CSE 421
Introduction to Algorithms

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Lecture 10, Winter 2024

Divide and Conquer

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Announcements

- Divide and Conquer and Recurrences
  - ~~Recurrence Techniques~~
  - Fast Matrix Multiplication
  - Counting Inversions (5.3)
  - Closest Pair (5.4)
  - Multiplication (5.5)
  - Quicksort and Median Finding
- Dynamic Programming
- Midterm, Friday, February 9

$$k^{\log_k n} = n$$

$$\log_k n = \frac{\log_2 n}{\log_2 k}$$

$$k^{\log_2 n} = n^{\log_2 k}$$

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}$$

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Recurrence Analysis

- Solution methods
  - Unrolling recurrence
  - Guess and verify

$\boxed{T(n) \leq T(3n/4) + T(n/5) + 20\ n}$

  - Plugging in to a "Master Theorem"
    - $T(n) = a\ T(n/b) + O(n^d)$
      - $T(n) = O(n^d)$ if $d > \log_b a$
      - $T(n) = O(n^d \log n)$ if $d = \log_b a$
      - $T(n) = O(n^{\log_b a})$ if $d < \log_b a$

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Recursive Matrix Multiplication

Multiply 2 x 2 Matrices:

$$\begin{vmatrix} r & s \\ t & u \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & g \\ f & h \end{vmatrix}$$

r = ae + bf

s = ag + bh

t = ce + df

u = cg + dh

A N x N matrix can be viewed as a 2 x 2 matrix with entries that are (N/2) x (N/2) matrices.

The recursive matrix multiplication algorithm recursively multiplies the (N/2) x (N/2) matrices and combines them using the equations for multiplying 2 x 2 matrices

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Recursive Matrix Multiplication

- How many recursive calls are made at each level?
  - 8, for the multiplication of n/2 X n/2 submatrices
- How much work in combining the results?
  - $O(n^2)$, for matrix addition and copying matrices
- What is the recurrence?
  - $T(n) = 8\ T(n/2) + n^2$; $T(2) = 1$;

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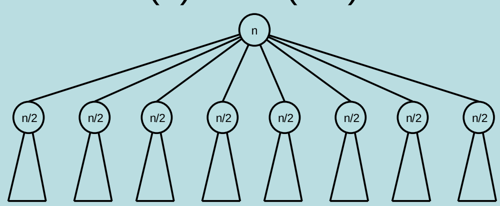
$T(n) = 8\ T(n/2) + n^2$



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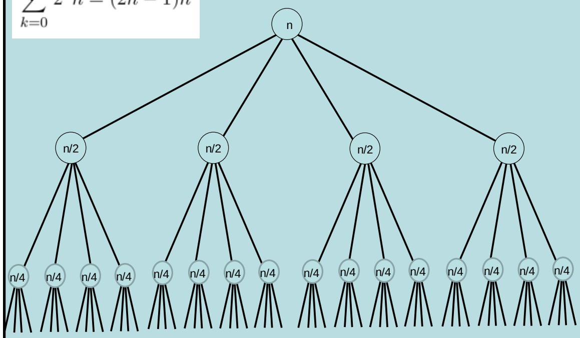
Total Work

$$\sum_{k=0}^{\log n} 2^k n = (2n-1)n$$

T(n) = 4T(n/2) + n



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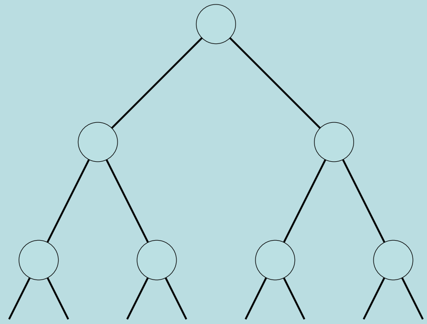
T(n) = 2T(n/2) + n^2



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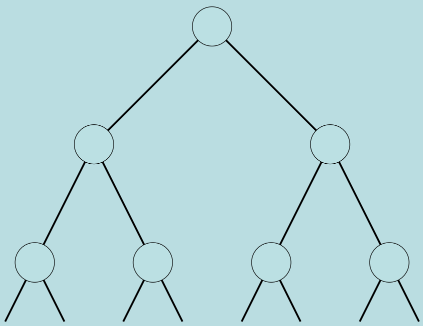
T(n) = 2T(n/2) + $n^{1/2}$



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Recurrences

- Three basic behaviors
  - Dominated by initial case
  - Dominated by base case
  - All cases equal – we care about the depth

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What you really need to know about recurrences

- Work per level changes geometrically with the level
- Geometrically increasing (x > 1)
  - The bottom level wins
- Geometrically decreasing (x < 1)
  - The top level wins
- Balanced (x = 1)
  - Equal contribution

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Classify the following recurrences (Increasing, Decreasing, Balanced)

- T(n) = n + 5T(n/8)
- T(n) = n + 9T(n/8)
- T(n) = n^2 + 4T(n/2)
- T(n) = n^3 + 7T(n/2)
- T(n) = $n^{1/2}$ + 3T(n/4)

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Strassen's Algorithm

Multiply 2 x 2 Matrices:

$$\begin{vmatrix} r & s \\ t & u \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} e & g \\ f & h \end{vmatrix}$$

$r = p_1 + p_2 - p_4 + p_6$

$s = p_4 + p_5$

$t = p_6 + p_7$

$u = p_2 - p_3 + p_5 - p_7$

Where:

$p_1 = (b - d)(f + h)$

$p_2 = (a + d)(e + h)$

$p_3 = (a - c)(e + g)$

$p_4 = (a + b)h$

$p_5 = a(g - h)$

$p_6 = d(f - e)$

$p_7 = (c + d)e$

From AHU 1974

Recurrence for Strassen's Algorithm

- $T(n) = 7\ T(n/2) + cn^2$
- What is the runtime?

$\log_2 7 = 2.8073549221$

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Strassen's Algorithm

- Treat n x n matrices as 2 x 2 matrices of n/2 x n/2 submatrices
- Use Strassen's trick to multiply 2 x 2 matrices with 7 multiplies
- Base case standard multiplication for single entries
- Recurrence: $T(n) = 7\ T(n/2) + cn^2$
- Solution is $O(7^{\log n}) = O(n^{\log 7})$ which is about $O(n^{2.807})$
- Practical for n ~ 64
- Standard trick – switch to normal algorithm for small values of n

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Divide and Conquer Algorithms

- Split into sub problems
- Recursively solve the problem
- Combine solutions

- Make progress in the split and combine stages
  - Quicksort – progress made at the split step
  - Mergesort – progress made at the combine step
- D&C Algorithms
  - Strassen's Algorithm – Matrix Multiplication
  - Inversions
  - Median
  - Closest Pair
  - Integer Multiplication
  - FFT

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Inversion Problem

- Let $a_1, \ldots a_n$ be a permutation of 1 . . n
- (a_i, a_j) is an inversion if $i < j$ and $a_i > a_j$

4, 6, 1, 7, 3, 2, 5

- Problem: given a permutation, count the number of inversions
- This can be done easily in $O(n^2)$ time
  - Can we do better?

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Application

- Counting inversions can be use to measure how close ranked preferences are
  - People rank 20 movies, based on their rankings you cluster people who like that same type of movie

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Counting Inversions

11	12	4	1	7	2	3	15	9	5	16	8	6	13	10	14

Count inversions on lower half

Count inversions on upper half

Count the inversions between the halves

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Count the Inversions

5   2   3   1

11 12 4 1 | 7 2 3 15 | 9 5 16 8 | 6 13 10 14

8   6

15   10

11 12 4 1 7 2 3 15 | 9 5 16 8 6 13 10 14

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11 12 4 1 7 2 3 15 9 5 16 8 6 13 10 14

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Problem – how do we count inversions between sub problems in O(n) time?

- Solution – Count inversions while merging

1 2 3 4 7 11 12 15   5 6 8 9 10 13 14 16

Standard merge algorithm – add to inversion count when an element is moved from the upper array to the solution

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Use the merge algorithm to count inversions

1 4 11 12   2 3 7 15

5 8 9 16   6 10 13 14

Indicate the number of inversions for each element detected when merging

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Inversions

- Counting inversions between two sorted lists
  - O(1) per element to count inversions

x x x x x x x x   y y y y y y y y

z z z z z z z z z z z z z z z z

- Algorithm summary
  - Satisfies the "Standard recurrence"
  - $T(n) = 2\,T(n/2) + cn$

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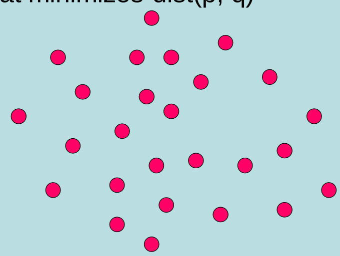
Closest Pair Problem (2D)

- Given a set of points find the pair of points p, q that minimizes dist(p, q)



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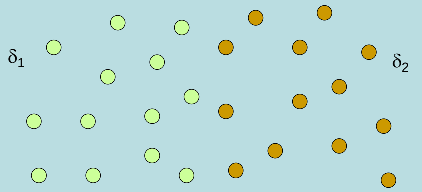
Divide and conquer

- If we solve the problem on two subsets, does it help? (Separate by median x coordinate)



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Packing Lemma

Suppose that the minimum distance between points is at least δ, what is the maximum number of points that can be packed in a ball of radius δ?

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Combining Solutions

- Suppose the minimum separation from the sub problems is δ
- In looking for cross set closest pairs, we only need to consider points with δ of the boundary
- How many cross border interactions do we need to test?

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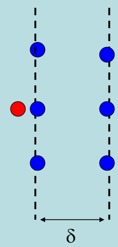
A packing lemma bounds the number of distances to check



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Details

- Preprocessing: sort points by y
- Merge step
  - Select points in boundary zone
  - For each point in the boundary
    - Find highest point on the other side that is at most δ above
    - Find lowest point on the other side that is at most δ below
    - Compare with the points in this interval (there are at most 6)

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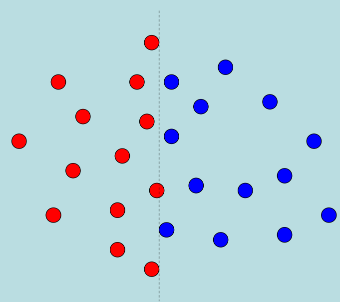
Identify the pairs of points that are compared in the merge step following the recursive calls



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Algorithm run time

- After preprocessing:
  - $T(n) = cn + 2\,T(n/2)$

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Integer Arithmetic

```
   9715480283945084383094856701043643845790217965702956767
+  1242431098234099057329075097179898430928779579277597977
```

Runtime for standard algorithm to add two n digit numbers:

```
   2095067093034680994318596846868779409766717133476767930
X  5920175091777634709677679342929097012308956679993010921
```

Runtime for standard algorithm to multiply two n digit numbers:

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Recursive Multiplication Algorithm (First attempt)

$x = x_1\, 2^{n/2} + x_0$

$y = y_1\, 2^{n/2} + y_0$

$xy = (x_1\, 2^{n/2} + x_0)\,(y_1\, 2^{n/2} + y_0)$

$\quad = x_1 y_1\, 2^n + (x_1 y_0 + x_0 y_1) 2^{n/2} + x_0 y_0$

Recurrence:

Run time:

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Simple algebra

$x = x_1\, 2^{n/2} + x_0$

$y = y_1\, 2^{n/2} + y_0$

$xy = x_1 y_1\, 2^n + (x_1 y_0 + x_0 y_1)\, 2^{n/2} + x_0 y_0$

$p = (x_1 + x_0)(y_1 + y_0) = x_1 y_1 + x_1 y_0 + x_0 y_1 + x_0 y_0$

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Karatsuba's Algorithm

Multiply n-digit integers x and y

Let $x = x_1\, 2^{n/2} + x_0$ and $y = y_1\, 2^{n/2} + y_0$
Recursively compute
$a = x_1 y_1$
$b = x_0 y_0$
$p = (x_1 + x_0)(y_1 + y_0)$
Return $a2^n + (p - a - b)2^{n/2} + b$

Recurrence: $T(n) = 3T(n/2) + cn$

$\log_2 3 = 1.58496250073\ldots$

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