Affine transformations

Brian Curless
CSE 457
Spring 2014

Reading

Required:

- Angel 3.1, 3.7-3.11

Further reading:

- Angel, the rest of Chapter 3
- Foley, et al, Chapter 5.1-5.5.
- David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics*, 2nd Ed., McGraw-Hill, New York, 1990, Chapter 2.

Geometric transformations

Geometric transformations will map points in one space to points in another: $(x', y', z') = \boldsymbol{f}(x, y, z)$.

These transformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.

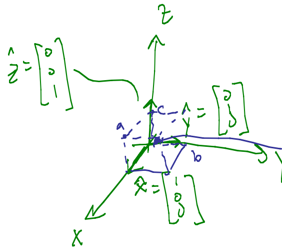
We'll focus on transformations that can be represented easily with matrix operations.

Vector representation

We can represent a **point**, $\mathbf{p} = (x,y)$, in the plane or $\mathbf{p}=(x,y,z)$ in 3D space

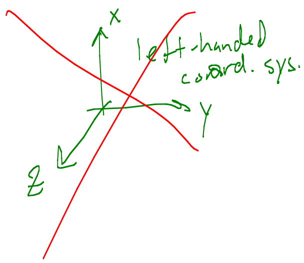
- as column vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$
- as row vectors $\cancel{\begin{bmatrix} x & y \end{bmatrix}}$ $\cancel{\begin{bmatrix} x & y & z \end{bmatrix}}$

Canonical axes



right-hand rule

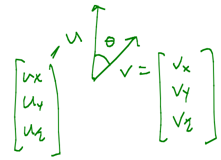
$$a\hat{x} + b\hat{y} + c\hat{z} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$



left-handed coord. sys.

$v^T = [v_x \ v_y \ v_z]$

Vector length and dot products



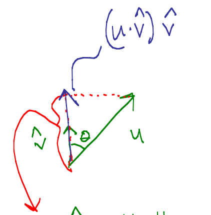
$v \cdot v = \|v\|^2$
$= v^T v$

$\|v\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

$u \cdot v \equiv u_x v_x + u_y v_y + u_z v_z$ ← scalar

→ $u \cdot v = v \cdot u$? yes!

$u^T v = [u_x \ u_y \ u_z]\begin{bmatrix}v_x\\v_y\\v_z\end{bmatrix} = u \cdot v$? yes!



$u \cdot \hat{v} = \|u\| \cos\theta$

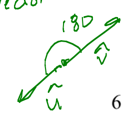
→ $u \cdot v = \|u\| \|v\| \cos\theta$

$u \cdot v = 0 \Rightarrow u \perp v$ or $\|u\|$ or $\|v\| = 0$

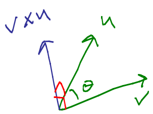
$\hat{u} = \frac{u}{\|u\|}$ $\quad \|\hat{u}\| = 1$ ← normalized vector, unit vector

$\hat{u} \cdot \hat{v} = \cos\theta$

$\hat{u} \cdot \hat{v} = -1 \Rightarrow$



Vector cross products

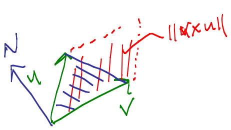


$$v \times u \equiv \det \begin{bmatrix}\hat{x} & \hat{y} & \hat{z}\\ v_x & v_y & v_z\\ u_x & u_y & u_z\end{bmatrix} = \hat{x}(v_y u_z - v_z u_y) + \hat{y}(v_z u_x - v_x u_z) + \hat{z}(v_x u_y - v_y u_x)$$

$$= \begin{bmatrix} v_y u_z - v_z u_y \\ v_z u_x - v_x u_z \\ v_x u_y - v_y u_x \end{bmatrix}$$

$\hat{x} \times \hat{y} = \hat{z}$
(right-handed coord system)

$(v \times u) \cdot v = 0$
$(v \times u) \cdot u = 0$
$v \times u = -u \times v$



$\|v \times u\| = \|u\| \|v\| |\sin\theta|$
$= \text{Area}(\square_{u,v})$

$\text{Area}(\Delta_{u,v}) = \frac{\|v \times u\|}{2}$

$N(\Delta_{u,v}) \sim v \times u$

$u \sim v \quad u = \alpha v$
$\Rightarrow u \times v = 0$

→ $(AB)^T = B^T A^T$

$M^{-1} M = I$

$(AB)^{-1}(AB) = I$
$(AB)^{-1} A B B^{-1} = I \cdot B^{-1}$
$(AB)^{-1} A A^{-1} = B^{-1} \cdot A^{-1}$
$(AB)^{-1} = B^{-1} A^{-1}$
[A, B are invertible matrices]

Representation, cont.

We can represent a **2-D transformation** M by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If **p** is a column vector, M goes on the left:

$p' = Mp \quad (p')^T = (Mp)^T$
$(p')^T = p^T M^T$

$$\mathbf{p}' = M\mathbf{p}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

If **p** is a row vector, M^T goes on the right:

$$\mathbf{p}' = \mathbf{p}M^T$$

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} ax+by & cx+dy \end{bmatrix}$$

We will use **column vectors**.

Two-dimensional transformations

Here's all you get with a 2 x 2 transformation matrix *M*:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So:

$$x' = ax + by$$
$$y' = cx + dy$$

We will develop some intimacy with the elements *a, b, c, d*…

Identity

Suppose we choose *a=d=1, b=c=0:*

- Gives the **identity** matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

- Doesn't move the points at all

$$x' = x$$
$$y' = y$$

Scaling

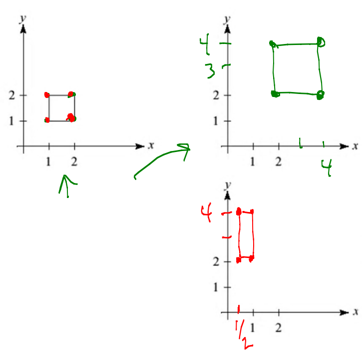
Suppose we set *b=c=0*, but let *a* and *d* take on any *positive* value:

- Gives a **scaling** matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Provides **differential (non-uniform) scaling** in *x* and *y*:

$$\left.\begin{aligned} x' &= ax \\ y' &= dy \end{aligned}\right\}$$



$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$x' = 2x$$
$$y' = 2y$$

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$$

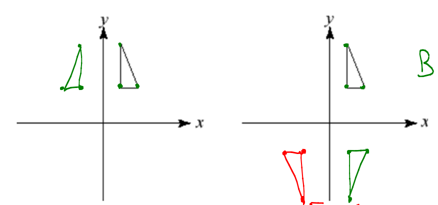
$$x' = \tfrac{1}{2}x$$
$$y' = 2y$$

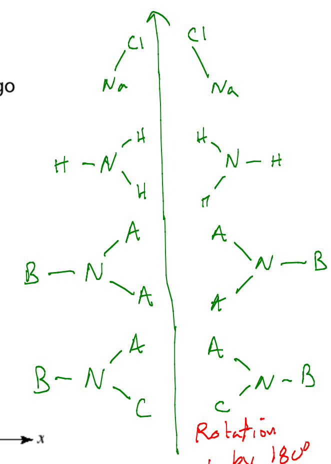
Reflection

Suppose we keep *b=c=0*, but let either *a* or *d* go negative.

Examples:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$





Rotation by 180°

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Chiral center

Shear

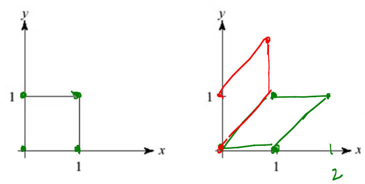
Now let's leave $a=d=1$ and experiment with b. . . .

The matrix

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

gives:

$$\left.\begin{aligned} x' &= x + by \\ y' &= y \end{aligned}\right\}$$



$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$$

$$x' = x + y$$
$$y' = y$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

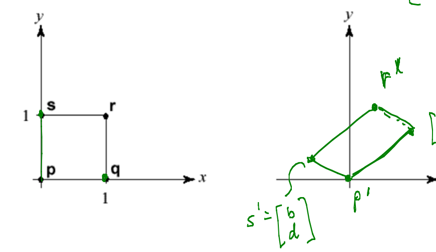
Effect on unit square

Let's see how a general 2 x 2 transformation M affects the unit square:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}[\mathbf{p} \;\; \mathbf{q} \;\; \mathbf{r} \;\; \mathbf{s}] = [\mathbf{p'} \;\; \mathbf{q'} \;\; \mathbf{r'} \;\; \mathbf{s'}]$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{bmatrix}$$

$$= \begin{bmatrix} a \\ c \end{bmatrix} + \begin{bmatrix} b \\ d \end{bmatrix} = q' + s'$$



$$\begin{bmatrix} a \\ c \end{bmatrix} = q'$$

$$s' = \begin{bmatrix} b \\ d \end{bmatrix}$$

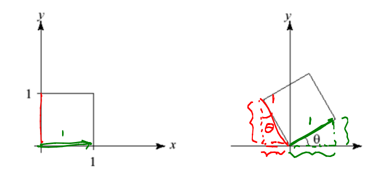
Effect on unit square, cont.

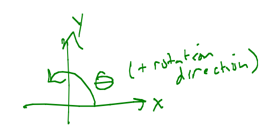
Observe:

- Origin invariant under M
- M can be determined just by knowing how the corners (1,0) and (0,1) are mapped
- a and d give x- and y-scaling
- b and c give x- and y-shearing

Rotation

From our observations of the effect on the unit square, it should be easy to write down a matrix for "rotation about the origin":





$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Thus,

$$M = R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Limitations of the 2 x 2 matrix

A 2 x 2 linear transformation matrix allows

- Scaling
- Rotation
- Reflection
- Shearing

Q: What important operation does that leave out?

Translation

Homogeneous coordinates

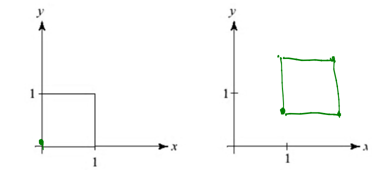
Idea is to loft the problem up into 3-space, adding a third component to every point:

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Adding the third "*w*" component puts us in **homogenous coordinates**.

And then transform with a 3 x 3 matrix:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = T(\mathbf{t}) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$$

. . . gives **translation**!

Anatomy of an affine matrix

The addition of translation to linear transformations gives us **affine transformations**.

In matrix form, 2D affine transformations always look like this:

$$M = \begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{c|c} A_{2\times 2} & \mathbf{t} \\ \hline 0 \quad 0 & 1 \end{array}\right]$$

2D affine transformations always have a bottom row of [0 0 1].

An "affine point" is a "linear point" with an added *w*-coordinate which is always 1:

$$\mathbf{p}_{\text{aff}} = \begin{bmatrix} \mathbf{p}_{\text{lin}} \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Applying an affine transformation gives another affine point:

$$M\mathbf{p}_{\text{aff}} = \begin{bmatrix} A\mathbf{p}_{\text{lin}} + \mathbf{t} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + t_x \\ cx + dy + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \\ 1 \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ 0 \end{bmatrix} = \begin{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} \\ 1 \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ 0 \end{bmatrix}$$

$$p_{lin} = \begin{bmatrix} x \\ y \end{bmatrix}$$

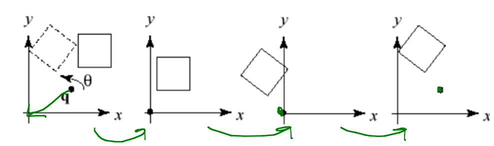
Rotation about arbitrary points

Until now, we have only considered rotation about the origin.

With homogeneous coordinates, you can specify a rotation, q, about any point $\mathbf{q} = [q_x \; q_y]^T$ with a matrix:

$T(\cdot)$ - translate

$R(\cdot)$ - rotate



$$M \neq T(-q) \cdot R(\theta) \cdot T(q)$$

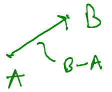
1. Translate **q** to origin
2. Rotate
3. Translate back

$$\underbrace{T(q)\, R(\theta)\, T(-q)} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Note: Transformation order is important!!

$$M = T(q) R(\theta) T(-q)$$

Points and vectors



$\begin{bmatrix} B_x \\ B_y \\ 1 \end{bmatrix} - \begin{bmatrix} A_x \\ A_y \\ 1 \end{bmatrix} = \begin{bmatrix} B_x - A_x \\ B_y - A_y \\ 0 \end{bmatrix}$

Vectors have an additional coordinate of *w*=0. Thus, a change of origin has no effect on vectors.

Q: What happens if we multiply a vector by an affine matrix?

$\begin{bmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} = \begin{bmatrix} a v_x + b v_y \\ c v_x + d v_y \\ 0 \end{bmatrix}$

These representations reflect some of the rules of affine operations on points and vectors:

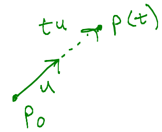
vector + vector	→	vector
scalar · vector	→	vector
point - point	→	vector
point + vector	→	point
point + point	→	chaos

$aP + bQ \to$ point if $a+b=1$

$a\begin{bmatrix} P_x \\ P_y \\ 1 \end{bmatrix} + b\begin{bmatrix} Q_x \\ Q_y \\ 1 \end{bmatrix} = \begin{bmatrix} aP_x + bQ_x \\ aP_y + bQ_y \\ a+b \end{bmatrix}$

One useful combination of affine operations is:

$$\mathbf{p}(t) = \mathbf{p}_o + t\mathbf{u}$$



Q: What does this describe?

$t \in [-\infty, \infty] \Rightarrow$ line

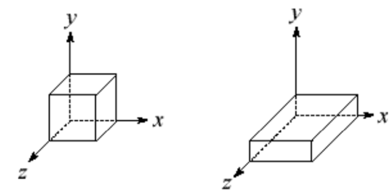
$t \in [0, \infty) \Rightarrow$ half-line or ray

Basic 3-D transformations: scaling

Some of the 3-D transformations are just like the 2-D ones.

For example, scaling:

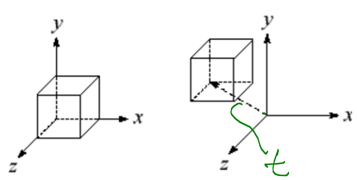
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Translation in 3D

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$t = \begin{bmatrix} t_x \\ t_y \\ t_z \\ 0 \end{bmatrix}$



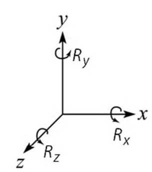
Rotation in 3D (cont'd)

These are the rotations about the canonical axes:

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 & 0 \\ \sin\gamma & \cos\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Use right hand rule

A general rotation can be specified in terms of a product of these three matrices. How else might you specify a rotation?

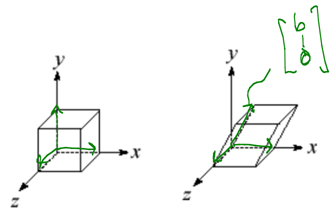
Rotation about a direction

Quaternions ... equivalent to

Shearing in 3D

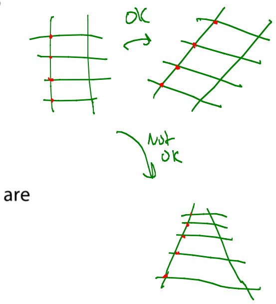
Shearing is also more complicated. Here is one example:

$$\begin{bmatrix}x'\\y'\\z'\\1\end{bmatrix}=\begin{bmatrix}1&b&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\\1\end{bmatrix}$$



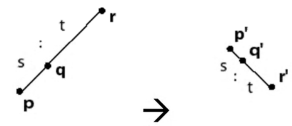
We call this a shear with respect to the x-z plane.

Properties of affine transformations



Here are some useful properties of affine transformations:

- Lines map to lines
- Parallel lines remain parallel
- Midpoints map to midpoints (in fact, ratios are always preserved)



$$\text{ratio} = \frac{\|\mathbf{pq}\|}{\|\mathbf{qr}\|} = \frac{s}{t} = \frac{\|\mathbf{p'q'}\|}{\|\mathbf{q'r'}\|}$$

Affine transformations in OpenGL

OpenGL maintains a “modelview” matrix that holds the current transformation **M.**

The modelview matrix is applied to points (usually vertices of polygons) before drawing.

It is modified by commands including:

- `glLoadIdentity()` **M** ← **I**
  – set **M** to identity
- `glTranslatef(`t_x`,` t_y`,` t_z`)` **M** ← **MT**
  – translate by (t_x, t_y, t_z)
- `glRotatef(`θ`, x, y, z)` **M** ← **MR**
  – rotate by angle θ about axis (x, y, z)
- `glScalef(`s_x`,` s_y`,` s_z`)` **M** ← **MS**
  – scale by (s_x, s_y, s_z)

Note that OpenGL adds transformations by *postmultiplication* of the modelview matrix.

Summary

What to take away from this lecture:

- All the names in boldface.
- How points and transformations are represented.
- How to compute lengths, dot products, and cross products of vectors, and what their geometrical meanings are.
- What all the elements of a 2 x 2 transformation matrix do and how these generalize to 3 x 3 transformations.
- What homogeneous coordinates are and how they work for affine transformations.
- How to concatenate transformations.
- The mathematical properties of affine transformations.