CSE 421
Introduction to Algorithms

Lecture 13, Winter 2024

Dynamic Programming

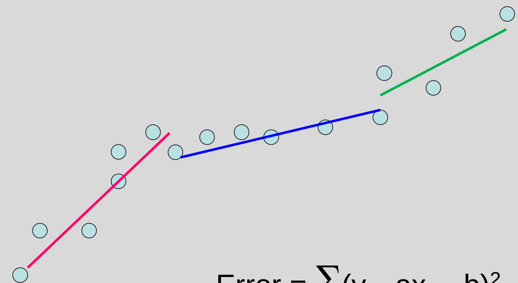
Announcements

- Dynamic Programming Reading:
  - 6.1-6.2, Weighted Interval Scheduling
  - 6.3 Segmented Least Squares
  - 6.4 Knapsack and Subset Sum
  - 6.6 String Alignment
  - 6.8 Shortest Paths (Bellman-Ford)
  - 6.9 Negative cost cycles
- Midterm, Friday, Feb 9
  - Material through 6.3 and HW 5
  - Feb 8 Section will be Midterm review

Dynamic Programming

- Key ideas
  - Express solution in terms of a polynomial number of sub problems
  - Order sub problems to avoid recomputation

Optimal linear interpolation



Error = $\sum(y_i - ax_i - b)^2$

Optimal interpolation with k segments

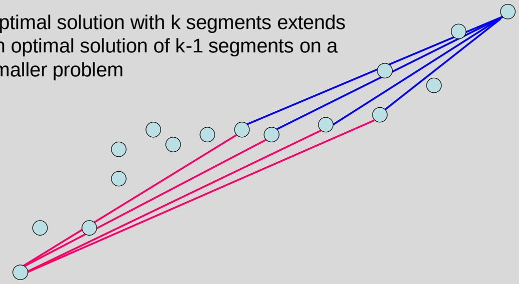
- Optimal segmentation with three segments
  - $\text{Min}_{i,j}\{E_{1,i} + E_{i,j} + E_{j,n}\}$
  - $O(n^2)$ combinations considered
- Generalization to k segments leads to considering $O(n^{k-1})$ combinations

$\text{Opt}_k[\,j\,]$: Minimum error approximating $p_1 \ldots p_j$ with k segments

How do you express $\text{Opt}_k[\,j\,]$ in terms of $\text{Opt}_{k-1}[1],\ldots,\text{Opt}_{k-1}[\,j\,]$?

$Opt_k[\,j\,] = \min_i \{\, Opt_{k-1}[\,i\,] + E_{i,j} \,\}$ for $0 < i < j$

Optimal solution with k segments extends an optimal solution of k-1 segments on a smaller problem



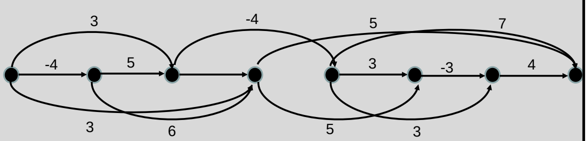
Optimal multi-segment interpolation

Compute Opt[k, j] for 0 < k < j < n

```
for j := 1 to n
   Opt[ 1, j] = E1,j;
for k := 2 to n-1
   for j := 2 to n
       t := E1,j
       for i := 1 to j -1
           t = min (t, Opt[k-1, i ] + Ei,j)
       Opt[k, j] = t
```


Shortest Paths in Linear Graphs

- A directed graph with edge weights on the vertex set V = {1, 2, . . . N} is linear is all edges (i, j) ϵ E satisfy i < j



Dynamic Programming for Shortest Paths in Linear Graphs

- D[j] = dist(1, j)
- What is the optimization equation?

How many different ways can I walk to work?

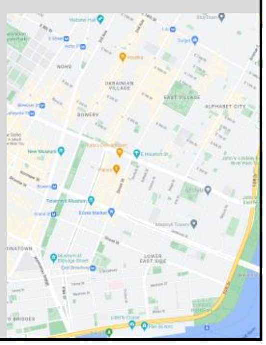
Only taking “efficient” routes

Make the problem discrete

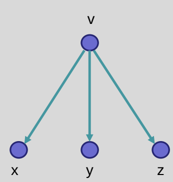
Directed Graph model: Intersections and streets

Assume the graph is a directed acyclic graph (DAG)

Problem: compute the number of paths from vertex h to vertex w



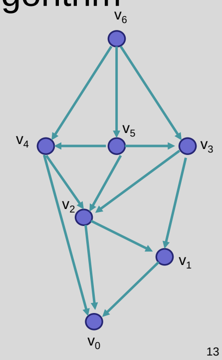
P[v]: Number of paths from v to v_0



How do you compute P[v] if you know P[x], P[y], and P[z]?

Recursive Algorithm

```
PC(v){
   if (v == v0)
      return 1;
   count = 0;
   foreach (w in N+(v)){
      count = count + PC(w);
   }
   return count;
}
```



Ordering the vertices

How do you order the vertices of a DAG such that if there is an edge from v to w, w comes before v in the ordering?

Path Counting

G=(V,E) is an n node directed acyclic graph, with v_{n-1}, v_{n-2}, . . ., v_1, v_0 a topological order of the vertices. An array is computed giving the number of paths from each vertex to v_0.

```
CountPaths(G, P){
   P[0] = 1;
   for (i = 1 to n-1){
      P[i] = 0;
      foreach (w in N+(vi)){
          P[i] = P[i] + P[w];
   }
}
```


Typesetting

- Layout text on a page to optimize readability and aesthetic measures
- Skilled profession replaced by computing
- Goal – give text a uniform appearance which is primarily done by choosing line breaks to balance white space
  - Interword spacing can stretch or shrink
  - Hyphenation is sometimes available

Optimal line breaking
The LaTeX algorithm

Element distinctness has been a particular focus of lower bound analysis. The first time-space tradeoff lower bounds for the problem apply to structured algorithms. Borodin et al. [13] gave a time-space tradeoff lower bound for computing ED on *comparison* branching programs of $T \in \Omega(n^{3/2}/S^{1/2})$ and, since $S \geq \log_2 n$, $T \in \Omega(n^{3/2}\sqrt{\log n}/S)$. Yao [32] improved this to a near-optimal $T \in \Omega(n^{2-\epsilon(n)}/S)$, where $\epsilon(n) = 5/(\ln n)^{1/2}$. Since these lower bounds apply to the average case for randomly ordered inputs, by Yao's lemma, they also apply to randomized comparison branching programs. These bounds also trivially apply to all frequency moments since, for $k \neq 1$, $ED(x) = n$ iff $F_k(x) = n$. This near-quadratic lower bound seemed to suggest that the complexity of ED and F_k should closely track that of sorting.

Optimal Line Breaking

- Words have length w_i, line length L
- Penalty related to white space or overflow of the line
  - Quadratic measure often used
- Pen(i, j): Penalty for putting w_i, w_{i+1},…,w_j on the same line
- Opt[m]: minimum penalty for ending a line with w_m

The quick brown
fox jumped over
the lazy dog.

The quick brown
fox jumped
over the lazy dog.

Pen("The quick brown") = 1
Pen("fox jumped over") = 2
Pen("fox jumped") = 8
Pen("the lazy dog") = 6
Pen("over the lazy dog.") = 4

Pen(i, j): Penalty for putting $w_i, w_{i+1}, \ldots, w_j$ on the same line

Optimal Line Breaking

Optimal score for ending a line with w_m

$$Opt[m] = \min_i \{ Opt[i] + Pen(i+1,m)\} \text{ for } 0 < i < m$$

For words $w_1, w_2, \ldots, w_n$, we compute Opt[n] to find the optimal layout

Optimal Line Breaking

```
Opt[0] = 0;
for m = 1 to n {
    Find i that minimizes Opt [ i ] + Pen(i+1,m);
    Opt[m] = Opt [ i ] + Pen(i+1,m);
    Pred[m] = i;
}
```