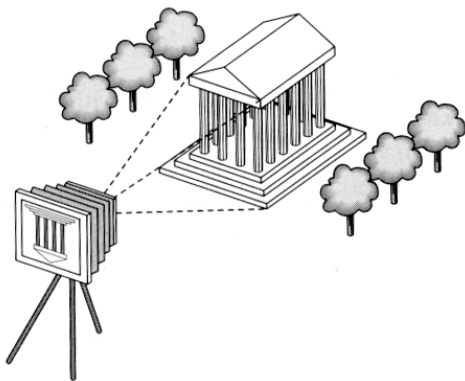# Projections

Reading

Required:

- Watt, Section 5.2.2 – 5.2.4.

Further reading:

- Foley, et al, Chapter 5.6 and Chapter 6
- David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics*, 2nd Ed., McGraw-Hill, New York, 1990, Chapter 2.

The 3D synthetic camera model

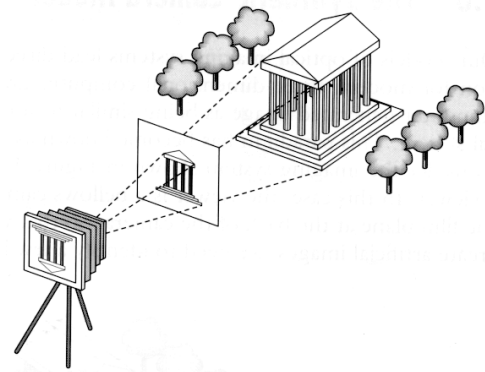The **synthetic camera model** is a paradigm for creating images of 3D geometry.

It involves two components, specified *independently:*

- objects (a.k.a. **geometry**)
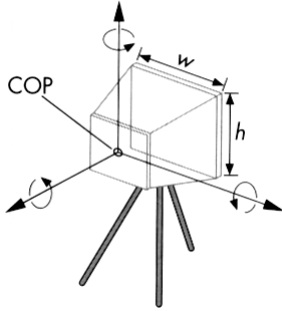- viewer (a.k.a. **camera**)

Imaging with the synthetic camera

The image is rendered onto an **image plane** or **projection plane** (usually in front of the camera).

Projectors emanate from the **center of projection** (COP) at the center of the lens (or pinhole).

The image of an object point P is at the intersection of the projector through P and the image plane.

Specifying a viewer

Camera specification requires four kinds of parameters:

◆ *Position:* the COP.

◆ *Orientation:* rotations about axes with origin at the COP.

◆ *Focal length:* determines the size of the image on the film plane, or the **field of view**.

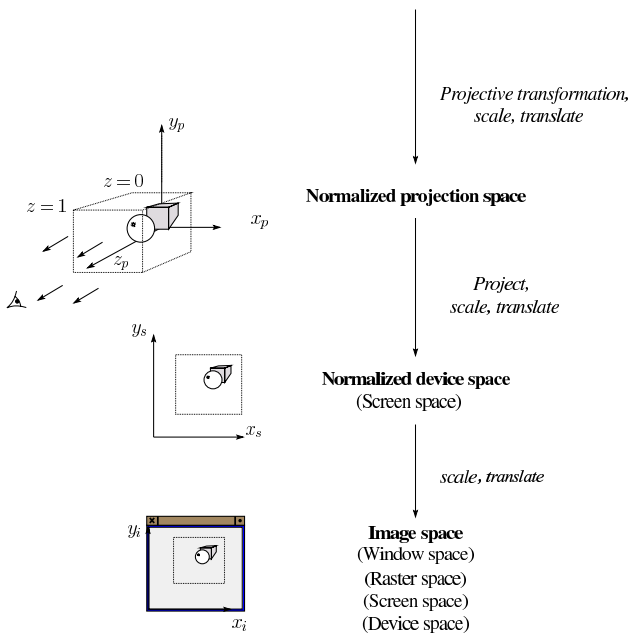◆ *Film plane:* its width and height, and possibly orientation.

3D Geometry Pipeline

Before being turned into pixels, a piece of geometry goes through a number of transformations...
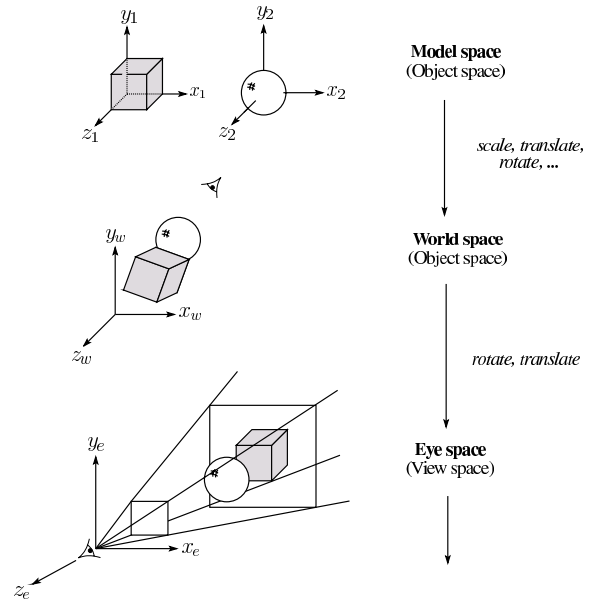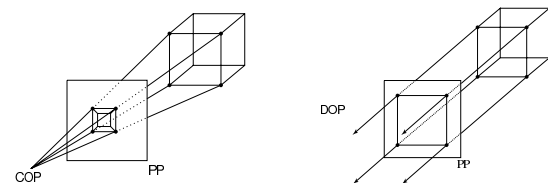
3D Geometry Pipeline (cont'd)

Projections

Projections transform points in n-space to m-space, where $m<n$.

In 3-D, we map points from 3-space to the **projection plane** (PP) along **projectors** emanating from the **center of projection** (COP):

The center of projection is exactly the same as the pinhole in a pinhole camera.

There are two basic types of projections:

◆ Perspective – distance from COP to PP finite

◆ Parallel – distance from COP to PP infinite

Parallel projections

For parallel projections, we specify a **direction of projection** (DOP) instead of a COP.

There are two types of parallel projections:

- **Orthographic projection** – DOP perpendicular to PP
- **Oblique projection** – DOP not perpendicular to PP

We can write orthographic projection onto the $z=0$ plane with a simple matrix.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Normally, we do not drop the z value right away. Why not?

Properties of parallel projection

Properties of parallel projection:

- Not realistic looking
- Good for exact measurements
- Are actually a kind of affine transformation
  - Parallel lines remain parallel
  - Angles not (in general) preserved
- Most often used in CAD, architectural drawings, etc., where taking exact measurement is important

Derivation of perspective projection

Consider the projection of a point onto the projection plane:
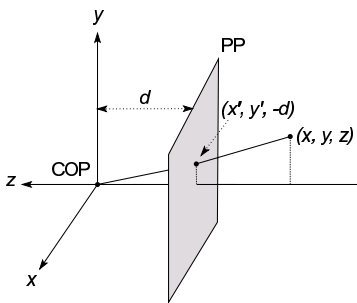
By similar triangles, we can compute how much the x and y coordinates are scaled:

[Note: Watt uses a left-handed coordinate system, and he looks down the $+z$ axis, so his PP is at $+d$.]

Homogeneous coordinates revisited

Remember how we said that affine transformations work with the last coordinate always set to one.

What happens if the coordinate is not one?

We divide all the coordinates by w:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \rightarrow \begin{bmatrix} x/w \\ y/w \\ z/w \\ 1 \end{bmatrix}$$

If $w = 1$, then nothing changes.

Sometimes we call this division step the "perspective divide."

Homogeneous coordinates and perspective projection

Now we can re-write the perspective projection as a matrix equation:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ -z/d \end{bmatrix}$$

After division by w, we get:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -\dfrac{x}{z}d \\ -\dfrac{y}{z}d \\ 1 \end{bmatrix}$$

Again, projection implies dropping the z coordinate to give a 2D image, but we usually keep it around a little while longer.
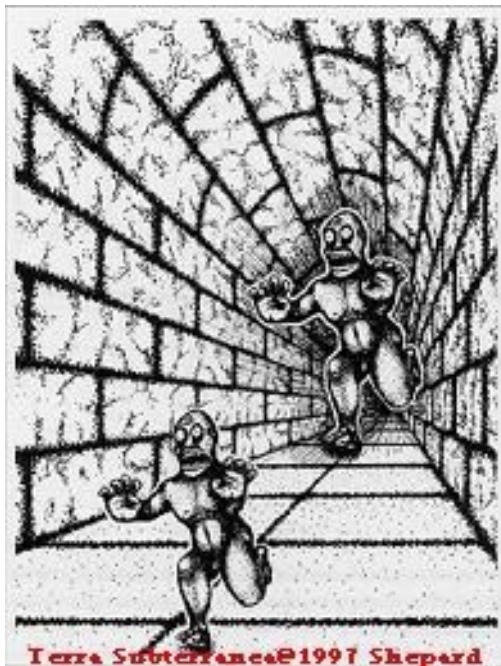
Projective normalization

After applying the perspective transformation and dividing by w, we are free to do a simple parallel projection to get the 2D image.
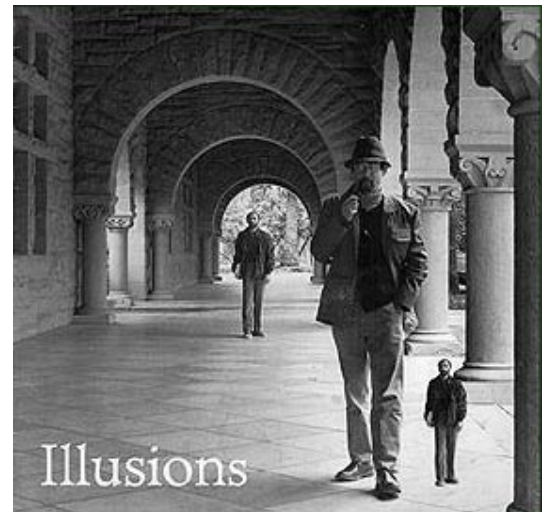
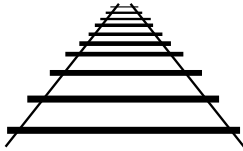What does this imply about the shape of things after the perspective transformation + divide?
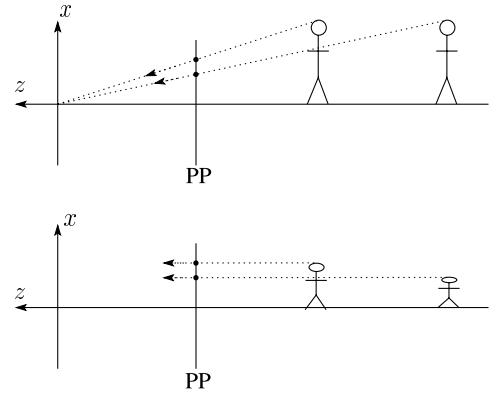
Vanishing points

What happens to two parallel lines that are not parallel to the projection plane?

Think of train tracks receding into the horizon...

The equation for a line is:

$$\mathbf{l} = \mathbf{p} + t\mathbf{v} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix}$$

After perspective transformation we get:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} p_x + tv_x \\ p_y + tv_y \\ -(p_z + tv_z)/d \end{bmatrix}$$

Vanishing points (cont'd)

Dividing by w:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} -\dfrac{p_x + tv_x}{p_z + tv_z} d \\ -\dfrac{p_y + tv_y}{p_z + tv_z} d \\ 1 \end{bmatrix}$$

Letting t go to infinity:

We get a point!

What happens to the line $\mathbf{l} = \mathbf{q} + t\mathbf{v}$?

Each set of parallel lines intersect at a **vanishing point** on the PP.

Q: How many vanishing points are there?

Principal vanishing points

If we define a set of **principal axes** in world coordinates, i.e., the x_w, y_w, and z_w axes, then it's possible to choose the viewpoint such that these axes will converge to different vanishing points.

The vanishing points of the principal axes are called the **principal vanishing points**. Example:
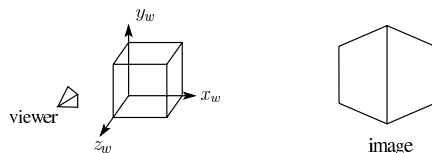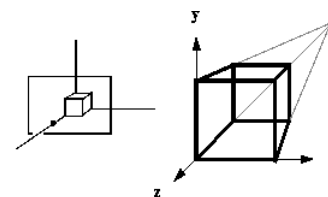

Perspective drawings are often classified by the number of principal vanishing points.

- One-point perspective – simplest to draw
- Two-point perspective – gives better impression of depth
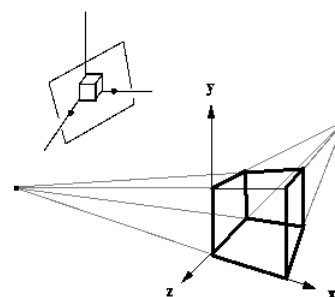- Three-point perspective – most difficult to draw

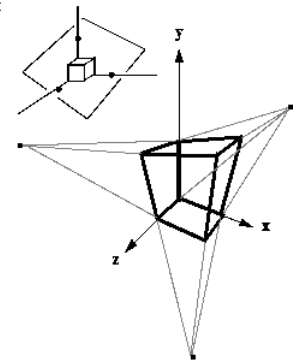All three types are equally simple with computer graphics.

N-point perspective

One Point Perspective
(z-axis vanishing point)

Two Point Perspective
z, and x-axis vanishing points

Three Point Perspective
(z, x, and y-axis vanishing points)

Clipping and the viewing frustum

The center of projection and the portion of the projection plane that map to the final image form an infinite pyramid. The sides of the pyramid are **clipping planes**.

Frequently, additional clipping planes are inserted to restrict the range of depths. These clipping planes are called the **near** and **far** or the **hither** and **yon** clipping planes.
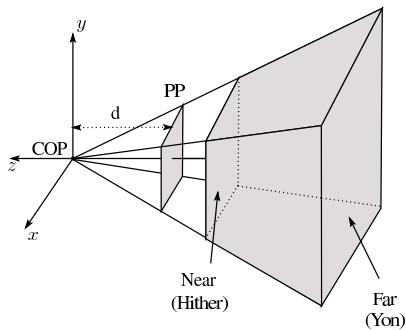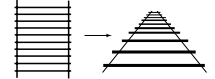

All of the clipping planes bound the the **viewing frustum**.

Properties of perspective projections

The perspective projection is an example of a **projective transformation**.

Here are some properties of projective transformations:

- Lines map to lines
- Parallel lines do <u>not</u> necessarily remain parallel
- Ratios are <u>not</u> preserved

One of the advantages of perspective projection is that size varies inversely with distance – looks realistic.
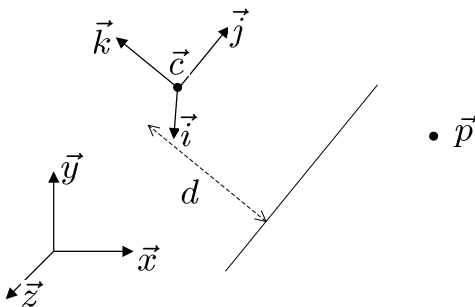
A disadvantage is that we can't judge distances as exactly as we can with parallel projections.

Q: Why did nature give us eyes that perform perspective projections?

Q: Do our eyes "see in 3D"?

General projections

Suppose you have a camera with COP $\vec{c}$, and X, Y, and Z axes are unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ respectively. How to compute the projection?

Summary

What to take away from this lecture:

- All the boldfaced words.
- The synthetic camera model
- An appreciation for the various coordinate systems used in computer graphics.
- How the perspective transformation works.
- How we use homogeneous coordinates to represent perspective projections.
- The classification of different types of projections.
- The concepts of vanishing points and one-, two-, and three-point perspective.
- The mathematical properties of projective transformations.
- General projections