CSE 421
Introduction to Algorithms

Lecture 18

Winter 2024

Network Flow, Part 2

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Outline

- ~~Network flow definitions~~
- ~~Flow examples~~
- ~~Augmenting Paths~~
- ~~Residual Graph~~
- ~~Ford Fulkerson Algorithm~~
- Cuts
- Maxflow-MinCut Theorem
- Worst Case Runtime for FF
- Improving Runtime bounds
  - Capacity Scaling
  - Fully Polynomial Time Algorithms

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Network Flow Definitions

- Flowgraph: Directed graph with distinguished vertices s (source) and t (sink)
- Capacities on the edges, $c(e) \geq 0$
- Problem, assign flows f(e) to the edges such that:
  - $0 \leq f(e) \leq c(e)$
  - Flow is conserved at vertices other than s and t
    - Flow conservation: flow going into a vertex equals the flow going out
  - The flow leaving the source is a large as possible

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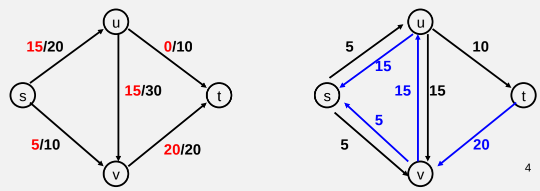
Residual Graph

- Flow graph showing the remaining capacity
- Flow graph G, Residual Graph G_R
  - G: edge e from u to v with capacity c and flow f
  - G_R: edge e' from u to v with capacity c – f
  - G_R: edge e'' from v to u with capacity f



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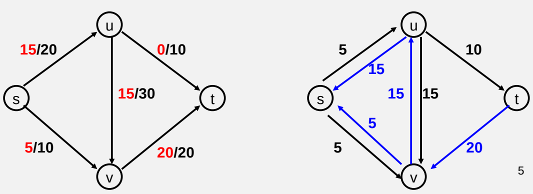
Augmenting Path Algorithm

- Augmenting path in residual graph
  - Vertices $v_1, v_2, \ldots, v_k$
    - $v_1 = s$, $v_k = t$
    - Possible to add b units of flow between v_j and v_{j+1} for j = 1 … k-1



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Ford-Fulkerson Algorithm (1956)

while not done

Construct residual graph G_R

Find an s-t path P in G_R with capacity b > 0

Add b units of flow along path P in G

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Runtime Analysis

- Assume the capacities are integers*
- Let C be the sum of edge capacities leaving s
- The total flow F is at most C
- Every iteration increases flow by at least 1, so there are at most C iterations
- Cost per iteration is O(m+n)
- Runtime is O(C(m+n))

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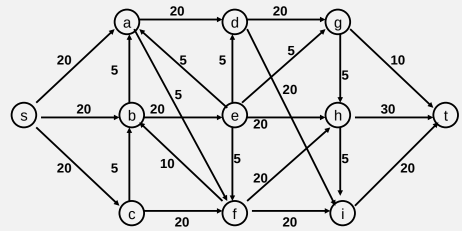
* This is actually a very important assumption, but we are not going to explore this rabbit hole

Flow Example



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Cuts in a graph

- Cut: Partition of V into disjoint sets S, T with s in S and t in T.
- Cap(S,T): sum of the capacities of edges from S to T
- Flow(S,T): net flow out of S
  - Sum of flows out of S minus sum of flows into S

- Flow(S,T) ≤ Cap(S,T)

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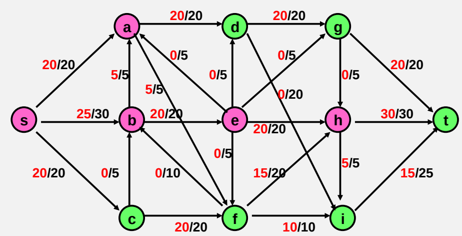
What is Cap(S,T) and Flow(S,T)

S={s, a, b, e, h}, T = {c, f, i, d, g, t}



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What is Cap(S,T) and Flow(S,T)

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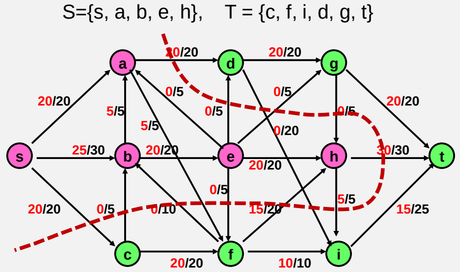

Cap(S,T) = 95, Flow(S,T) = 80 – 15 = 65

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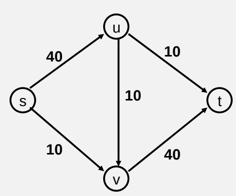
Minimum value cut



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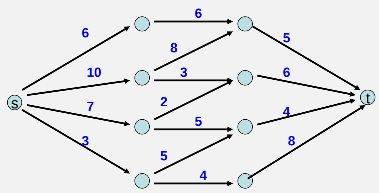
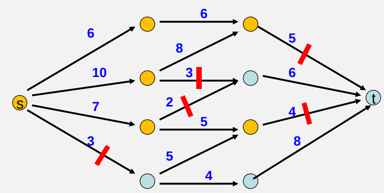
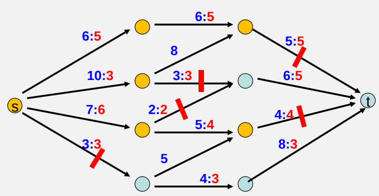
Find a minimum value cut



Find a minimum value cut

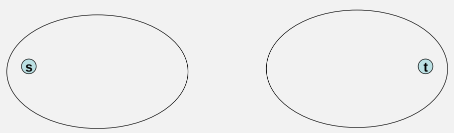


Find a minimum value cut

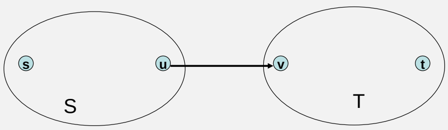


MaxFlow – MinCut Theorem

- There exists a flow which has the same value as the minimum cut
- Proof: Consider a flow where the residual graph has no s-t path with positive capacity
- Let S be the set of vertices in G_R reachable from s with paths of positive capacity



Let S be the set of vertices in G_R reachable from s with paths of positive capacity



What can we say about the flows and capacity between u and v?

Max Flow - Min Cut Theorem

- Ford-Fulkerson algorithm finds a flow where the residual graph is disconnected, hence FF finds a maximum flow.
- If we want to find a minimum cut, we begin by looking for a maximum flow.

History

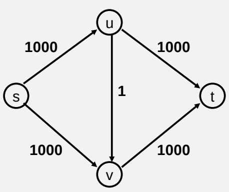
- Ford / Fulkerson studied network flow in the context of the Soviet Rail Network

Ford Fulkerson Runtime

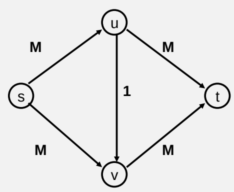
- Cost per phase X number of phases
- Phases
  - Capacity leaving source: C
  - Add at least one unit per phase
- Cost per phase
  - Build residual graph: O(m)
  - Find s-t path in residual: O(m)

Performance

- The worst case performance of the Ford-Fulkerson algorithm is horrible



Improving path selection



Better methods of finding augmenting paths

- Find the maximum capacity augmenting path
  - $O(m^2 \log(C))$ time algorithm for network flow
- Find the shortest augmenting path
  - $O(m^2 n)$ time algorithm for network flow
- Find a blocking flow in the residual graph
  - $O(mn \log n)$ time algorithm for network flow

Polynomial Time Algorithms

- Input of size n, runtime $T(n) = O(n^k)$
- Input size measures
  - Bits of input
  - Number of data items
- Maximum item size C
  - $O(Cn^k)$: Exponential
  - $O(n^k \log C)$: Polynomial
  - $O(n^k)$: Fully polynomial

Capacity Scaling Algorithm

- Choose $\Delta = 2^k$ such that all edges in G_R have capacity less than 2Δ

while $\Delta \geq 1$

    while there is a path P in G_R with capacity Δ

        Add Δ units of flow along path P in G

        Update G_R

    $\Delta = \Delta / 2$

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Edmonds-Karp: Easier analysis than Max Capacity First

Analysis

- If capacities are integers, then graph is disconnected when $\Delta = ½$
- If largest edge capacity is C, then there are at most log C outer phases
- At the start of each outer phase, the flow is within $2m\Delta$ of the maximum
  - So there are at most 2m inner phases for each Δ

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Shortest Augmenting Path

- Find augmenting paths by BFS

for k = 1 to n

    while there is a path P in G_R of length k and capacity b > 0

        Add b units of flow along path P in G

        Update G_R

- Need to show:
  - The length of the shortest augmenting path is non-decreasing
  - Each while loop finds at most m paths

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Analysis

- Augmenting along shortest path from s to t does not decrease distance from s to t

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Analysis

- The distance from s to t must increase in G_R after m augmentations by shortest paths

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Improving the shortest augmenting path algorithm

- Find a blocking flow in one phase to increase the length of augmenting paths
  - Dinitz (Ефим Абрамович Диниц) Algorithm
  - $O(n^2m)$
- Dynamic Trees to decrease cost per augmentation
  - O(nm log n)

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