CSE 431: Introduction to Theory of Computation Spring 2005

Sample Final

Instructions: The exam is on 100 points. There are **five** questions, each worth 20 points. You have 1 hour and 50 minutes to answer all the questions. You are allowed to use one two-sided (at least 11pt font) notes sheet.

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1. For each of the following assertions, state whether they are True, False, or Open according to our current state of knowledge of computability and complexity theory, as described in class. You do *not* have to justify your answer choice.

   (a) $HAMPATH \leq_P PATH$

   (b) $PATH \leq_P HAMPATH$

   (c) The intersection of a decidable language with a Turing-recognizable language is always decidable.

   (d) 0*1* is PSPACE-complete.

   (e) NL = coNL

   (f) P = NP $\cap$ coNP

   (g) All languages in P can decided in $\mathsf{SPACE}(\log^2 n)$.

   (h) There exists an undecidable language L for which $L \leq_m \overline{L}$.

   (i) If P = NP, every language in NP is NP-complete.

   (j) P contains all context-free languages.

2. Define $POLY_{\mathrm{TM}} = \{\langle M\rangle \mid M$ is a Turing machine that runs in polynomial time$\}$. Prove that $POLY_{\mathrm{TM}}$ is undecidable.
(Warning: $POLY_{\mathrm{TM}}$ is **not** the language $\{\langle M\rangle \mid L(M) \in \mathrm{P}\}$.)

3. For a polynomial with integer coefficients in several variables, an integral root is an assignment of integers to the variables so that the polynomial evaluates to 0. For example, the polynomial $2x_1^2x_2 - x_1^3x_2^2$ has an integral root $x_1 = 2$ and $x_2 = 1$, where as the polynomial $x_1^2 + x_2^2 + 1$ has no integral root. Consider the computational problem of determining, given a polynomial with integer coefficients, whether it has an integral root. We can capture this problem by the language:

$$ROOT = \{\langle p\rangle \mid p \text{ is a polynomial in several variables having an integral root}\} .$$

In his famous address at the International Congress of Mathematicians in Paris in 1900, David Hilbert posed 23 mathematical problems as challenges for the 20th century, the 10'th problem of which, re-stated in the language of computability theory, was to give a decider for $ROOT$. We now know, however, that $ROOT$ is in fact undecidable, so no algorithm exists to tell if a polynomial in several variables has an integral root.

In this problem you are to prove a different property of $ROOT$, namely, that $ROOT$ is NP-hard.

(Hint: Try a reduction from 3SAT or SUBSETSUM. In either case, first think about how you can ensure that any potential root must take 0/1 values. The fact that z^2 is always non-negative for real z is very helpful for this problem.)

4. A linear bounded automata (LBA) is a one-tape Turing machine that can only operate within the n input cells (assume that if the machine tries to move its head off either end of the input, the head stays where it is). LBA's are discussed in Section 5.1, pages 177-180, of Sipser's book.
   Let $A_{\text{LBA}} = \{\langle M, w\rangle \mid M \text{ is an LBA that accepts } w\}$.

   (a) Show that $A_{\text{LBA}} \in$ PSPACE. (Warning: Do not forget tomust argue that your proposed polynomial space algorithm always halts.)

   (b) Do we know whether $A_{\text{LBA}} \in$ P? Justify your answer.

5. For a positive integer n, the parity function $parity_n : \{0,1\}^n \to \{0,1\}$ outputs 1 if an odd number of 1s appear in the input variables. Show that $parity_n$ can be computed with $O(n)$ size ciruits (that have NOT gates, and AND/OR gates with fan-in two).
   (Hint: Use a recursive (Divide and conquer) approach)