CSE 421
Introduction to Algorithms

Lecture 12, Winter 2024

Dynamic Programming

Announcements

- Dynamic Programming Reading:
  - 6.1-6.2, Weighted Interval Scheduling
  - 6.4 Knapsack and Subset Sum
  - 6.6 String Alignment
    - 6.7* String Alignment in linear space
  - 6.8 Shortest Paths (again)
  - 6.9 Negative cost cycles
    - How to make an infinite amount of money

Dynamic Programming

- The most important algorithmic technique covered in CSE 421
- Key ideas
  - Express solution in terms of a polynomial number of sub problems
  - Order sub problems to avoid recomputation

Recursion vs Iteration

```
Factorial(n){
    if (n <= 1)
        return 1;
    else
        return n*Factorial(n-1);
}
```

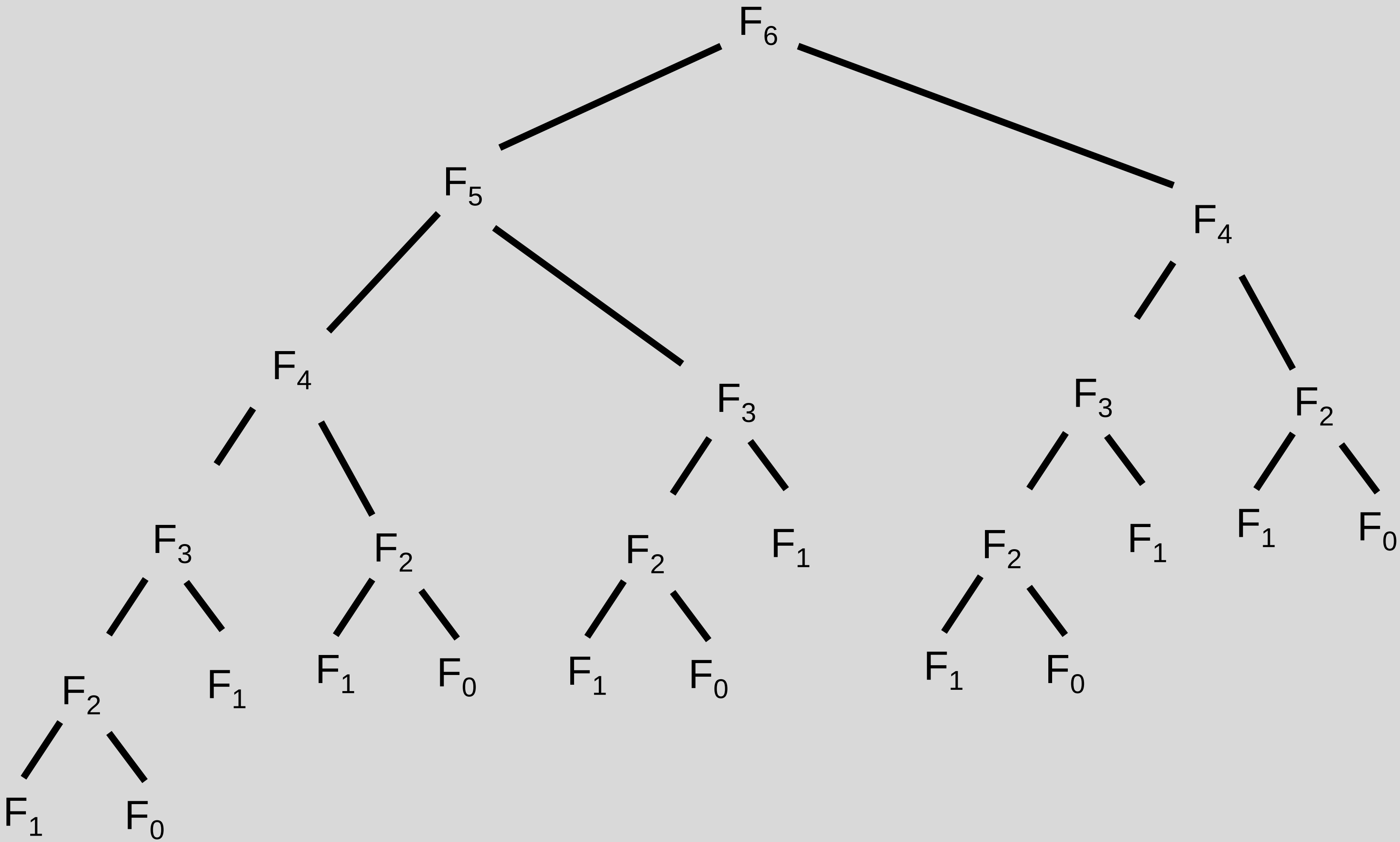
```
Factorial(n){
    v = 1;
    for (i = 2; i <= n; i++)
        v = v*i
    return v;
}
```

Counting Rabbits

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, . . .

$$F_0 = 0; \quad F_1 = 1; \quad F_n = F_{n-1} + F_{n-2}$$

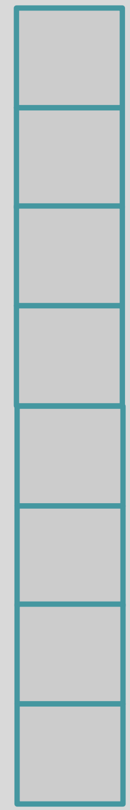
```
Fib(n){
    if (n =  0)
         return 0;
    else if (n = 1)
         return 1;
    else
         return Fib(n-1) + Fib(n-2);
}
```



F6
F5
F4
F4
F3
F3
F2
F3
F2
F2
F1
F2
F1
F1
F0
F2
F1
F1
F0
F1
F0
F1
F0
F1
F0

Fibonacci with Memoization

```
Fib(n){
    if (n ==  0)
        return 0;
    else if (n == 1)
        return 1;
    else
        return Fib(n-1) + Fib(n-2);
}
```



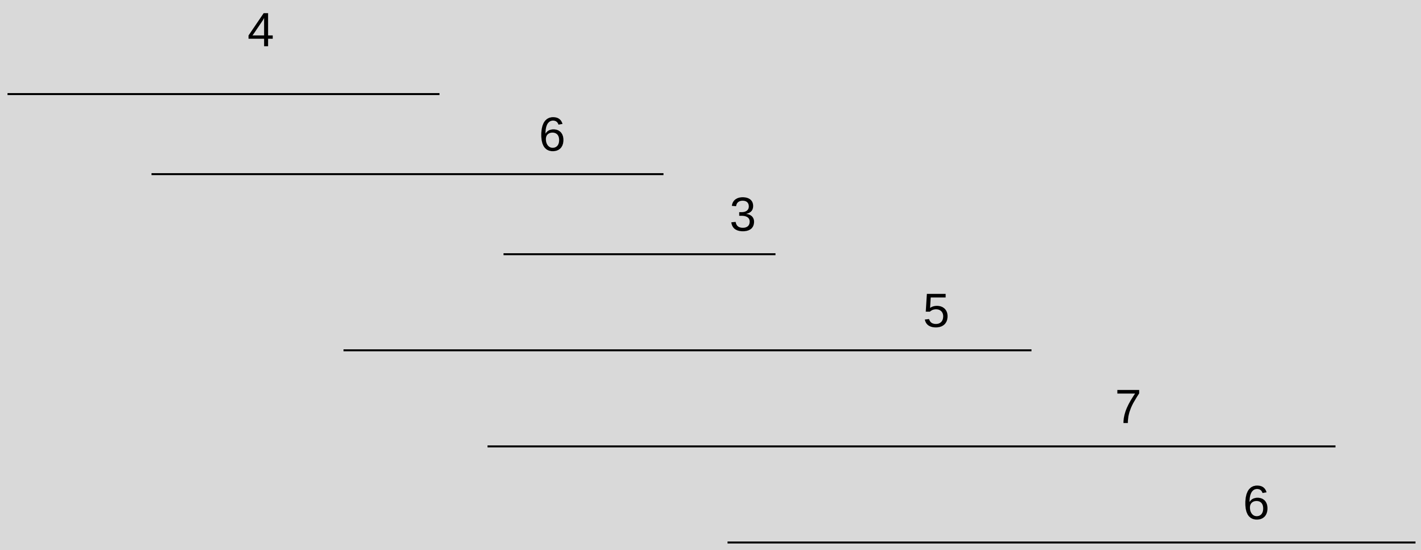
Reordering computation

```
Fib(n){
    int[ ] F = new [n+1]

    F[0] = 0;
    F[1] = 1;
    for (i = 2; i <= n; i++)
        F[i] = F[i-1] + F[i-2];
    return F[n];
}
```

Intervals sorted by end time

Dynamic Programming

- Weighted Interval Scheduling
- Given a collection of intervals $I_1,\ldots,I_n$ with weights $w_1,\ldots,w_n$, choose a maximum weight set of non-overlapping intervals



Intervals sorted by end time

Optimality Condition

- Opt[j] is the maximum weight independent set of intervals $I_1, I_2, \ldots, I_j$
- Opt[j] = max(Opt[j – 1], w_j + Opt[p[j]])
  - Where p[j] is the index of the last interval which finishes before I_j starts

Algorithm

```
MaxValue(j) =
   if j = 0 return 0
   else
        return max( MaxValue(j-1),
                    w_j + MaxValue(p[ j ]))
```

Worst case run time: 2^n

A better algorithm

M[j] initialized to -1 before the first recursive call for all j

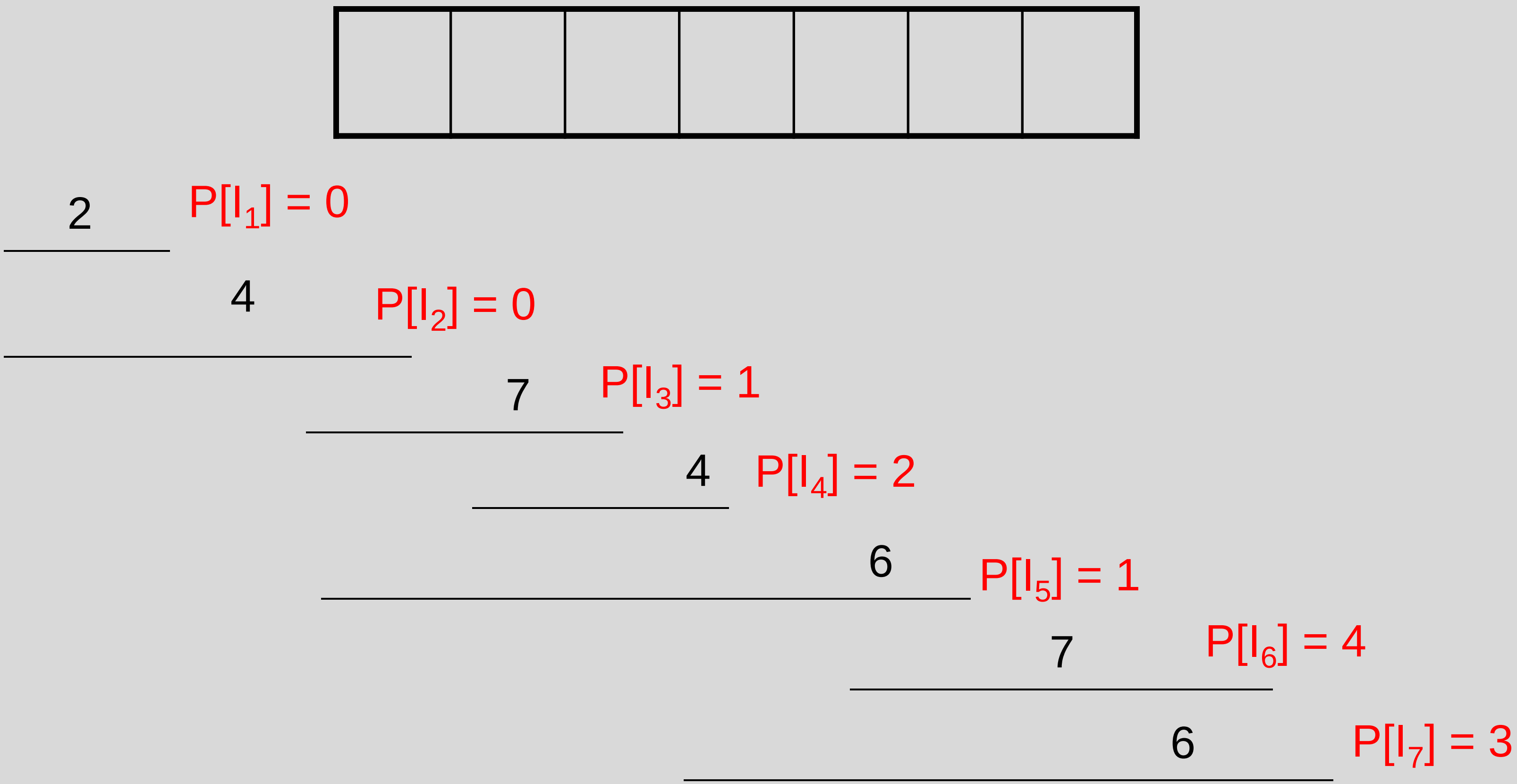
```
MaxValue(j) =
    if j = 0 return 0;
    else if M[ j ] != -1 return M[ j ];
    else {
        M[ j ] = max(MaxValue(j-1), w_j + MaxValue(p[ j ]));
        return M[ j ];
    }
```

Iterative Algorithm

```
MaxValue(n){
    int[ ] M = new int[n+1];

    M[0] = 0;

    for (int i = 1; i <= n; i++){
        M[ j ] = max(M[j-1], w_j + M[p[ j ]]);
    }

    return M[n];
}
```

Fill in the array with the Opt values

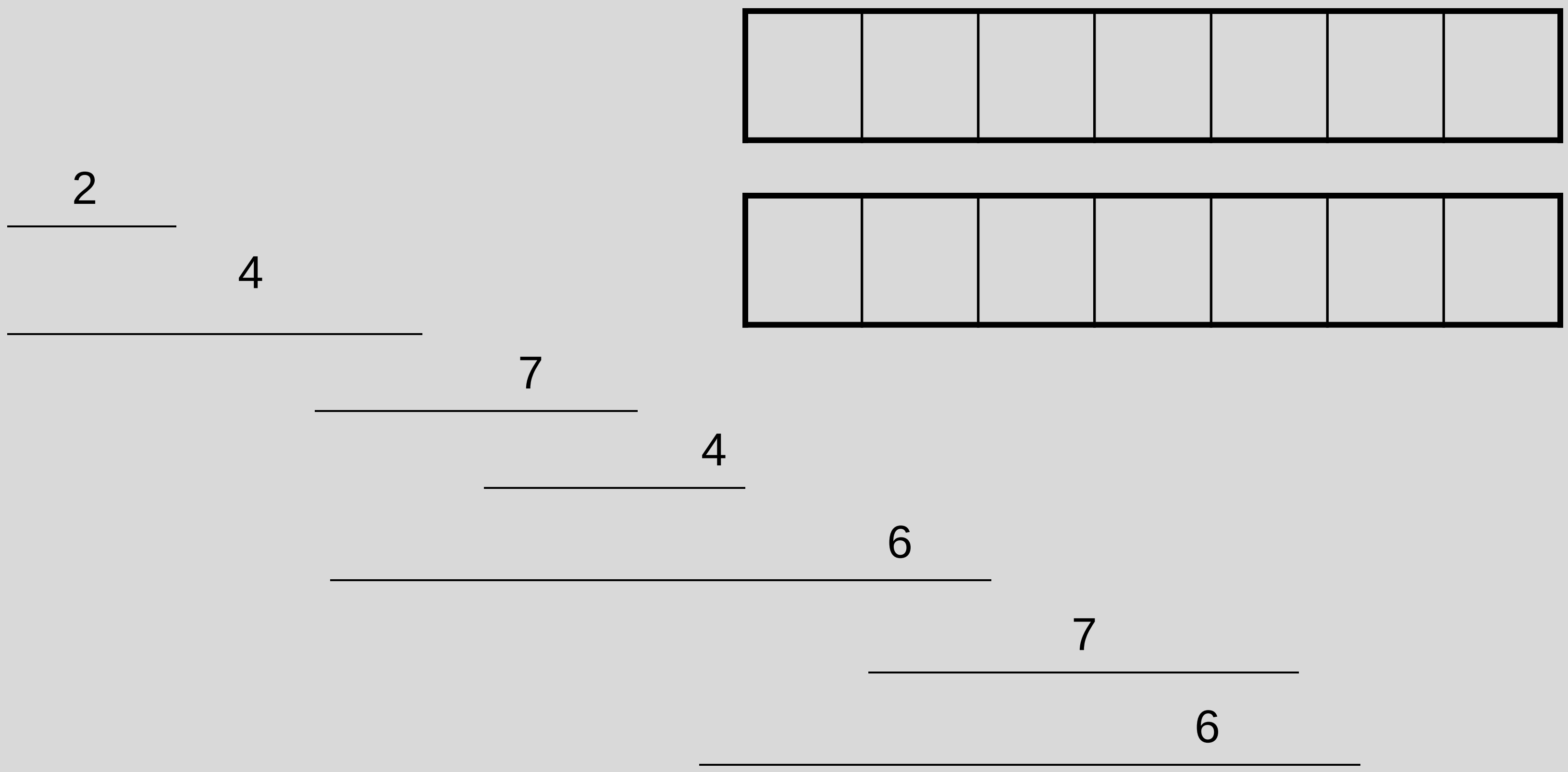
$$Opt[\ j\ ] = \max\ (Opt[\ j - 1],\ w_j + Opt[\ p[\ j\ ]\ ])$$



Computing the solution

Opt[j] = max (Opt[j – 1], w_j + Opt[p[j]])

Record which case is used in Opt computation



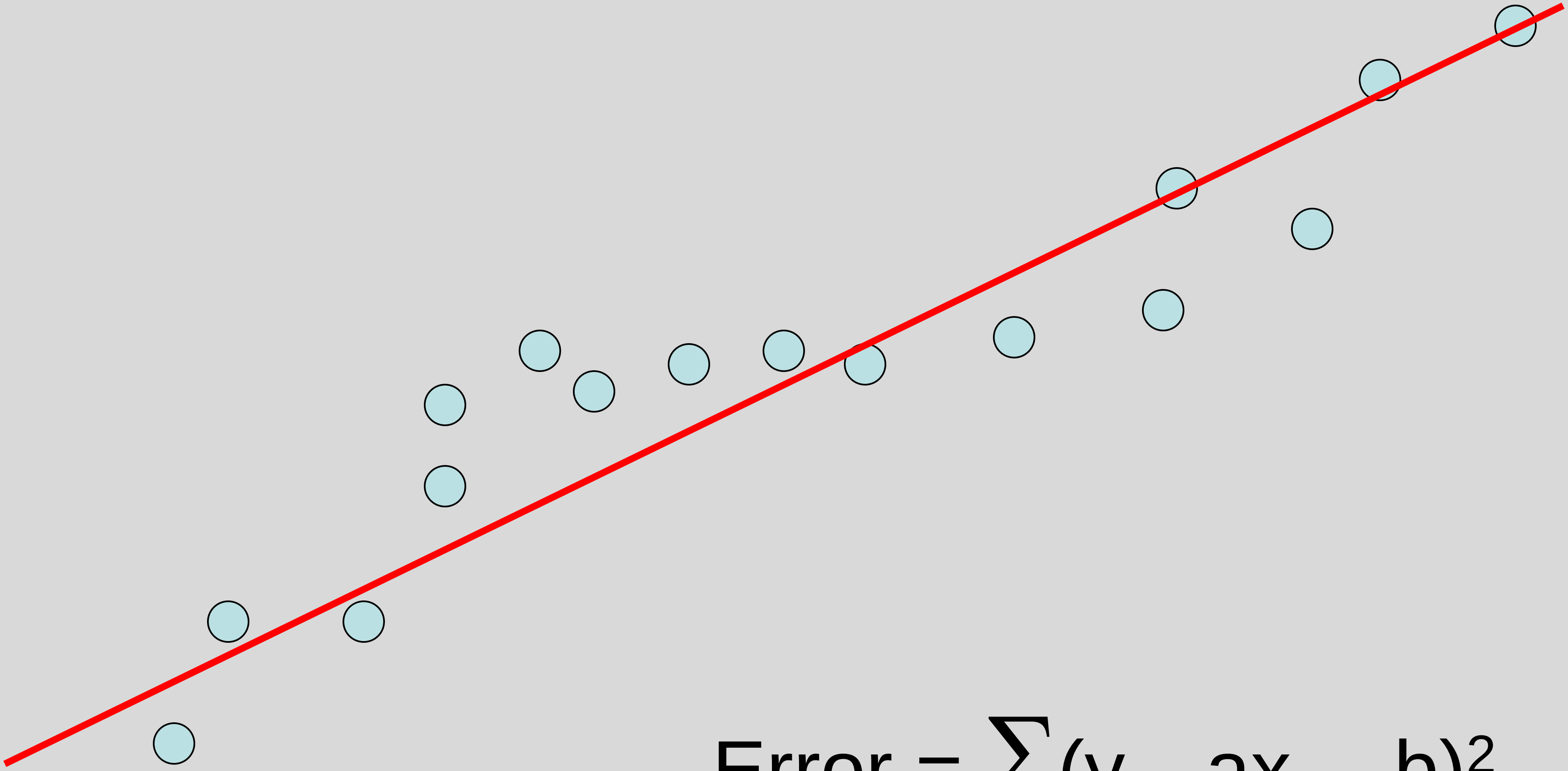
Iterative Algorithm

```
int[] M = new int[n+1];
char[] R = new char[n+1];

M[0] = 0;
for (int j = 1; j < n+1; j++){
        v1 = M[j-1];
        v2 = W[j] + M[P[j]];
        if (v1 > v2) {
                M[j] = v1;
                R[j] = 'A';
        }
        else {
                M[j] = v2;
                R[j] = 'B';
        }
}
```

Algorithm Summary

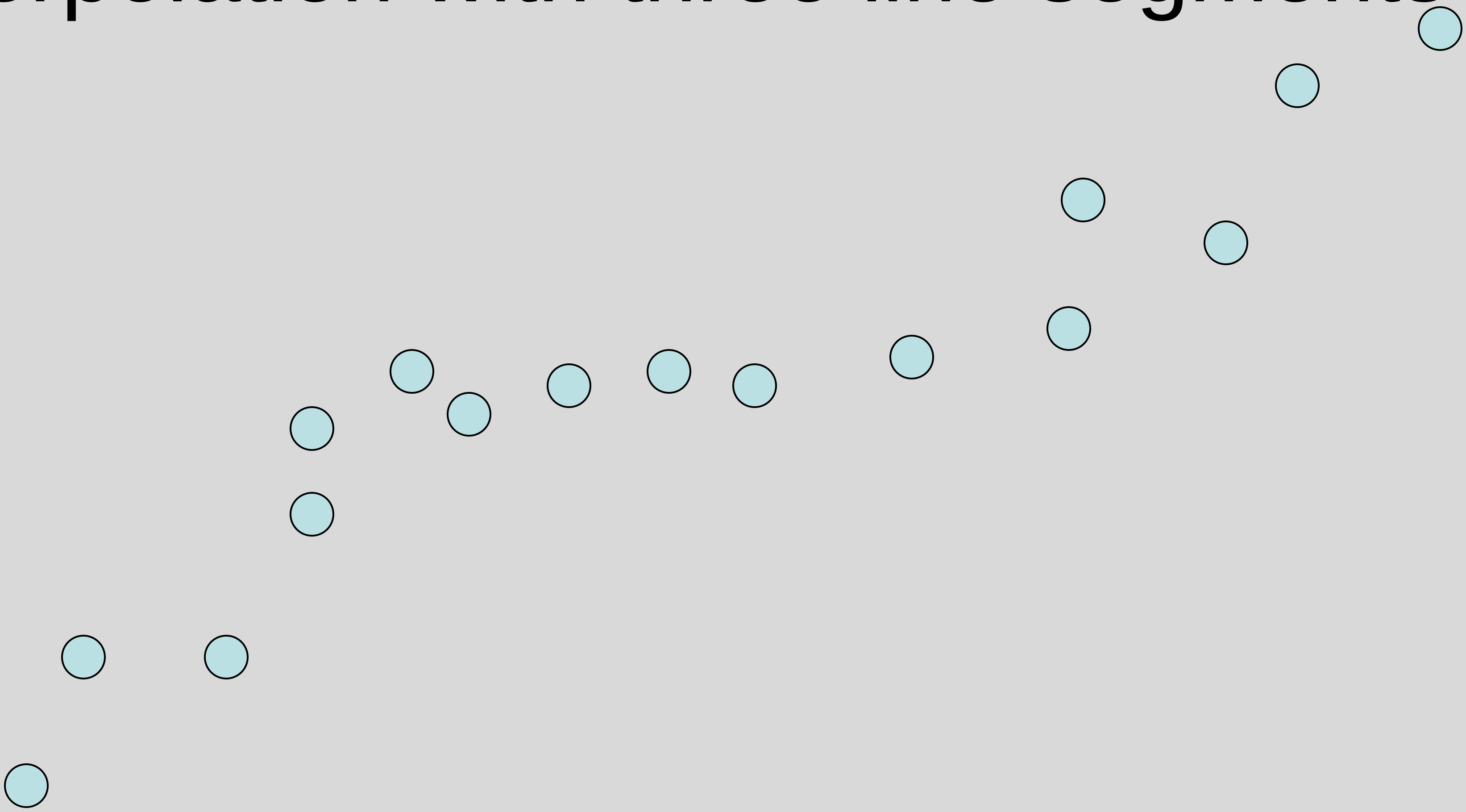
- O(n) time algorithm for finding maximum weight independent set of intervals
- Key idea: Creating an Opt function to express optimal set of $I_1, I_2, \ldots I_k$ in terms of optimal set of $I_1, I_2, \ldots I_{k-1}$
- Organize computation to avoid recomputation

Optimal linear interpolation

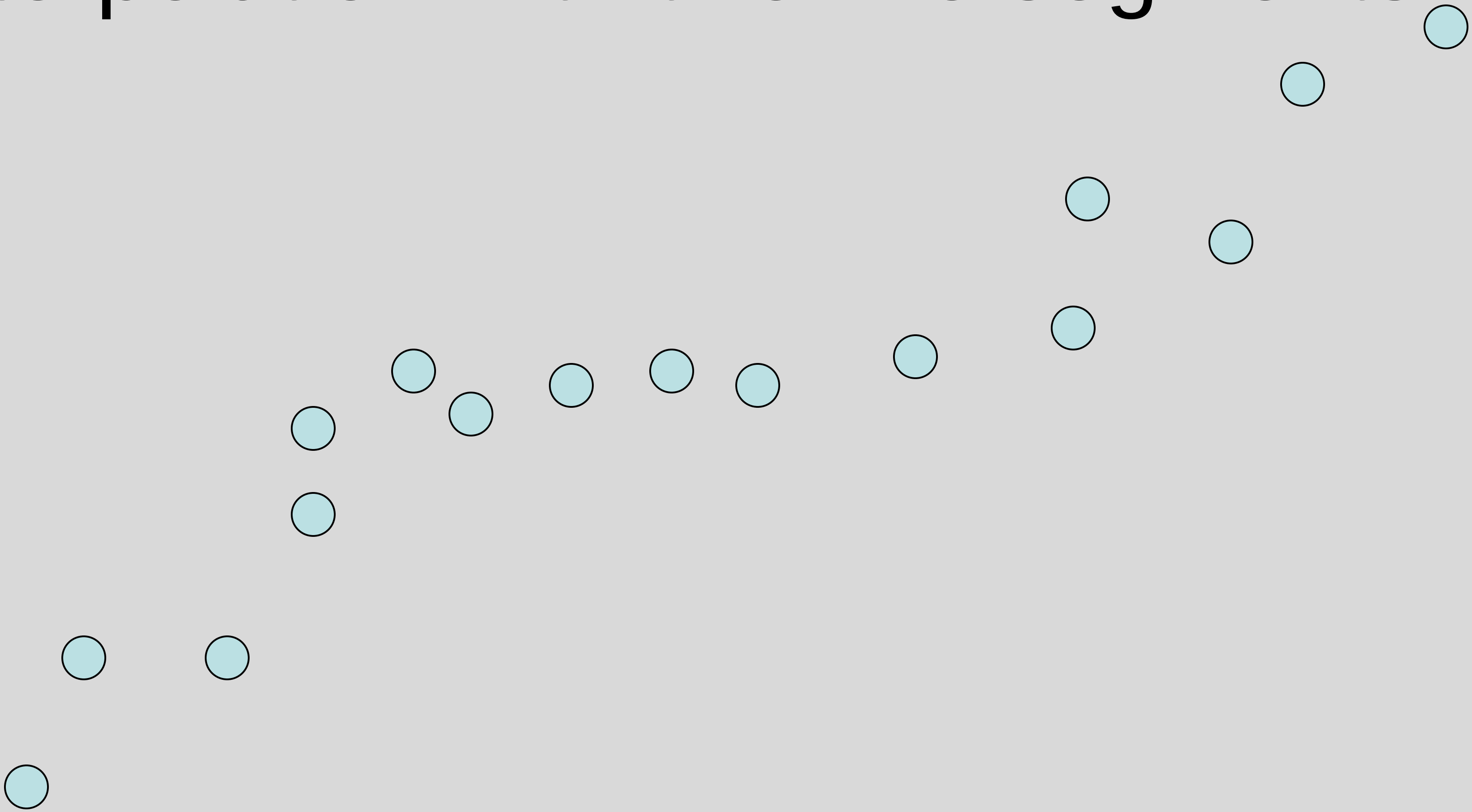


$$\text{Error} = \sum (y_i - ax_i - b)^2$$

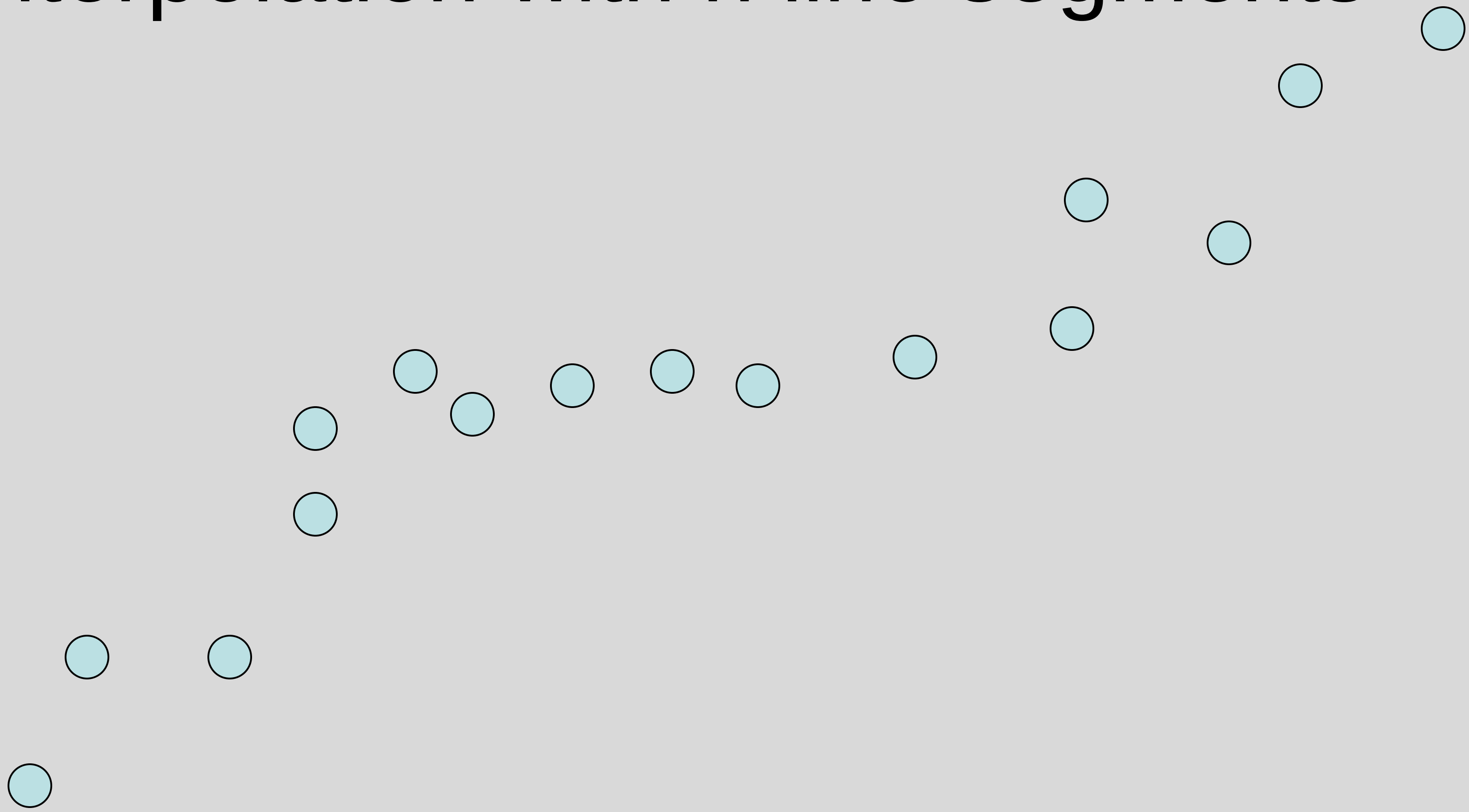
What is the optimal linear interpolation with three line segments



What is the optimal linear interpolation with two line segments

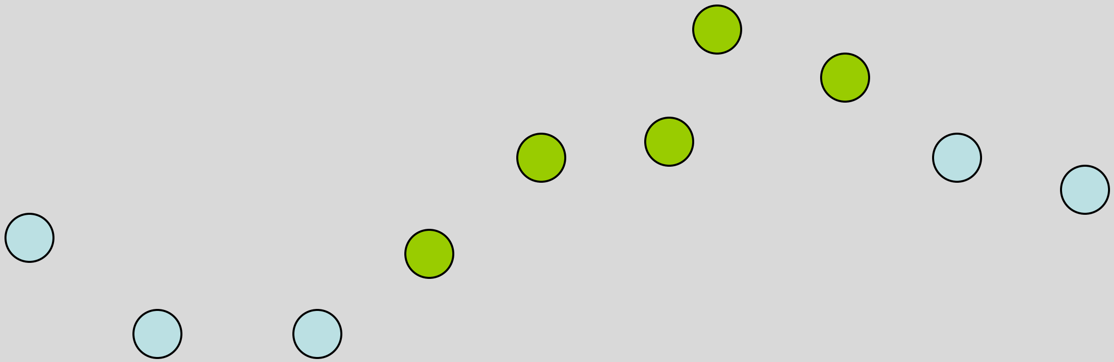


What is the optimal linear interpolation with n line segments



Notation

- Points $p_1, p_2, \ldots, p_n$ ordered by x-coordinate ($p_i = (x_i, y_i)$)
- $E_{i,j}$ is the least squares error for the optimal line interpolating $p_i, \ldots p_j$



Optimal interpolation with two segments

- Give an equation for the optimal interpolation of $p_1,\ldots,p_n$ with two line segments

- $E_{i,j}$ is the least squares error for the optimal line interpolating $p_i, \ldots p_j$

Optimal interpolation with k segments

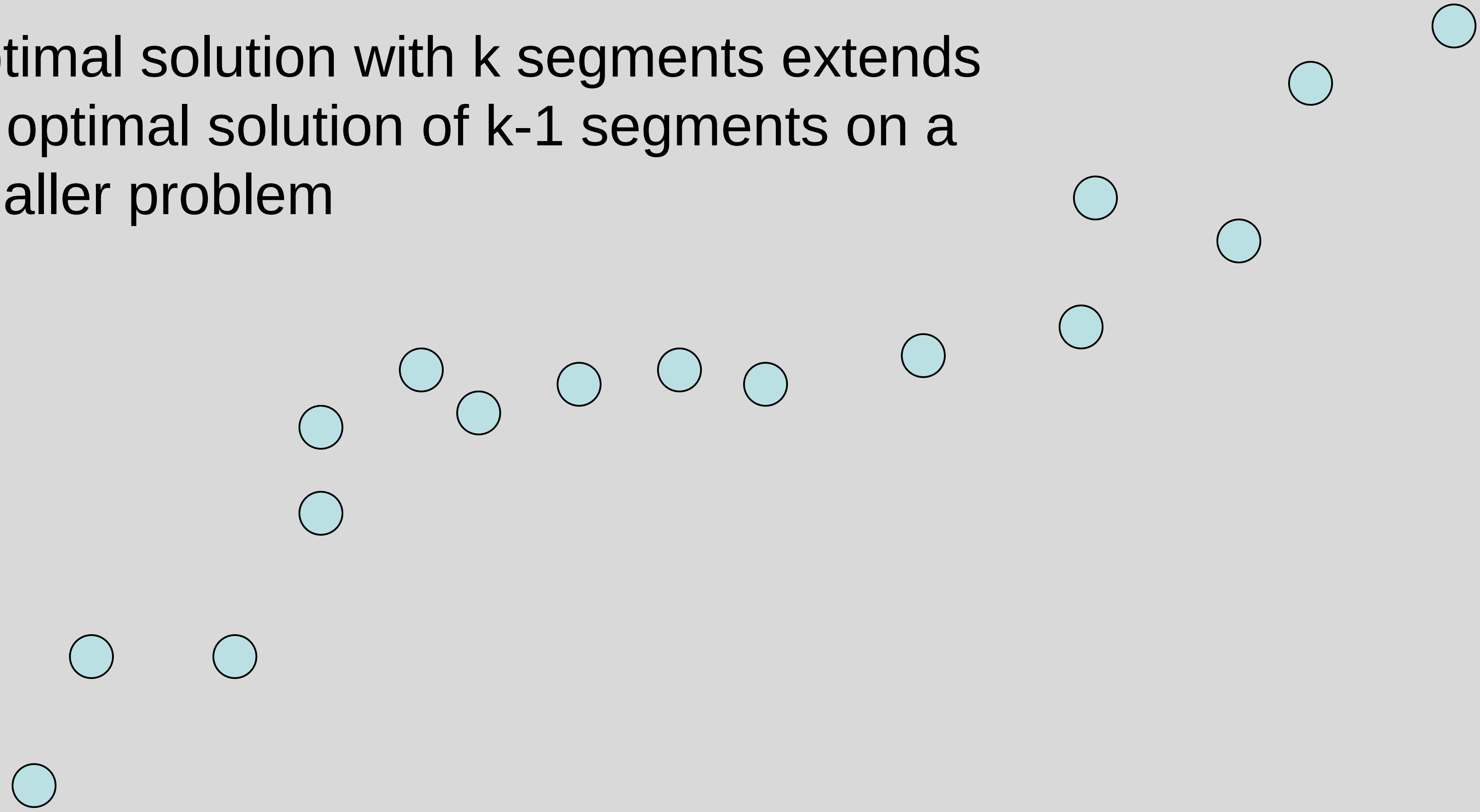
- Optimal segmentation with three segments
  - $\text{Min}_{i,j}\{E_{1,i} + E_{i,j} + E_{j,n}\}$
  - $O(n^2)$ combinations considered
- Generalization to k segments leads to considering $O(n^{k-1})$ combinations

$Opt_k[\, j\,]$: Minimum error approximating $p_1 \ldots p_j$ with k segments

How do you express $Opt_k[\, j\,]$ in terms of $Opt_{k-1}[1], \ldots, Opt_{k-1}[\, j\,]$?

Optimal sub-solution property

Optimal solution with k segments extends an optimal solution of k-1 segments on a smaller problem



Optimal multi-segment interpolation

Compute Opt[k, j] for $0 < k < j < n$

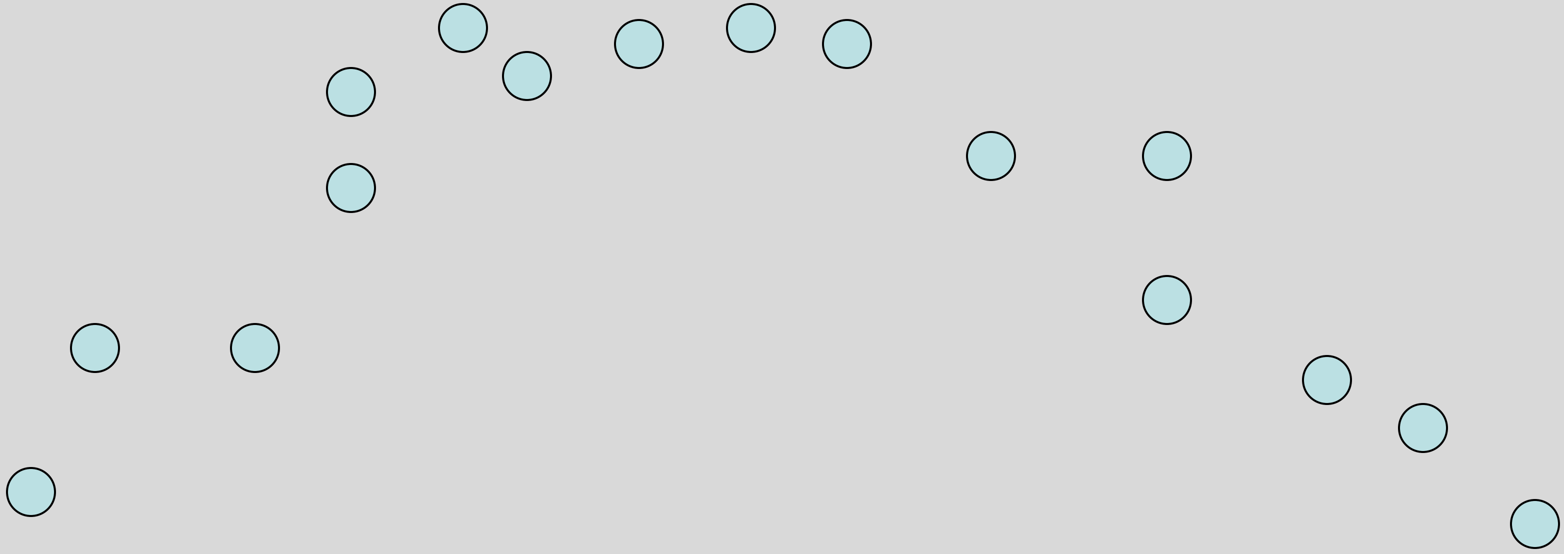
```
for j := 1 to n
    Opt[ 1, j] = E_{1,j};
for k := 2 to n-1
    for j := 2 to n
        t := E_{1,j}
        for i := 1 to j -1
            t = min (t, Opt[k-1, i ] + E_{i,j})
        Opt[k, j] = t
```

Determining the solution

- When Opt[k,j] is computed, record the value of i that minimized the sum
- Store this value in a auxiliary array
- Use to reconstruct solution

Variable number of segments

- Segments not specified in advance
- Penalty function associated with segments
- Cost = Interpolation error + C x #Segments



Penalty cost measure

- $Opt[\,j\,] = \min(E_{1,j}, \min_i(Opt[\,i\,] + E_{i,j} + P))$