T(n/8)$

- $T(n) = n + 9T(n/8)$

- $T(n) = n^2 + 4T(n/2)$

- $T(n) = n^3 + 7T(n/2)$

- $T(n) = n^{1/2} + 3T(n/4)$