CSE 421
Introduction to Algorithms

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Lecture 10, Winter 2024

Divide and Conquer

Announcements

- Divide and Conquer and Recurrences
  - ~~Recurrence Techniques~~
  - Fast Matrix Multiplication
  - Counting Inversions (5.3)
  - Closest Pair (5.4)
  - Multiplication (5.5)
  - Quicksort and Median Finding
- Dynamic Programming
- Midterm, Friday, February 9

$$k^{\log_k n} = n$$

$$\log_k n = \frac{\log_2 n}{\log_2 k}$$

$$k^{\log_2 n} = n^{\log_2 k}$$

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}$$

Recurrence Analysis

- Solution methods
  - Unrolling recurrence
  - Guess and verify

$$\boxed{T(n) \leq T(3n/4) + T(n/5) + 20\ n}$$

  - Plugging in to a "Master Theorem"
    - $T(n) = a\ T(n/b) + O(n^d)$
      - $T(n) = O(n^d)$ if $d > \log_b a$
      - $T(n) = O(n^d \log n)$ if $d = \log_b a$
      - $T(n) = O(n^{\log_b a})$ if $d < \log_b a$

Recursive Matrix Multiplication

Multiply 2 x 2 Matrices:

$$\begin{vmatrix} r & s \\ t & u \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & g \\ f & h \end{vmatrix}$$

$r = ae + bf$

$s = ag + bh$

$t = ce + df$

$u = cg + dh$

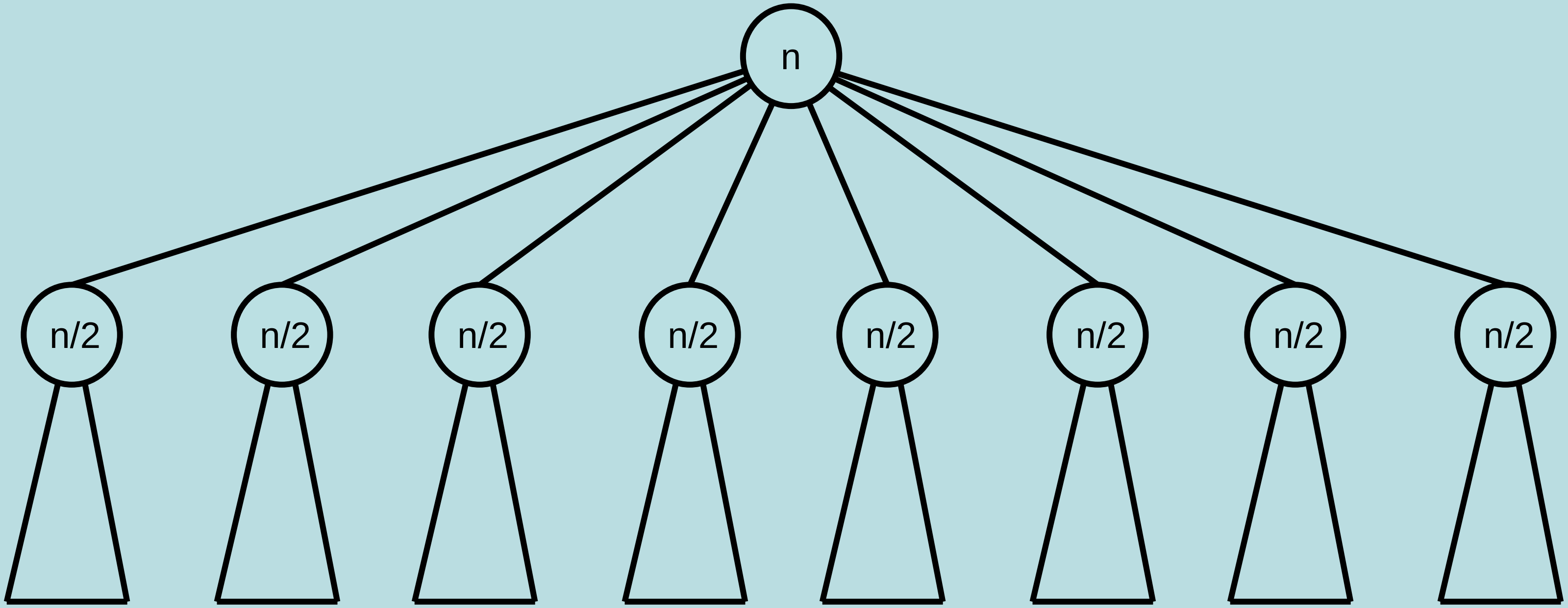
A N x N matrix can be viewed as a 2 x 2 matrix with entries that are (N/2) x (N/2) matrices.

The recursive matrix multiplication algorithm recursively multiplies the (N/2) x (N/2) matrices and combines them using the equations for multiplying 2 x 2 matrices

Recursive Matrix Multiplication

- How many recursive calls are made at each level?
  - 8, for the multiplication of n/2 X n/2 submatrices
- How much work in combining the results?
  - $O(n^2)$, for matrix addition and copying matrices
- What is the recurrence?
  - $T(n) = 8\ T(n/2) + n^2$; $T(2) = 1$;

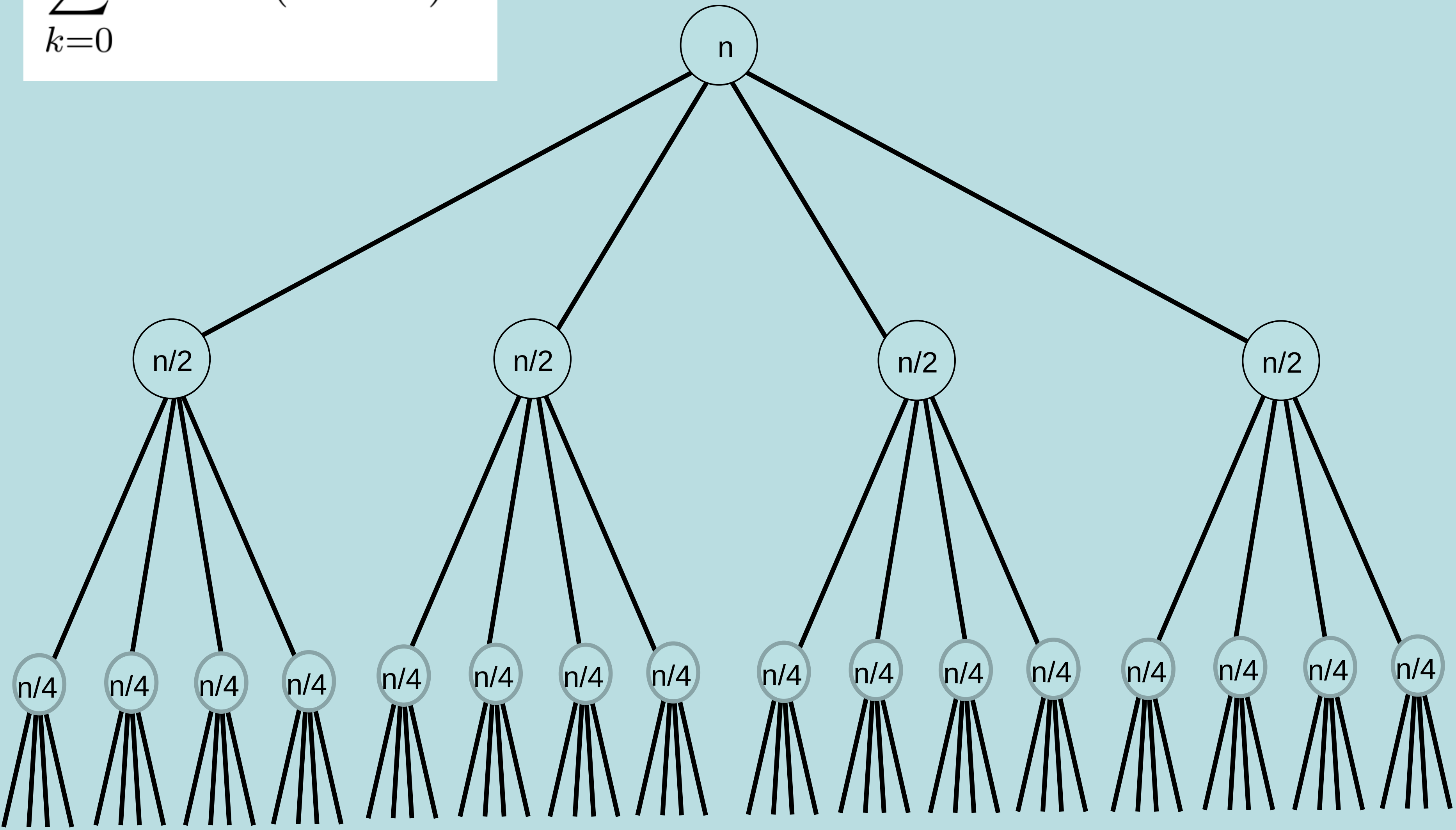
$T(n) = 8\,T(n/2) + n^2$



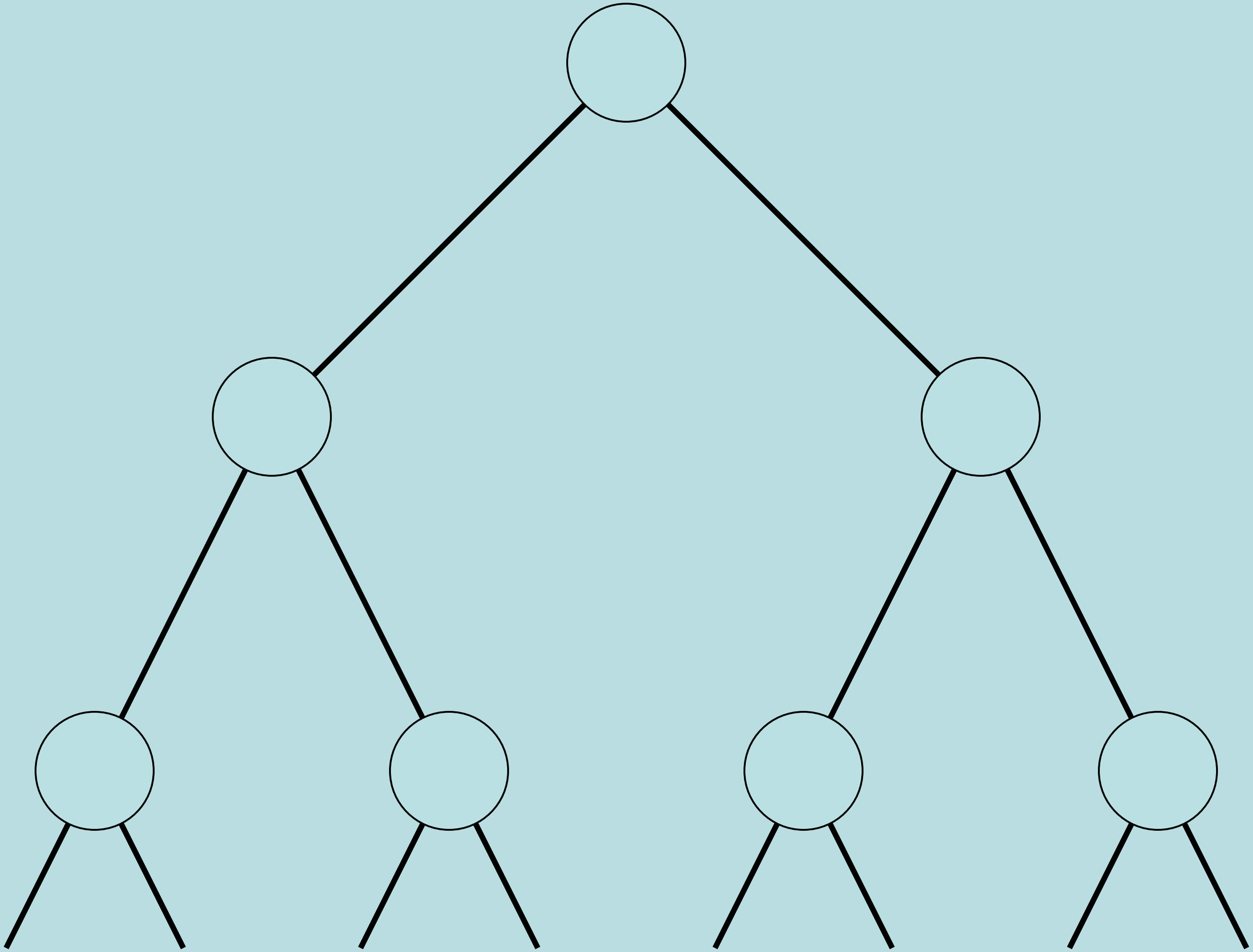
Total Work

$$\sum_{k=0}^{\log n} 2^k n = (2n-1)n$$

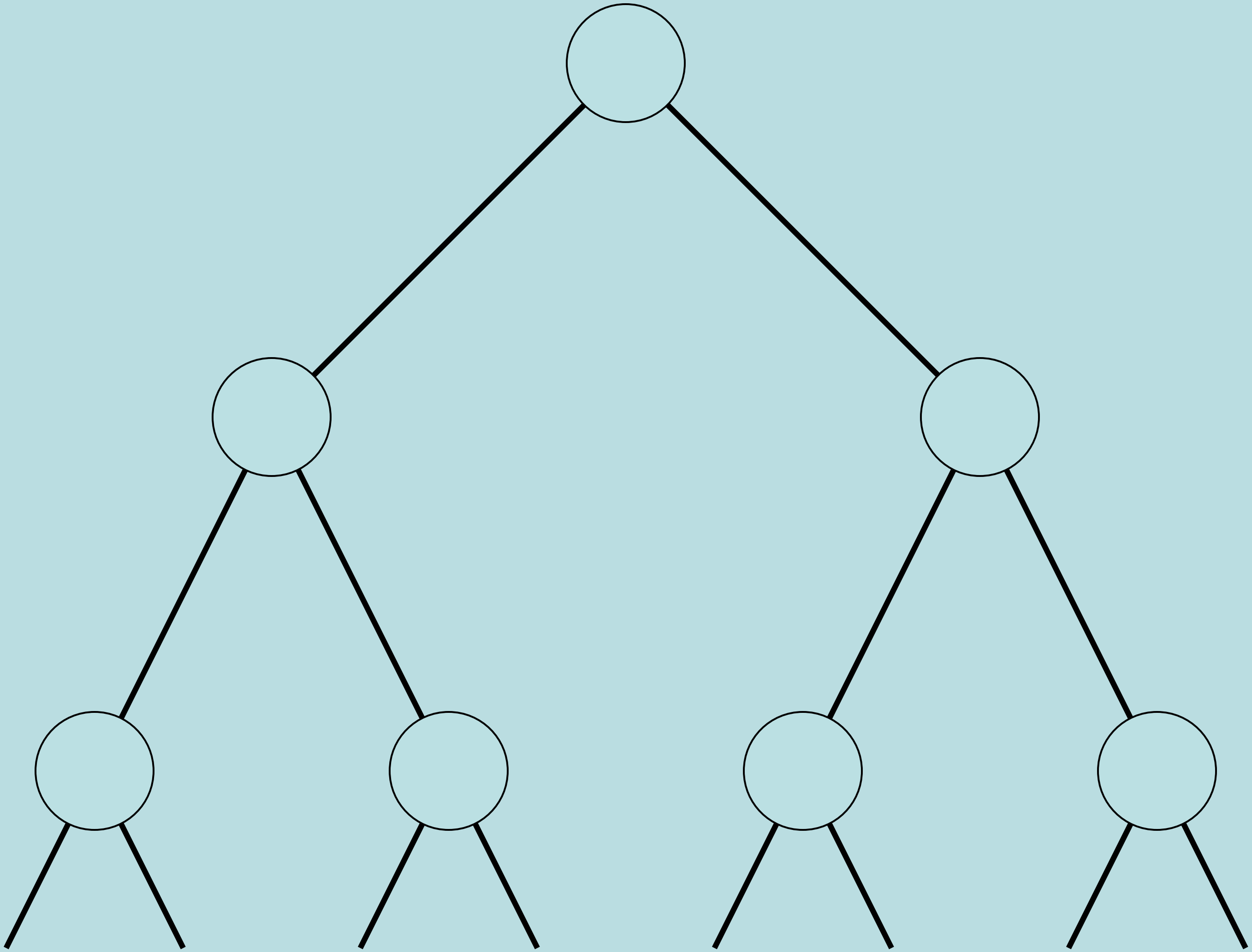
T(n) = 4T(n/2) + n



$T(n) = 2T(n/2) + n^2$



$$T(n) = 2T(n/2) + n^{1/2}$$



Recurrences

- Three basic behaviors
  - Dominated by initial case
  - Dominated by base case
  - All cases equal – we care about the depth

What you really need to know about recurrences

- Work per level changes geometrically with the level
- Geometrically increasing ($x > 1$)
  - The bottom level wins
- Geometrically decreasing ($x < 1$)
  - The top level wins
- Balanced ($x = 1$)
  - Equal contribution

Classify the following recurrences (Increasing, Decreasing, Balanced)

- $T(n) = n + 5T(n/8)$
- $T(n) = n + 9T(n/8)$
- $T(n) = n^2 + 4T(n/2)$
- $T(n) = n^3 + 7T(n/2)$
- $T(n) = n^{1/2} + 3T(n/4)$

Strassen's Algorithm

Multiply 2 x 2 Matrices:

$$\begin{vmatrix} r & s \\ t & u \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & g \\ f & h \end{vmatrix}$$

$r = p_1 + p_2 – p_4 + p_6$

$s = p_4 + p_5$

$t = p_6 + p_7$

$u = p_2 - p_3 + p_5 - p_7$

Where:

$p_1 = (b – d)(f + h)$

$p_2 = (a + d)(e + h)$

$p_3 = (a – c)(e + g)$

$p_4 = (a + b)h$

$p_5 = a(g – h)$

$p_6 = d(f – e)$

$p_7 = (c + d)e$

From AHU 1974

Recurrence for Strassen's Algorithm

- $T(n) = 7\ T(n/2) + cn^2$
- What is the runtime?

$\log_2 7 = 2.8073549221$

Strassen's Algorithm

- Treat n x n matrices as 2 x 2 matrices of n/2 x n/2 submatrices
- Use Strassen's trick to multiply 2 x 2 matrices with 7 multiplies
- Base case standard multiplication for single entries
- Recurrence: $T(n) = 7\,T(n/2) + cn^2$
- Solution is $O(7^{\log n}) = O(n^{\log 7})$ which is about $O(n^{2.807})$
- Practical for n ~ 64
- Standard trick – switch to normal algorithm for small values of n

Divide and Conquer Algorithms

- Split into sub problems
- Recursively solve the problem
- Combine solutions

- Make progress in the split and combine stages
  - Quicksort – progress made at the split step
  - Mergesort – progress made at the combine step
- D&C Algorithms
  - Strassen’s Algorithm – Matrix Multiplication
  - Inversions
  - Median
  - Closest Pair
  - Integer Multiplication
  - FFT

Inversion Problem

- Let $a_1, \ldots a_n$ be a permutation of $1 \ldots n$
- (a_i, a_j) is an inversion if $i < j$ and $a_i > a_j$

4, 6, 1, 7, 3, 2, 5

- Problem: given a permutation, count the number of inversions
- This can be done easily in $O(n^2)$ time
  - Can we do better?

Application

- Counting inversions can be use to measure how close ranked preferences are
  - People rank 20 movies, based on their rankings you cluster people who like that same type of movie

Counting Inversions

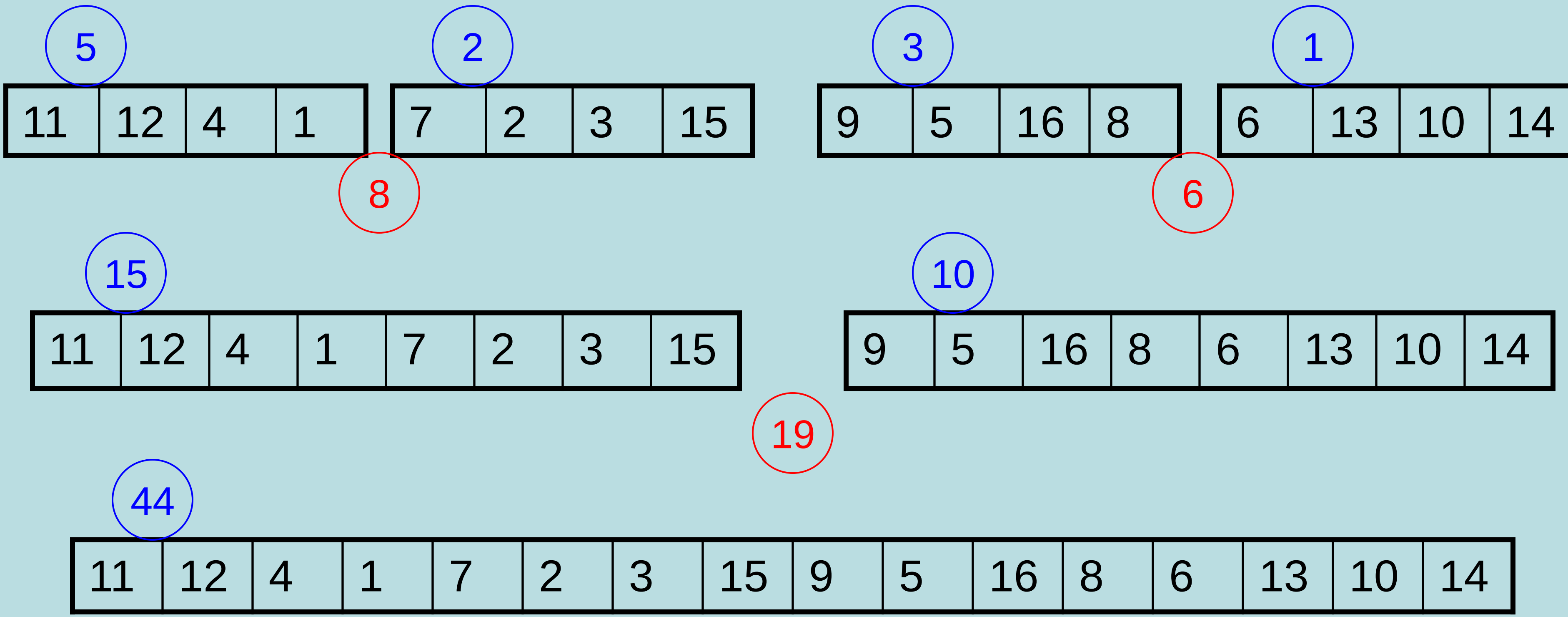
11	12	4	1	7	2	3	15	9	5	16	8	6	13	10	14

Count inversions on lower half

Count inversions on upper half

Count the inversions between the halves

Count the Inversions



Problem – how do we count inversions between sub problems in O(n) time?

- Solution – Count inversions while merging

1	2	3	4	7	11	12	15

5	6	8	9	10	13	14	16

Standard merge algorithm – add to inversion count when an element is moved from the upper array to the solution

Use the merge algorithm to count inversions

1	4	11	12

2	3	7	15

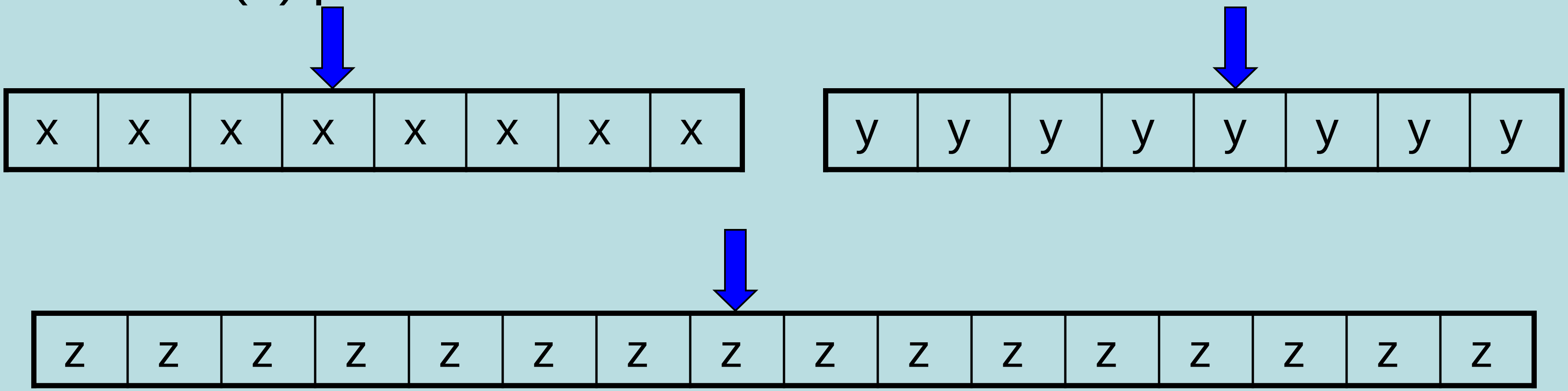
5	8	9	16

6	10	13	14

Indicate the number of inversions for each element detected when merging

Inversions

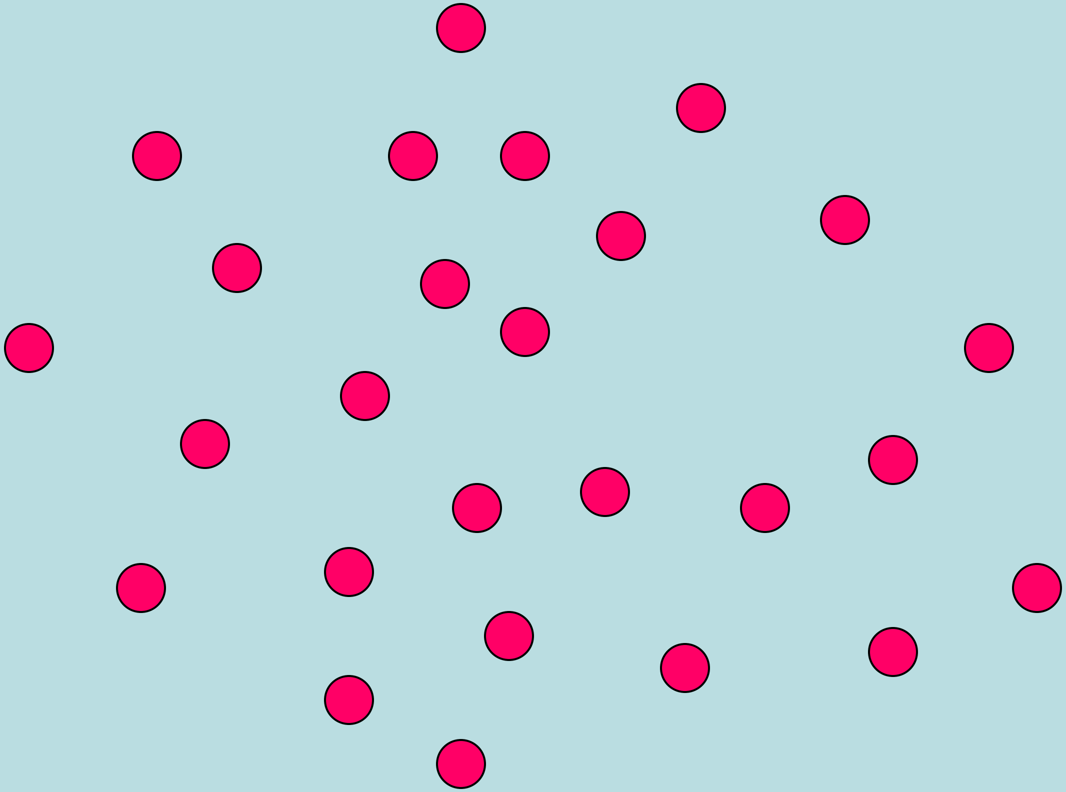
- Counting inversions between two sorted lists
  - O(1) per element to count inversions



- Algorithm summary
  - Satisfies the “Standard recurrence”
  - $T(n) = 2\, T(n/2) + cn$

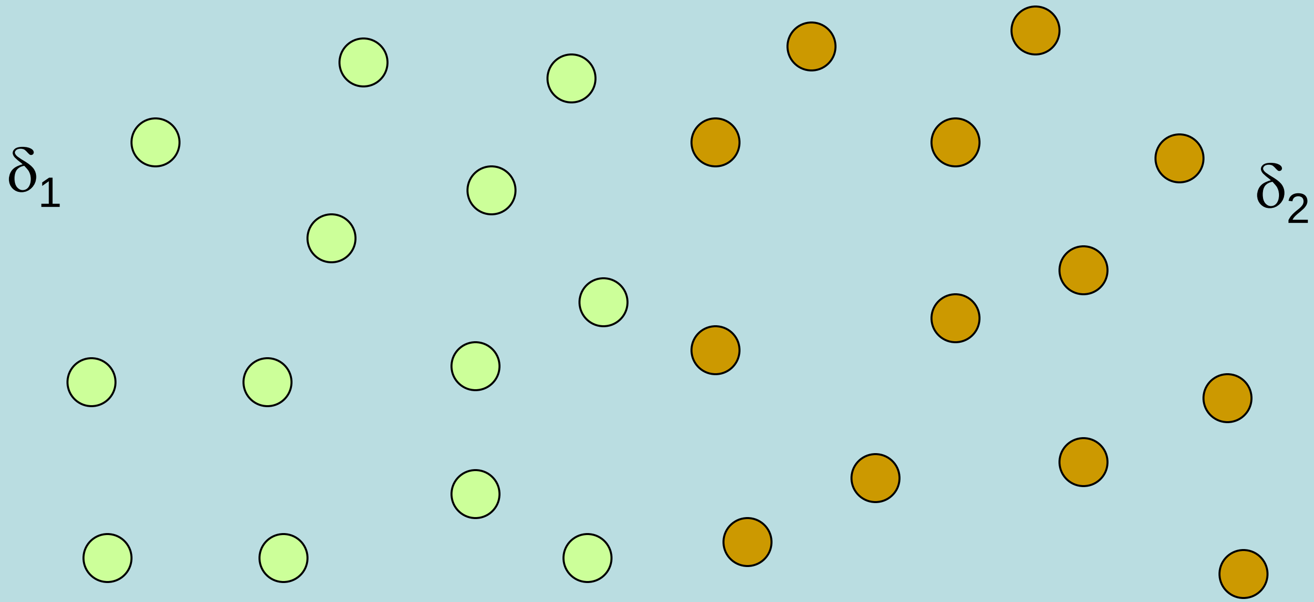
Closest Pair Problem (2D)

- Given a set of points find the pair of points p, q that minimizes dist(p, q)



Divide and conquer

- If we solve the problem on two subsets, does it help? (Separate by median x coordinate)



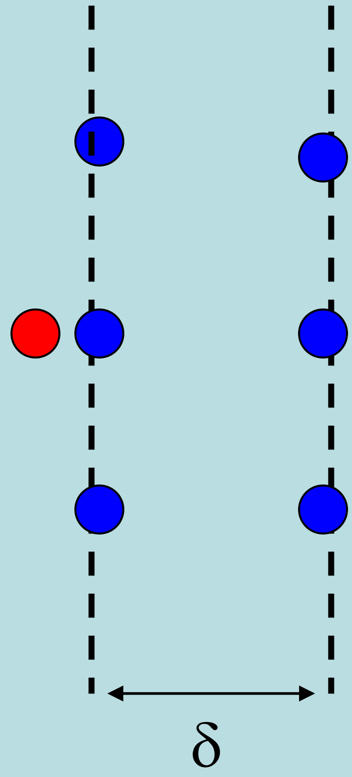
Packing Lemma

Suppose that the minimum distance between points is at least δ, what is the maximum number of points that can be packed in a ball of radius δ?

Combining Solutions

- Suppose the minimum separation from the sub problems is δ
- In looking for cross set closest pairs, we only need to consider points with δ of the boundary
- How many cross border interactions do we need to test?

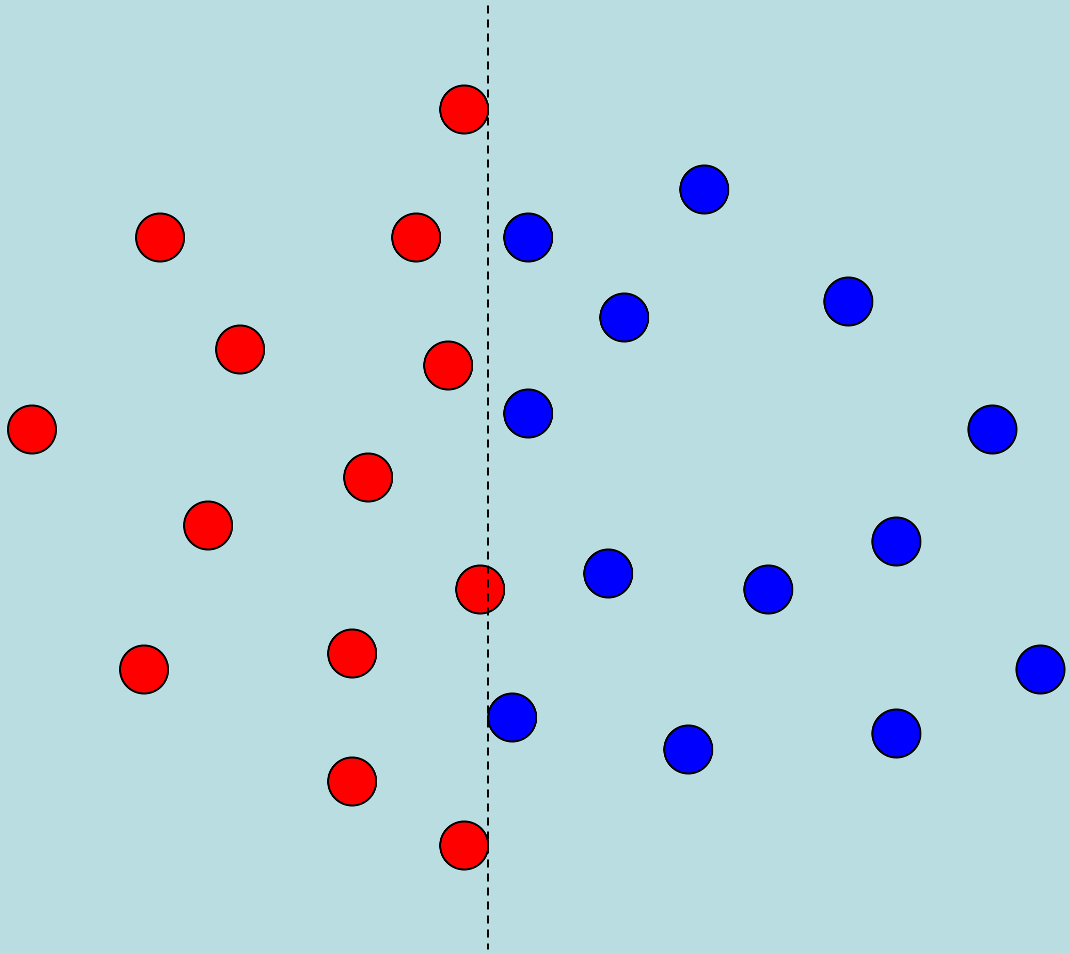
A packing lemma bounds the number of distances to check



Details

- Preprocessing: sort points by y
- Merge step
  - Select points in boundary zone
  - For each point in the boundary
    - Find highest point on the other side that is at most δ above
    - Find lowest point on the other side that is at most δ below
    - Compare with the points in this interval (there are at most 6)

Identify the pairs of points that are compared in the merge step following the recursive calls



Algorithm run time

- After preprocessing:
  - $T(n) = cn + 2\,T(n/2)$

Integer Arithmetic

```
    9715480283945084383094856701043643845790217965702956767
+   1242431098234099057329075097179898430928779579277597977
```

Runtime for standard algorithm to add two n digit numbers:

```
    2095067093034680994318596846868779409766717133476767930
X   5920175091777634709677679342929097012308956679993010921
```

Runtime for standard algorithm to multiply two n digit numbers:

Recursive Multiplication Algorithm (First attempt)

$x = x_1 2^{n/2} + x_0$

$y = y_1 2^{n/2} + y_0$

$xy = (x_1 2^{n/2} + x_0)(y_1 2^{n/2} + y_0)$

$= x_1 y_1 2^n + (x_1 y_0 + x_0 y_1) 2^{n/2} + x_0 y_0$

Recurrence:

Run time:

Simple algebra

$x = x_1\, 2^{n/2} + x_0$

$y = y_1\, 2^{n/2} + y_0$

$xy = x_1 y_1\, 2^n + (x_1 y_0 + x_0 y_1)\, 2^{n/2} + x_0 y_0$

$p = (x_1 + x_0)(y_1 + y_0) = x_1 y_1 + x_1 y_0 + x_0 y_1 + x_0 y_0$

Karatsuba's Algorithm

Multiply n-digit integers x and y

Let $x = x_1 2^{n/2} + x_0$ and $y = y_1 2^{n/2} + y_0$
Recursively compute

$a = x_1 y_1$

$b = x_0 y_0$

$p = (x_1 + x_0)(y_1 + y_0)$

Return $a2^n + (p - a - b)2^{n/2} + b$

Recurrence: $T(n) = 3T(n/2) + cn$

$\log_2 3 = 1.58496250073\ldots$