Lecture22

CSE 421
Introduction to
Algorithms

Lecture 22
NP-Completeness

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Announcements

- Read Chapter 8
- Old final exams posted on course homepage

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Algorithms vs. Lower bounds

- Algorithmic Theory
  - What we can compute
    - I can solve problem X with resources R
  - Proofs are almost always to give an algorithm that meets the resource bounds
- Lower bounds
  - How do we show that something can't be done?

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Theory of NP Completeness

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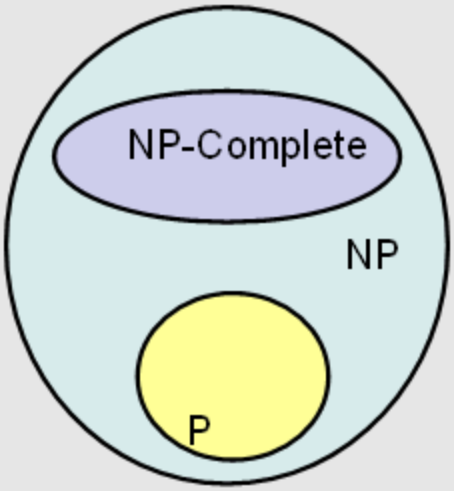
The Universe



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Polynomial Time

- P: Class of problems that can be solved in polynomial time
  - Corresponds with problems that can be solved efficiently in practice
  - Right class to work with "theoretically"

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Decision Problems

- Theory developed in terms of yes/no problems
  - Independent set
    - Given a graph G and an integer K, does G have an independent set of size at least K
  - Network Flow
    - Given a graph G with edge capacities, a source vertex s, and sink vertex t, and an integer K, does the graph have flow function with value at least K

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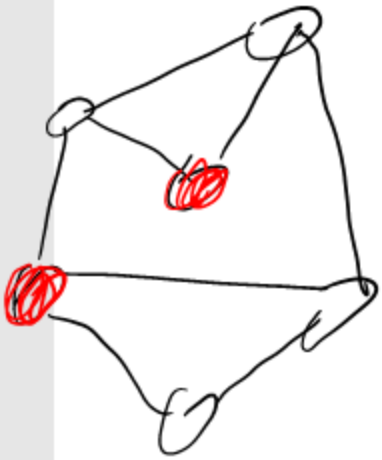

Definition of P

Decision problems with polynomial time algorithms

Problem	Description	Algorithm	Yes	No
MULTIPLE	Is x a multiple of y?	Grade school division	51, 17	51, 16
RELPRIME	Are x and y relatively prime?	Euclid's algorithm	34, 39	34, 51
PRIMES	Is x prime?	Agrawal, Kayal, Saxena (2002)	53	51
EDIT-DISTANCE	Is the edit distance between x and y less than 5?	Dynamic programming	niether neither	acgggt ttttta
LSOLVE	Is there a vector x that satisfies Ax = b?	Gaussian elimination	$\begin{bmatrix}0&1&1\\2&4&-2\\0&3&15\end{bmatrix}\cdot\begin{bmatrix}4\\2\\36\end{bmatrix}$	$\begin{bmatrix}1&0&0\\1&1&1\\0&1&1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}$

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What is NP?

- Problems solvable in non-deterministic polynomial time
- Problems where “yes” instances have polynomial time checkable certificates

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Non-deterministic Computation

- Non-deterministic finite automata
  - Multiple different next states
  - Accept a string if some set of choices get to an accept state

JMP A, B, C

- Non-deterministic computer
  - Add a non-deterministic GOTO statement (choose between multiple statements)
  - Accept if some computation reaches an accept state

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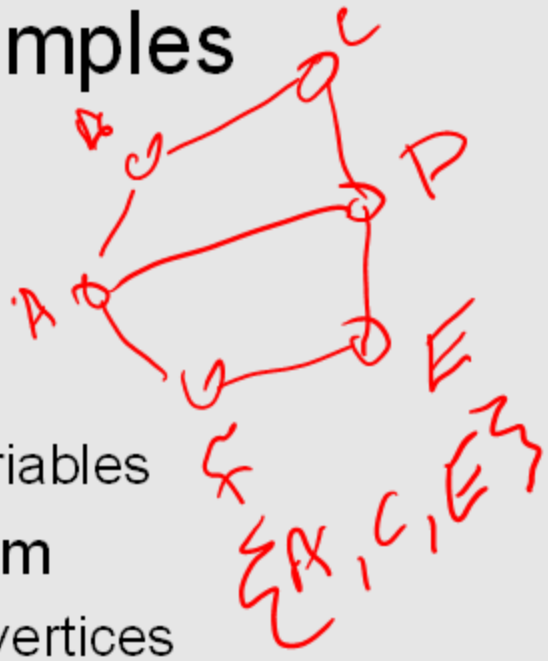
Certificate examples

- **Independent set of size K**
  - The Independent Set
- **Satifisfiable formula**
  - Truth assignment to the variables
- **Hamiltonian Circuit Problem**
  - A cycle including all of the vertices
- **K-coloring a graph**
  - Assignment of colors to the vertices



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Certifiers and Certificates: 3-Satisfiability

SAT: Does a given CNF formula have a satisfying formula

Certificate: An assignment of truth values to the n boolean variables

Certifier: Check that each clause has at least one true literal,

instance s

$$(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (\overline{x_1} \vee \overline{x_3} \vee \overline{x_4})$$

certificate t

$$x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ x_4 = 1$$

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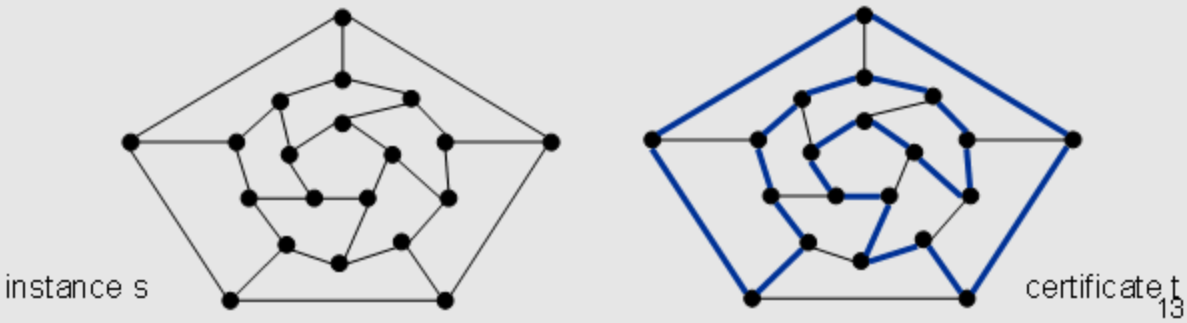
Certifiers and Certificates: Hamiltonian Cycle

HAM-CYCLE. Given an undirected graph G = (V, E), does there exist a simple cycle C that visits every node?

Certificate. A permutation of the n nodes.

Certifier. Check that the permutation contains each node in V exactly once, and that there is an edge between each pair of adjacent nodes in the permutation.



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Polynomial time reductions

- Y is Polynomial Time Reducible to X
  - Solve problem Y with a polynomial number of computation steps and a polynomial number of calls to a black box that solves X
  - Notations: $Y <_P X$

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Composability Lemma

- If $X <_P Y$ and $Y <_P Z$ then $X <_P Z$

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Lemmas

- Suppose $Y <_P X$. If X can be solved in polynomial time, then Y can be solved in polynomial time.
- Suppose $Y <_P X$. If Y cannot be solved in polynomial time, then X cannot be solved in polynomial time.

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NP-Completeness

- A problem X is NP-complete if
  - X is in NP
  - For every Y in NP, $Y <_P X$
- X is a "hardest" problem in NP
- If X is NP-Complete, Z is in NP and $X <_P Z$
  - Then Z is NP-Complete

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Cook's Theorem

- The Circuit Satisfiability Problem is NP-Complete

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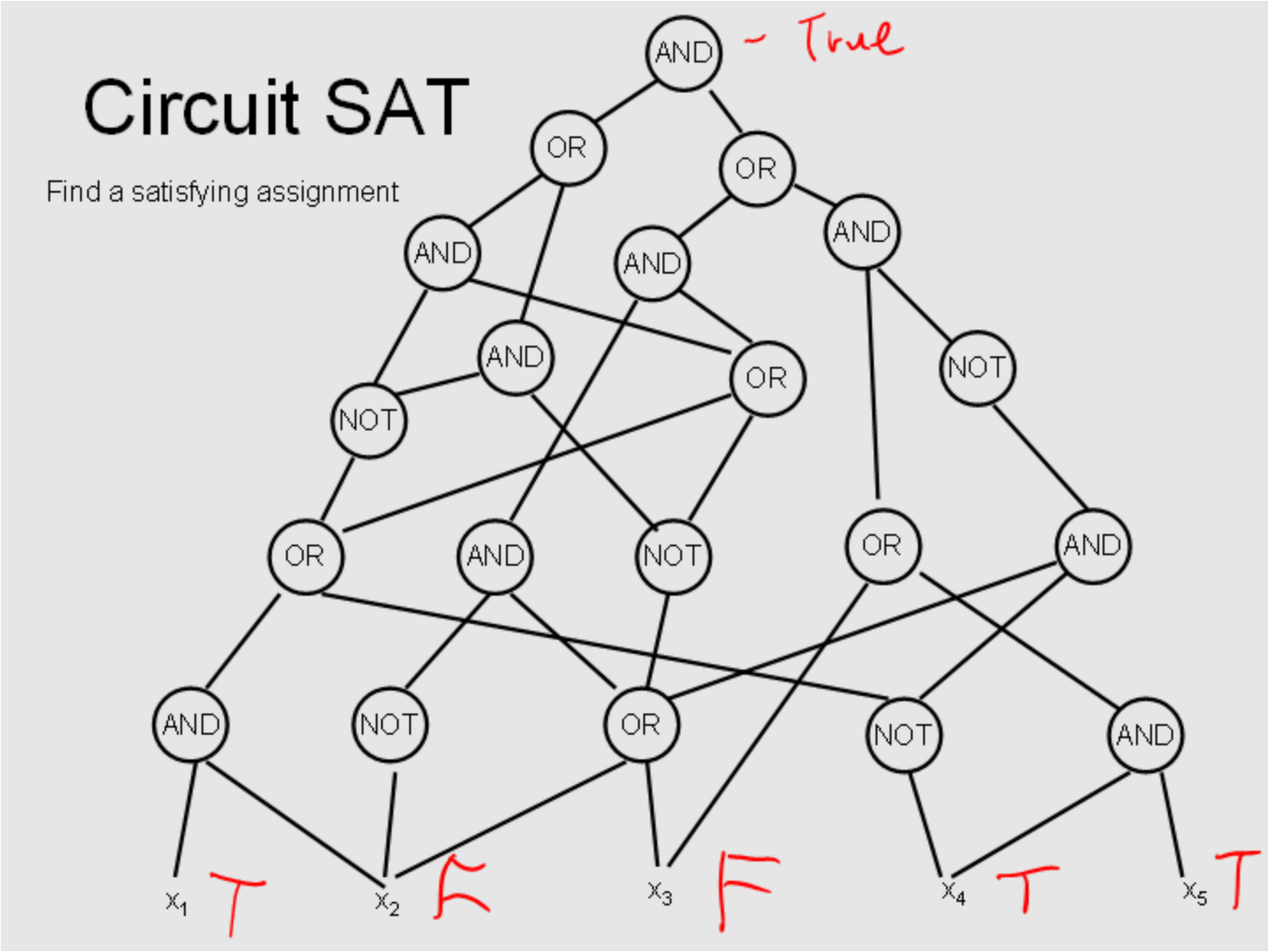
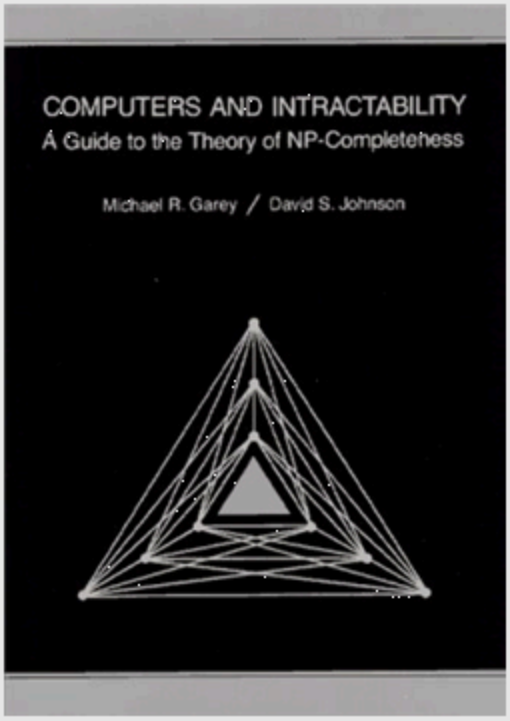

Circuit SAT
Find a satisfying assignment
AND
OR
OR
AND
AND
AND
AND
OR
NOT
NOT
OR
AND
NOT
OR
AND
AND
NOT
OR
NOT
AND
x_1
x_2
x_3
x_4
x_5

Garey and Johnson



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History

Jack Edmonds

- Identified NP

Steve Cook

- Cook's Theorem – NP-Completeness

Dick Karp

- Identified the "standard" collection of NP-Complete Problems

Leonid Levin

- Independent discovery of NP-Completeness in USSR

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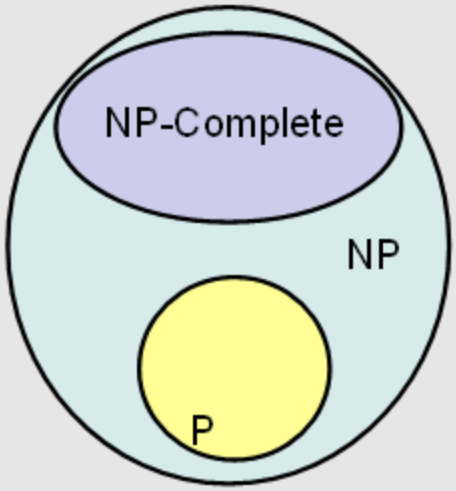
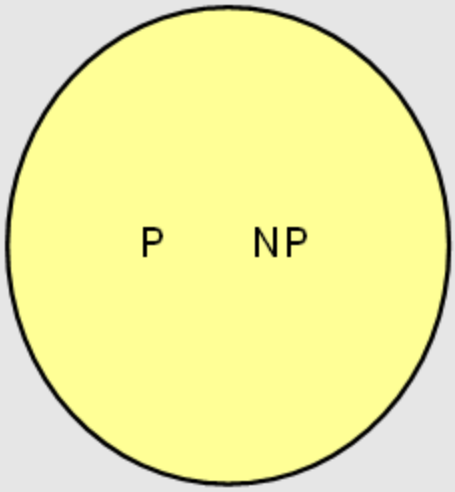
P vs. NP Question





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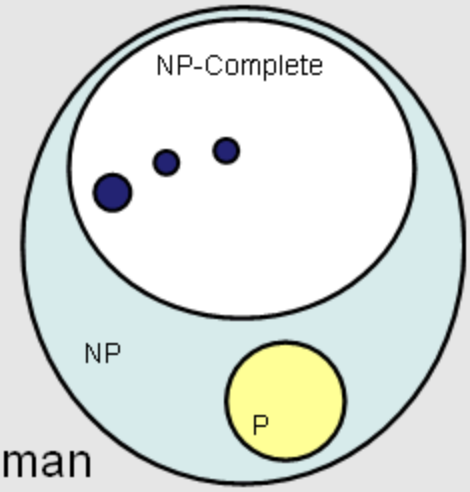
Populating the NP-Completeness Universe



- Circuit Sat $\leq_P$ 3-SAT
- 3-SAT $\leq_P$ Independent Set
- 3-SAT $\leq_P$ Vertex Cover
- Independent Set $\leq_P$ Clique
- 3-SAT $\leq_P$ Hamiltonian Circuit
- Hamiltonian Circuit $\leq_P$ Traveling Salesman
- 3-SAT $\leq_P$ Integer Linear Programming
- 3-SAT $\leq_P$ Graph Coloring
- 3-SAT $\leq_P$ Subset Sum
- Subset Sum $\leq_P$ Scheduling with Release times and deadlines

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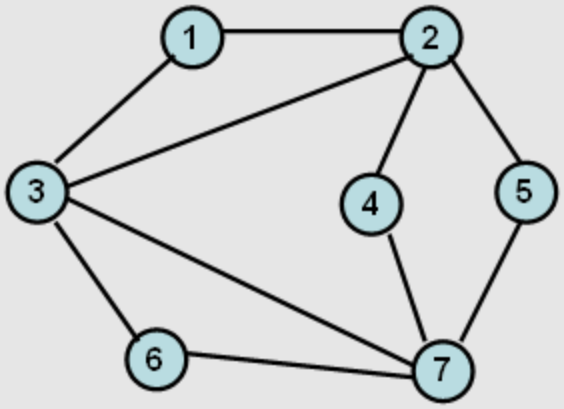
Sample Problems

- Independent Set
  - Graph G = (V, E), a subset S of the vertices is independent if there are no edges between vertices in S



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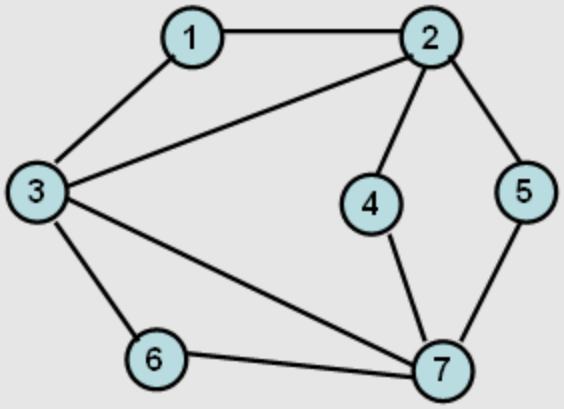
Vertex Cover

- Vertex Cover
  - Graph G = (V, E), a subset S of the vertices is a vertex cover if every edge in E has at least one endpoint in S



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Cook's Theorem

- The Circuit Satisfiability Problem is NP-Complete

- Circuit Satisfiability
  - Given a boolean circuit, determine if there is an assignment of boolean values to the input to make the output true

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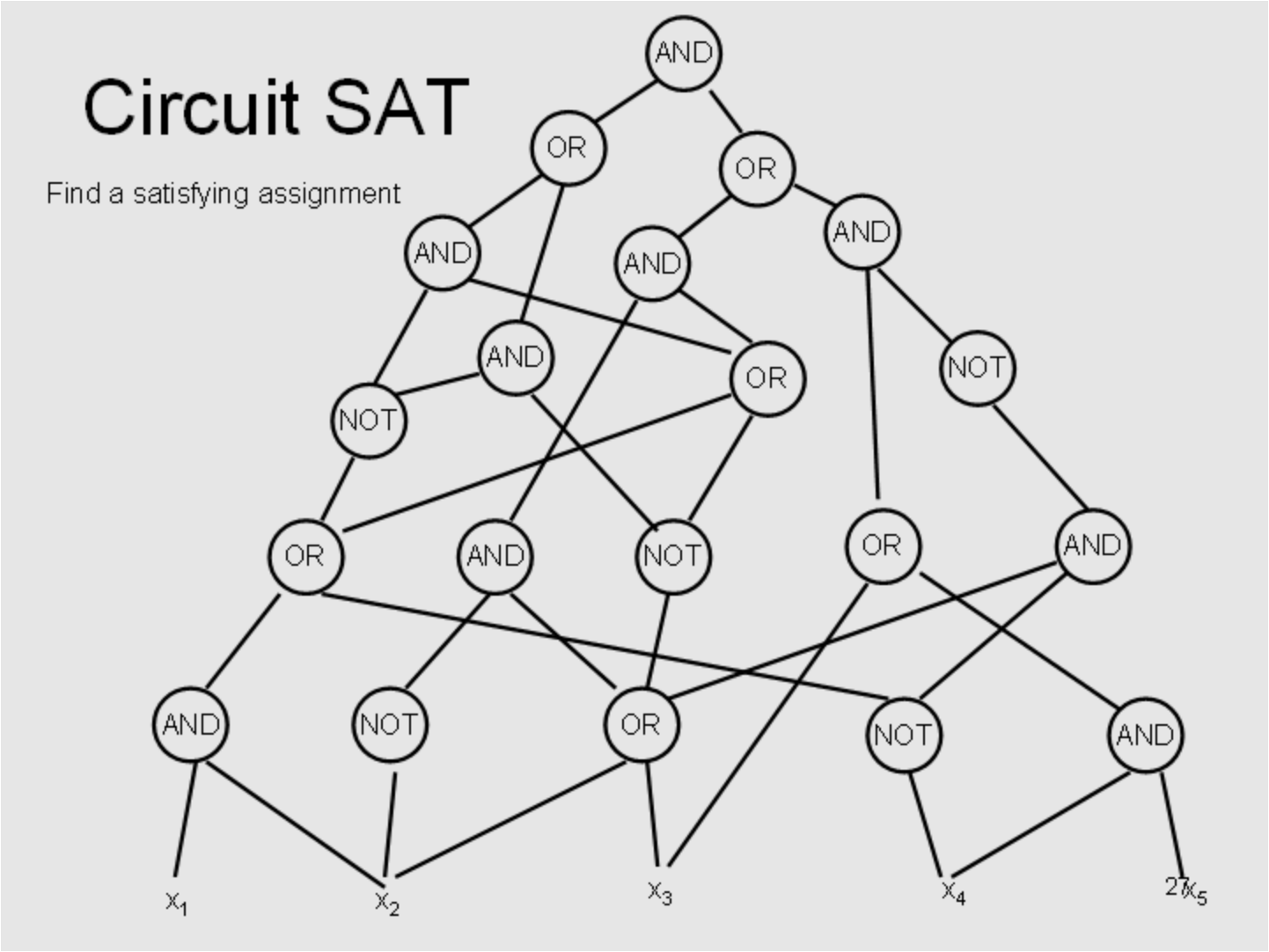
Circuit SAT
Find a satisfying assignment
AND
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NOT
OR
AND
AND
NOT
OR
NOT
AND
x_1
x_2
x_3
x_4
x_5

Proof of Cook's Theorem

- Reduce an arbitrary problem Y in NP to X
- Let A be a non-deterministic polynomial time algorithm for Y
- Convert A to a circuit, so that Y is a Yes instance iff and only if the circuit is satisfiable

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