Review of Eigenvectors and Eigenvalues

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Definition

The eigenvectors **x** and eigenvalues λ of a matrix A satisfy

A**x** = λ**x**

If A is an n x n matrix, then **x** is an n x 1 vector, and λ is a constant.

The equation can be rewritten as $(A - \lambda I)$ **x** = 0, where I is the n x n identity matrix.

Computing Eigenvalues

Since **x** is required to be nonzero, the eigenvalues must satisfy

$\det(A - \lambda I) = 0$

which is called the *characteristic equation*. Solving it for values of λ gives the eigenvalues of matrix A.

2 X 2 Example

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \qquad \text{so } A - \lambda I = \begin{bmatrix} 1 - \lambda & -2 \\ 3 & -4 - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (1 - \lambda)(-4 - \lambda) - (3)(-2)$$
$$= \lambda^2 + 3\lambda + 2$$

Set $\lambda^2 + 3\lambda + 2$ to 0

Then = λ = (-3 +/- sqrt(9-8))/2

So the two values of λ are -1 and -2.

Finding the Eigenvectors

Once you have the eigenvalues, you can plug them into the equation A**x** = λ**x** to find the corresponding sets of eigenvectors **x**.

$$\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{so} \quad \begin{aligned} x_1 - 2x_2 &= -x_1 \\ 3x_1 - 4x_2 &= -x_2 \end{aligned}$$

(1) $2x_1 - 2x_2 = 0$

(2) $3x_1 - 3x_2 = 0$

These equations are not independent. If you multiply (2) by 2/3, you get (1).

The simplest form of (1) and (2) is $x_1 - x_2 = 0$, or just $x_1 = x_2$.

Since $x_1 = x_2$, we can represent all eigenvectors for eigenvalue -1 as multiples of a simple basis vector:

$$E = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$, where t is a parameter.

So $[1\ 1]^T$, $[4\ 4]^T$, $[3000\ 3000]^T$ are all possible eigenvectors for eigenvalue -1.

For the second eigenvalue (-2) we get

$$\begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ so $$\begin{aligned} x_1 - 2x_2 &= -2x_1 \\ 3x_1 - 4x_2 &= -2x_2 \end{aligned}$$

(1)	$3x_1 - 2x_2 = 0$
(2)	$3x_1 - 2x_2 = 0$

so eigenvectors are of the form $t \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Generalization for 2 X 2 Matrices

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\lambda = \dfrac{(a + d) +/- \ \text{sqrt}[(a+d)^2 - 4(ad - bc)]}{2}$

The discriminant (the part under the square root), can be simplified to get $\text{sqrt}[(a-d)^2 + 4bc]$.

If b = c, this becomes $\text{sqrt}[(a-d)^2 + (2b)^2]$
Since the discriminant is the sum of 2 squares, it has real roots.

We will be seeing some 2 x 2 matrices where indeed b = c,
so we'll be guaranteed a real-valued solution for the eigenvalues.

Another observation we will use:

For 2 x 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$\lambda_1 + \lambda_2 = a + d$, which is called trace(A)
and
$\lambda_1\lambda_2 = ad - bc$, which is called det(A).

Finally, zero is an eigenvalue of A if and only if
A is singular and det(A) = 0.