Lecture17

CSE 421
Introduction to Algorithms

Lecture 17
Network Flow, Part 1

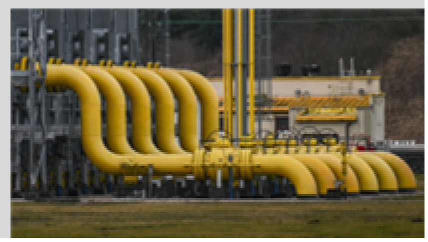
Announcements

- Network Flow Reading
  - 7.1-7.3, Network Flow Problem and Algorithms
  - 7.5-7.12, Network Flow Applications
- No class on Monday
  - Homework deadline shifting to Friday starting with HW 7

Outline

- Network flow definitions
- Flow examples
- Augmenting Paths
- Residual Graph
- Ford Fulkerson Algorithm
- Cuts
- Maxflow-MinCut Theorem
- Efficient Network Flow Algorithms

Duality



Network Flow Definitions

$c_e = c(u,v)$

- Capacity

f_e $f(u,v)$

- Source, Sink

- Capacity Condition $0 \leq f_e \leq c_e$

- Conservation Condition $v \neq s,t$

$$\sum_{u \in N^-(v)} f(u,v) = \sum_{w \in N^+(v)} f(v,w)$$

- Value of a flow

$$F = \sum_{v \in N^+(s)} f(s,v) = \sum_{u \in N^-(t)} f(u,t)$$

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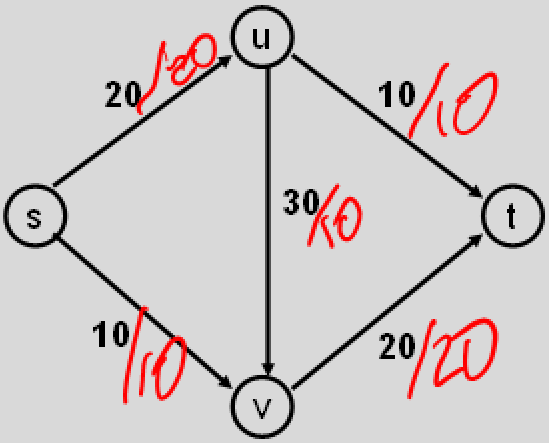
Flow Example



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Network Flow Definitions

- Flowgraph: Directed graph with distinguished vertices s (source) and t (sink)
- Capacities on the edges, $c(e) \geq 0$
- Problem, assign flows $f(e)$ to the edges such that:
  - $0 \leq f(e) \leq c(e)$
  - Flow is conserved at vertices other than s and t
    - Flow conservation: flow going into a vertex equals the flow going out
  - The flow leaving the source is a large as possible

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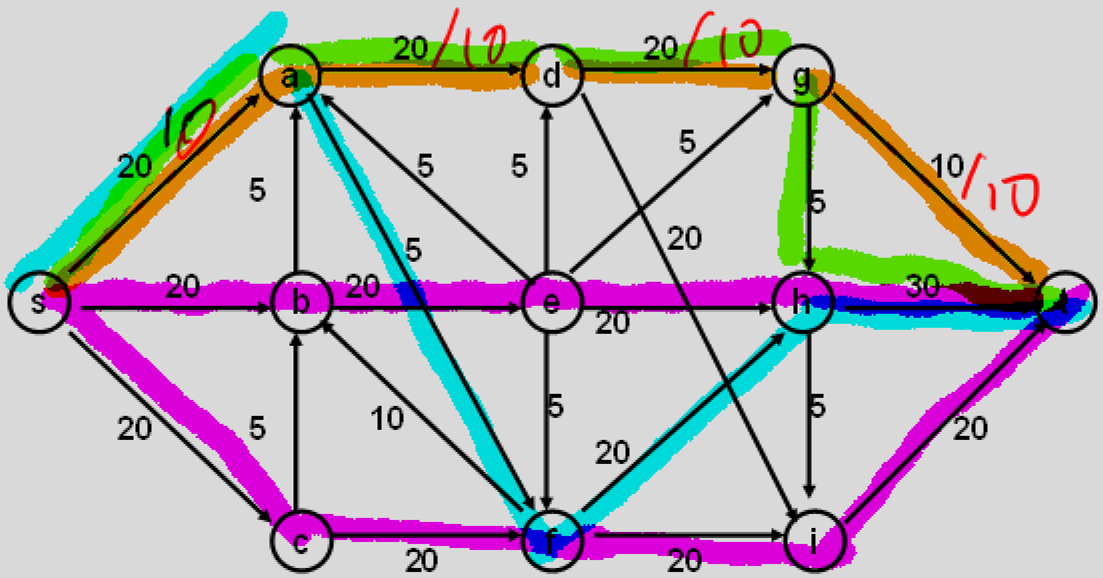
Flow Example



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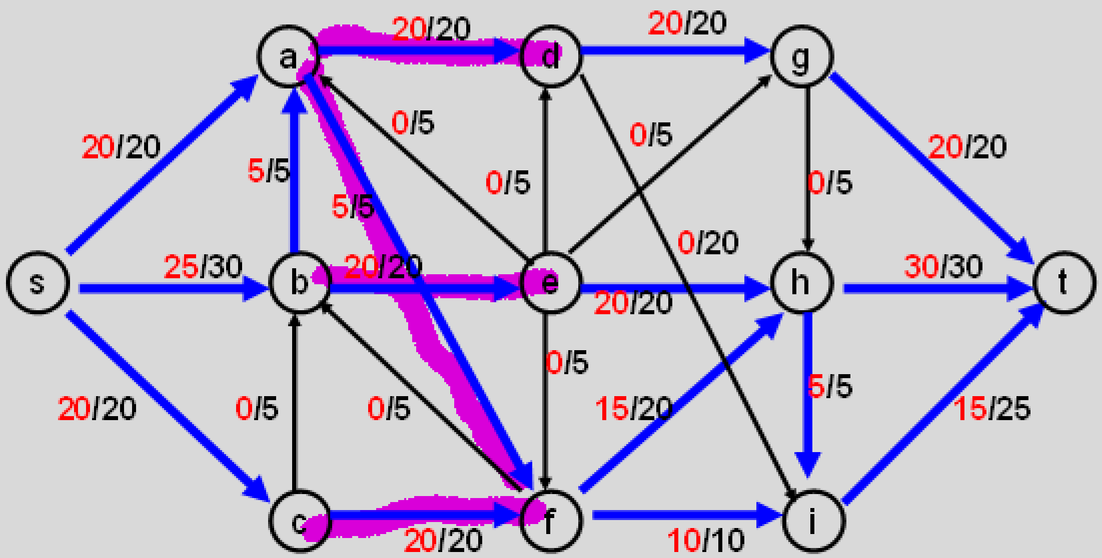
Find a maximum flow



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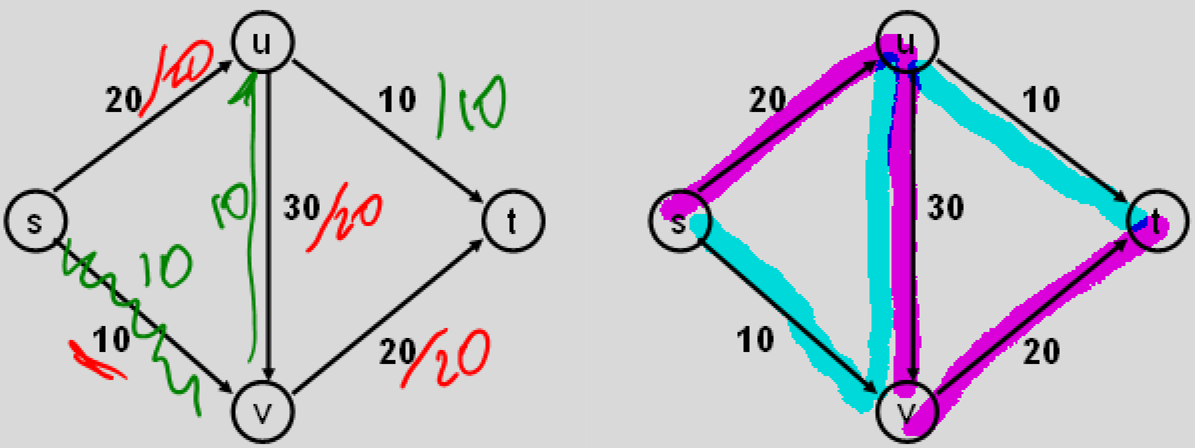
Flow Example



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Residual Graph

- Flow graph showing the remaining capacity
- Flow graph G, Residual Graph G_R
  - G: edge e from u to v with capacity c and flow f
  - G_R: edge e' from u to v with capacity c – f
  - G_R: edge e'' from v to u with capacity f

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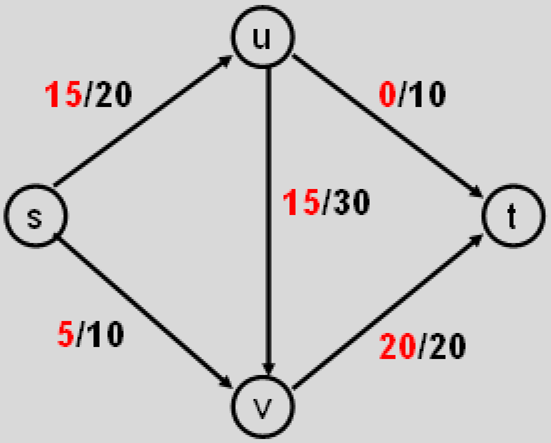
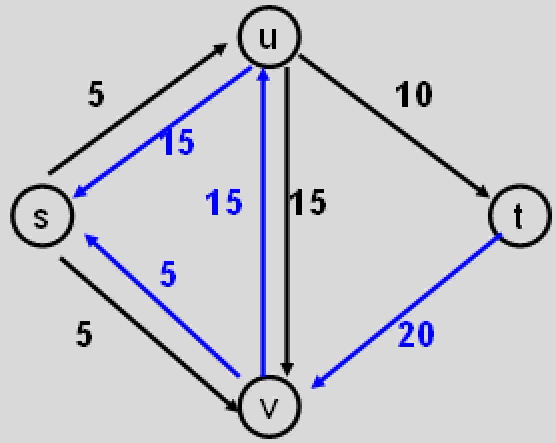
Flow assignment and the residual graph





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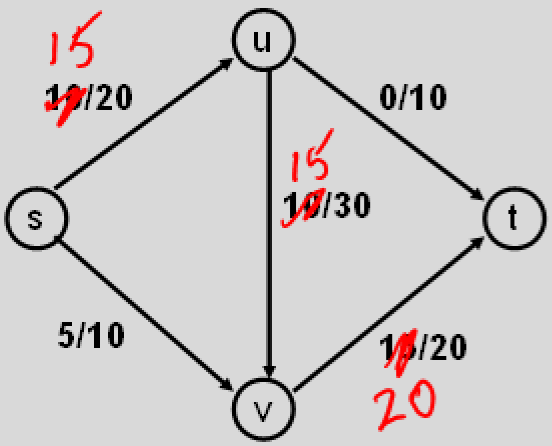
Augmenting Path Algorithm

- Augmenting path
  - Vertices $v_1, v_2, \ldots, v_k$
    - $v_1 = s$, $v_k = t$
    - Possible to add b units of flow between v_j and v_{j+1} for j = 1 … k-1



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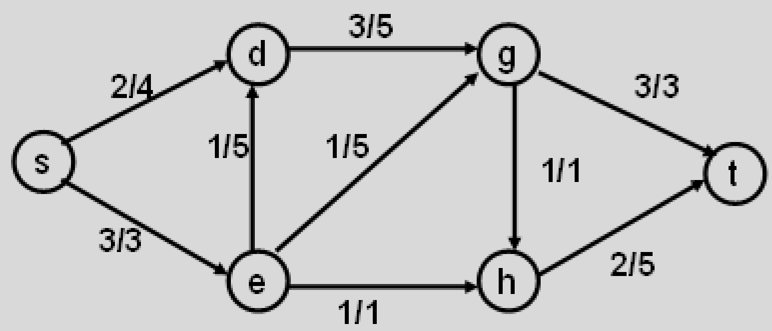
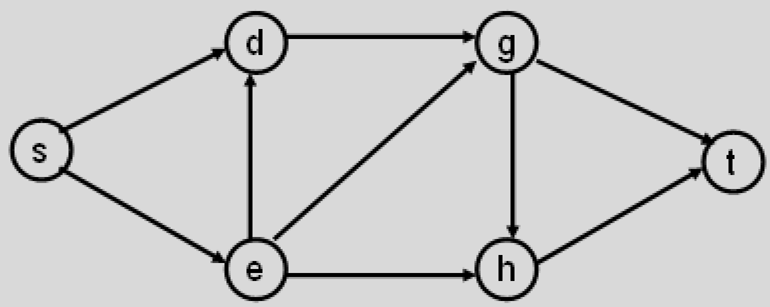
Build the residual graph



Residual graph:



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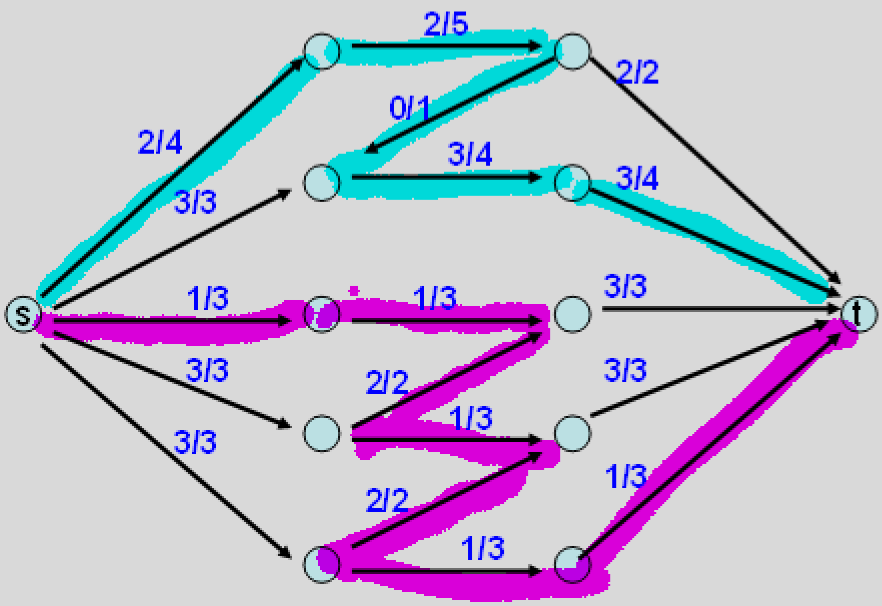
Find two augmenting paths



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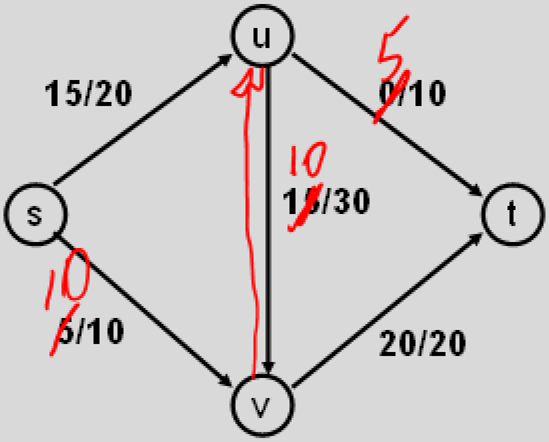
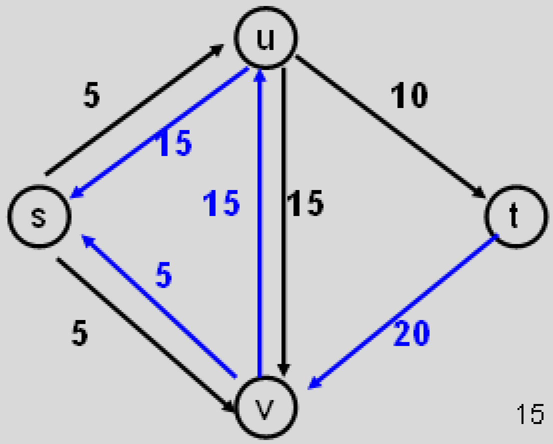
Augmenting Path Lemma

- Let P = $v_1, v_2, \ldots, v_k$ be a path from s to t with minimum capacity b in the residual graph
- b units of flow can be added along the path P in the flow graph





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Proof

- Add b units of flow along the path P
- What do we need to verify to show we have a valid flow after we do this?
  - capacity $0 \le f(e) \le c(e)$
  - Conservation

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O(C(m+n)) Integer capacities

Ford-Fulkerson Algorithm (1956)

while not done

O(n+m)

- Construct residual graph G_R
- Find an s-t path P in G_R with capacity b > 0
- Add b units along in G

If the sum of the capacities of edges leaving S is at most C, then the algorithm takes at most C iterations

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