Affine transformations

Brian Curless
CSE 457
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Reading

Required:

- Angel 4.1, 4.6-4.10

Further reading:

- Angel, the rest of Chapter 4
- Foley, et al, Chapter 5.1-5.5.
- David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics*, 2nd Ed., McGraw-Hill, New York, 1990, Chapter 2.

Geometric transformations

Geometric transformations will map points in one space to points in another: $(x', y', z') = \boldsymbol{f}(x, y, z)$.

These transformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.

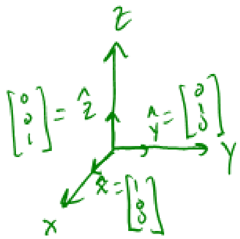
We'll focus on transformations that can be represented easily with matrix operations.

Vector representation

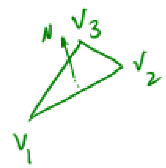
We can represent a **point**, **p** = (x,y), in the plane or p=(x,y,z) in 3D space

- as column vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$
- as row vectors $\begin{bmatrix} x & y \end{bmatrix}$ $\begin{bmatrix} x & y & z \end{bmatrix}$

Canonical axes



Right-handed coord. systems



Vector length and dot products

$$u = \begin{bmatrix}u_x\\u_y\\u_z\end{bmatrix} \qquad \|u\| = \sqrt{u_x^2 + u_y^2 + u_z^2} \qquad \hat{u} = \frac{u}{\|u\|} \text{ unit vector, direction vector}$$

$$v = \begin{bmatrix}v_x\\v_y\\v_z\end{bmatrix}$$

$$u \cdot v = u_x v_x + u_y v_y + u_z v_z$$

$$u \cdot v \overset{?}{=} v \cdot u \quad \text{True}$$

$$u \cdot u = \|u\|^2$$

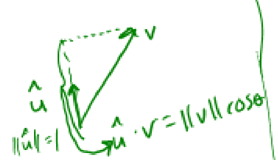
$$u \cdot v = \|u\|\|v\| \cos\theta$$

$$u \cdot v = 0 \Rightarrow \|u\| = 0 \text{ and/or } \|v\| = 0$$

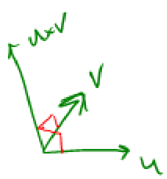
$$\text{and/or } \theta = 90° \text{ or } 270° \quad \perp \text{ Orthogonal}$$

$$\|u\| = \|v\| = 1 \Rightarrow u \cdot v = \cos\theta$$

$$u \cdot v = u^T v = \begin{bmatrix}u_x & u_y & u_z\end{bmatrix}\begin{bmatrix}v_x\\v_y\\v_z\end{bmatrix} = v^T u$$



Vector cross products



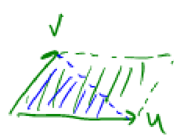
$$u \times v \equiv \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ u_x & u_y & u_z\\ v_x & v_y & v_z\end{vmatrix} = \hat{x}(u_y v_z - u_z v_y) \perp$$
$$\hat{y}(u_z v_x - u_x v_z) \perp$$
$$\hat{z}(u_x v_y - u_y v_x)$$

$$u \times v \overset{?}{=} v \times u \quad \text{False} \qquad u \times v = -v \times u$$

$$(u \times v) \cdot u = 0$$

$$\|u \times v\| = \|u\|\|v\| \sin\theta$$

$$u \times u = 0$$



$$\|u \times v\| = \text{Area } \text{▱}$$
$$= 2 \text{ Area } \triangle$$

Representation, cont.

$$A^{-1}A = I \qquad (AB)^T = B^T A^T$$

We can represent a **2-D transformation** M by a matrix

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

$$(AB)^{-1}(AB) = I$$
$$(AB)^{-1}AB = I$$
$$(AB)^{-1}A\cancel{B}\cancel{B^{-1}} = IB^{-1}$$
$$(AB)^{-1}\cancel{A}\cancel{A^{-1}} = B^{-1}A^{-1}$$
$$(AB)^{-1} = B^{-1}A^{-1}$$

If **p** is a column vector, M goes on the left:

$$\mathbf{p}' = M\mathbf{p}$$

$$\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}ax+by\\ cx+dy\end{bmatrix}$$

If **p** is a row vector, M^T goes on the right:

$$\mathbf{p}' = \mathbf{p}M^T$$

$$\begin{bmatrix}x' & y'\end{bmatrix} = \begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}a & c\\ b & d\end{bmatrix} = \begin{bmatrix}ax+by & cx+dy\end{bmatrix}$$

We will use **column vectors**.

Two-dimensional transformations

Here's all you get with a 2 x 2 transformation matrix *M*:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So:

$$x' = ax + by$$
$$y' = cx + dy$$

We will develop some intimacy with the elements *a, b, c, d…*

Identity

Suppose we choose $a=d=1$, $b=c=0$:

- Gives the **identity** matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Doesn't move the points at all

Scaling

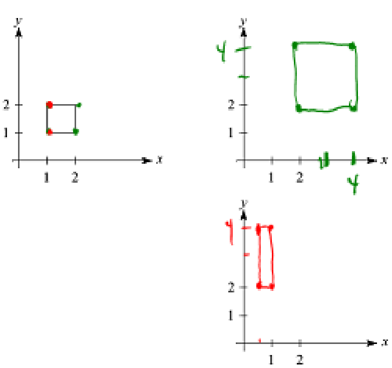
Suppose we set $b=c=0$, but let a and d take on any *positive* value:

- Gives a **scaling** matrix:

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

- Provides **differential (non-uniform) scaling** in x and y:

$$x' = ax$$
$$y' = dy$$



uniform scaling

a=d

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x/2 \\ 2y \end{bmatrix}$$

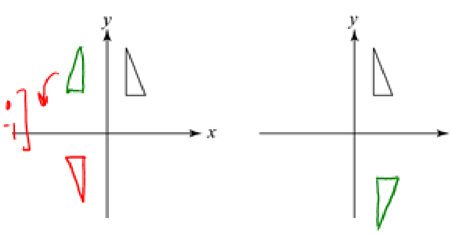
Mirror or reflection

Suppose we keep $b=c=0$, but let either a or d go negative.

Examples:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Shear

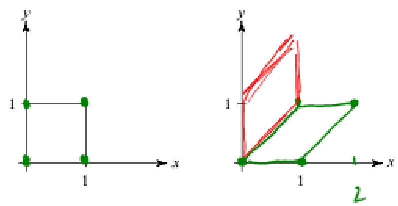
Now let's leave $a=d=1$ and experiment with b. . . .

The matrix

$$\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix}$$

gives:

$$x' = x + by$$
$$y' = y$$



$$\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}x+y\\ y\end{bmatrix}$$

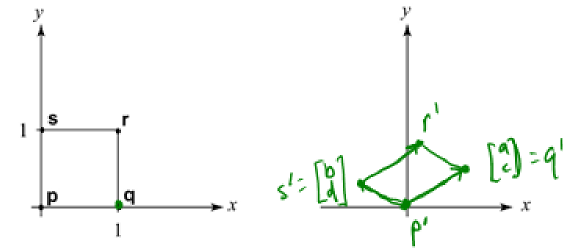
$$\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$$

Effect on unit square

Let's see how a general 2 x 2 transformation M affects the unit square:

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}[\mathbf{p}\ \ \mathbf{q}\ \ \mathbf{r}\ \ \mathbf{s}] = [\mathbf{p'}\ \ \mathbf{q'}\ \ \mathbf{r'}\ \ \mathbf{s'}]$$

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\end{bmatrix} = \begin{bmatrix}0 & a & a+b & b\\ 0 & c & c+d & d\end{bmatrix}$$



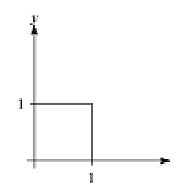
Effect on unit square, cont.

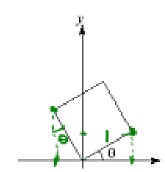
Observe:

- Origin invariant under M
- M can be determined just by knowing how the corners (1,0) and (0,1) are mapped
- a and d give x- and y-scaling
- b and c give x- and y-shearing

Rotation

From our observations of the effect on the unit square, it should be easy to write down a matrix for "rotation about the origin":





- $\begin{bmatrix}1\\ 0\end{bmatrix} \to \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$
- $\begin{bmatrix}0\\ 1\end{bmatrix} \to \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$

Thus,

$$M = R(\theta) = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

Limitations of the 2 x 2 matrix

A 2 x 2 linear transformation matrix allows

- Scaling
- Rotation
- Reflection
- Shearing

Q: What important operation does that leave out?

translation

Homogeneous coordinates

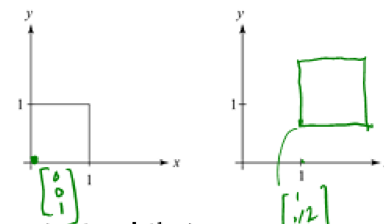
We can loft the problem up into 3-space, adding a third component to every point:

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Adding the third "w" component puts us in **homogenous coordinates**.

Then, transform with a 3 x 3 matrix:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = T(\mathbf{t}) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$

. . . gives **translation**!

Affine transformations

The addition of translation to linear transformations gives us **affine transformations**.

In matrix form, 2D affine transformations always look like this:

$$M = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \mathbf{t} \\ \hline 0 & 0 & 1 \end{array}\right]$$

2D affine transformations always have a bottom row of [0 0 1].

An "affine point" is a "linear point" with an added w-coordinate which is always 1:

$$\mathbf{p}_{\text{aff}} = \begin{bmatrix} \mathbf{p}_{\text{lin}} \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Applying an affine transformation gives another affine point:

$$M\mathbf{p}_{\text{aff}} = \begin{bmatrix} A\mathbf{p}_{\text{lin}} + \mathbf{t} \\ 1 \end{bmatrix}$$

$$M p_{aff} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & t \\ \hline 0 & 0 & 1 \end{array}\right] \begin{bmatrix} p_{lin} \\ 1 \end{bmatrix}$$

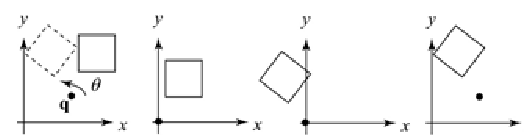
Rotation about arbitrary points

Until now, we have only considered rotation about the origin.

With homogeneous coordinates, you can specify a rotation, θ, about any point $\mathbf{q} = [q_x\ q_y\ 1]^T$ with a matrix:

$$T(v) = \begin{bmatrix} 1 & 0 & v_x \\ 0 & 1 & v_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



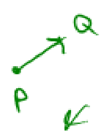
$$M_p \neq T(-q) \cdot R(\theta) \cdot T(q)\ p$$

1. Translate **q** to origin
2. Rotate
3. Translate back

$$M = T(q)\, R(\theta)\, T(-q)$$

<u>Note</u>: Transformation order is important!!

Points and vectors



Vectors have an additional coordinate of w=0. Thus, a change of origin has no effect on vectors.

$Q - P = \begin{bmatrix} Q_x \\ Q_y \\ 1 \end{bmatrix} - \begin{bmatrix} P_x \\ P_y \\ 1 \end{bmatrix} = \begin{bmatrix} Q_x - P_x \\ Q_y - P_y \\ 0 \end{bmatrix}$

Q: What happens if we multiply a vector by an affine matrix?

$$\left[\begin{array}{c|c} A & t \\ \hline 0\ 0 & 1 \end{array}\right]\begin{bmatrix} v_{lin} \\ 0 \end{bmatrix} = \begin{bmatrix} Av_{lin} \\ 0 \end{bmatrix}$$ vector

These representations reflect some of the rules of affine operations on points and vectors:

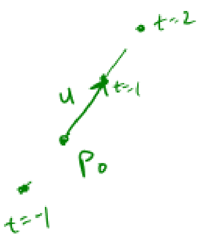
vector + vector	→	vector
scalar · vector	→	vector
point - point	→	vector
point + vector	→	point
point + point	→	chaos

$\alpha \cdot$ point + β point → point if $\alpha + \beta = 1$

→ vector if $\alpha + \beta = 0$

One useful combination of affine operations is:

$$\mathbf{p}(t) = \mathbf{p}_o + t\mathbf{u}$$



Q: What does this describe?

$t \in [-\infty, \infty] \Rightarrow$ line

$t \in [0, \infty] \Rightarrow$ ray (half line)

Basic 3-D transformations: scaling

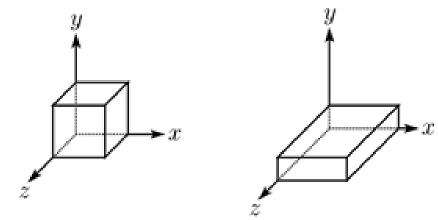
Some of the 3-D affine transformations are just like the 2-D ones.

In this case, the bottom row is always [0 0 0 1].

For example, scaling:

$S(s_x, s_y, s_z)$ $\quad S^{-1}(\,) = S\left(\frac{1}{s_x}, \frac{1}{s_y}, \frac{1}{s_z}\right)$

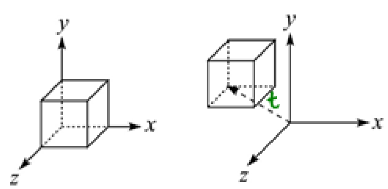
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Translation in 3D

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$T^{-1}(\vec{t}) = T(-\vec{t})$



Rotation in 3D

Rotation now has more possibilities in 3D:

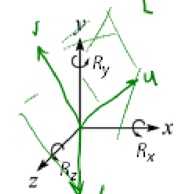
$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 & 0 \\ \sin\gamma & \cos\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R = \begin{bmatrix} u & v & w \end{bmatrix}$

$R^T = \begin{bmatrix} u^T \\ v^T \\ w^T \end{bmatrix}$



Use right hand rule

$R^T R = \begin{bmatrix} u^Tu & u^Tv & u^Tw \\ v^Tu & v^Tv & v^Tw \\ w^Tu & w^Tv & w^Tw \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R^T R = I$

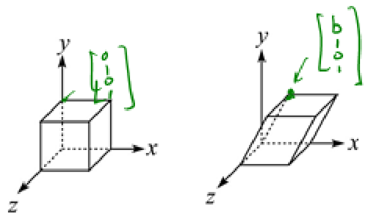
A general rotation can be specified in terms of a product of these three matrices. How else might you specify a rotation?

quaternions (q_x, q_y, q_z, q_w) .... rotation about an arbitrary axis $\hat{v}, \theta$ $\quad R^T = R^{-1}$

Shearing in 3D

Shearing is also more complicated. Here is one example:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

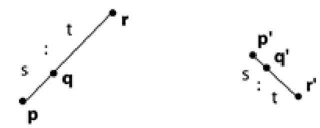


We call this a shear with respect to the x-z plane.

Properties of affine transformations

Here are some useful properties of affine transformations:

- Lines map to lines
- Parallel lines remain parallel
- Midpoints map to midpoints (in fact, ratios are always preserved)



$$\text{ratio} = \frac{\|\mathbf{pq}\|}{\|\mathbf{qr}\|} = \frac{s}{t} = \frac{\|\mathbf{p'q'}\|}{\|\mathbf{q'r'}\|}$$

Affine transformations in OpenGL

OpenGL maintains a "modelview" matrix that holds the current transformation **M.**

The modelview matrix is applied to points (usually vertices of polygons) before drawing.

It is modified by commands including:

- `glLoadIdentity()` $\quad \mathbf{M} \leftarrow \mathbf{I}$
  - set **M** to identity
- `glTranslatef(`t_x`, `t_y`, `t_z`)` $\quad \mathbf{M} \leftarrow \mathbf{MT}$
  - translate by (t_x, t_y, t_z)
- `glRotatef(θ, x, y, z)` $\quad \mathbf{M} \leftarrow \mathbf{MR}$
  - rotate by angle θ about axis (x, y, z)
- `glScalef(`s_x`, `s_y`, `s_z`)` $\quad \mathbf{M} \leftarrow \mathbf{MS}$
  - scale by (s_x, s_y, s_z)

Note that OpenGL adds transformations by *postmultiplication* of the modelview matrix.

Summary

What to take away from this lecture:

- All the names in boldface.
- How points and transformations are represented.
- How to compute lengths, dot products, and cross products of vectors, and what their geometrical meanings are.
- What all the elements of a 2 x 2 transformation matrix do and how these generalize to 3 x 3 transformations.
- What homogeneous coordinates are and how they work for affine transformations.
- How to concatenate transformations.
- The mathematical properties of affine transformations.