Affine transformations

CSE 457
Winter 2014

Reading

Required:

- Angel 3.1, 3.7-3.11

Further reading:

- Angel, the rest of Chapter 3
- Foley, et al, Chapter 5.1-5.5.
- David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics*, 2^{nd} Ed., McGraw-Hill, New York, 1990, Chapter 2.

Geometric transformations

Geometric transformations will map points in one space to points in another: $(x', y', z') = \boldsymbol{f}(x, y, z)$.

These transformations can be very simple, such as scaling each coordinate, or complex, such as non-linear twists and bends.

We'll focus on transformations that can be represented easily with matrix operations.

Vector representation

We can represent a **point**, $\mathbf{p} = (x,y)$, in the plane or $\mathbf{p}=(x,y,z)$ in 3D space

- as column vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

- as row vectors $\begin{bmatrix} x & y \end{bmatrix}$ $\begin{bmatrix} x & y & z \end{bmatrix}$

Canonical axes

Vector length and dot products

Vector cross products

Representation, cont.

We can represent a **2-D transformation** *M* by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If **p** is a column vector, *M* goes on the left:

$$\mathbf{p}' = M\mathbf{p}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

If **p** is a row vector, M^T goes on the right:

$$\mathbf{p}' = \mathbf{p}M^T$$

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

We will use **column vectors**.

Two-dimensional transformations

Here's all you get with a 2 x 2 transformation matrix *M*:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So:

$$x' = ax + by$$
$$y' = cx + dy$$

We will develop some intimacy with the elements *a, b, c, d*…

Identity

Suppose we choose *a=d=1, b=c=0:*

- Gives the **identity** matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Doesn't move the points at all

Scaling

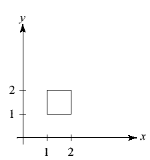
Suppose we set *b=c=0*, but let *a* and *d* take on any *positive* value:

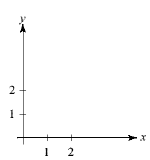
- Gives a **scaling** matrix:

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

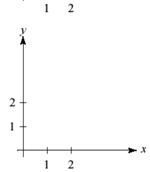
- Provides **differential (non-uniform) scaling** in *x* and *y*:

$$x' = ax$$
$$y' = dy$$





$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

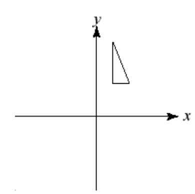


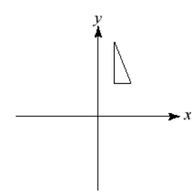
$$\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$$

Suppose we keep *b=c=0*, but let either *a* or *d* go negative.

Examples:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$





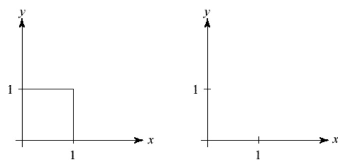
Now let's leave *a=d=1* and experiment with *b*. . . .

The matrix

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

gives:

$$x' = x + by$$
$$y' = y$$



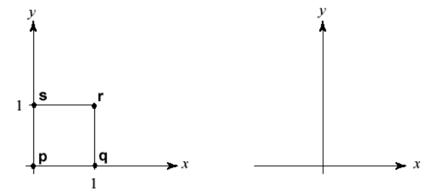
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Effect on unit square

Let's see how a general 2 x 2 transformation *M* affects the unit square:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q & r & s \end{bmatrix} = \begin{bmatrix} p' & q' & r' & s' \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{bmatrix}$$



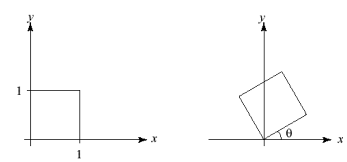
Effect on unit square, cont.

Observe:

- Origin invariant under *M*
- *M* can be determined just by knowing how the corners (1,0) and (0,1) are mapped
- *a* and *d* give *x*- and *y*-scaling
- *b* and *c* give *x*- and *y*-shearing

Rotation

From our observations of the effect on the unit square, it should be easy to write down a matrix for "rotation about the origin":



- $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow$
- $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow$

Thus,

$$M = R(\theta) = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

Limitations of the 2 x 2 matrix

A 2 x 2 linear transformation matrix allows

- Scaling
- Rotation
- Reflection
- Shearing

Q: What important operation does that leave out?

Affine transformations

In order to incorporate the idea that both the basis and the origin can change, we augment the linear space **u**, **v** with an origin **t**.

We call **u**, **v**, and **t** (basis and origin) a **frame** for an **affine space**.

Then, we can represent a change of frame as:

$$\mathbf{p}' = x \cdot \mathbf{u} + y \cdot \mathbf{v} + \mathbf{t}$$

This change of frame is also known as an **affine transformation**.

How do we write an affine transformation with matrices?

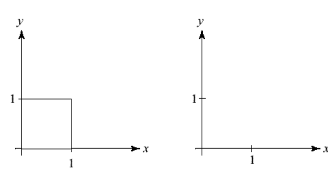
Homogeneous coordinates

Idea is to loft the problem up into 3-space, adding a third component to every point:

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

And then transform with a 3 x 3 matrix:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = T(\mathbf{t}) \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

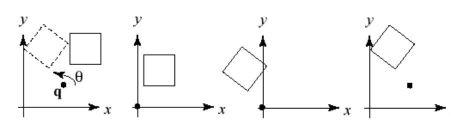


. . . gives **translation**!

Rotation about arbitrary points

Until now, we have only considered rotation about the origin.

With homogeneous coordinates, you can specify a rotation, θ, about any point $\mathbf{q} = [q_x \; q_y]^T$ with a matrix:



1. Translate **q** to origin
2. Rotate
3. Translate back

<u>Note</u>: Transformation order is important!!

Points and vectors

Vectors have an additional coordinate of *w*=0.
Thus, a change of origin has no effect on vectors.

Q: What happens if we multiply a vector by an affine matrix?

These representations reflect some of the rules of affine operations on points and vectors:

$$
\begin{aligned}
&\text{vector} + \text{vector} \rightarrow \\
&\text{scalar} \cdot \text{vector} \rightarrow \\
&\text{point} - \text{point} \rightarrow \\
&\text{point} + \text{vector} \rightarrow \\
&\text{point} + \text{point} \rightarrow
\end{aligned}
$$

One useful combination of affine operations is:

$$\mathbf{p}(t) = \mathbf{p}_o + t\mathbf{u}$$

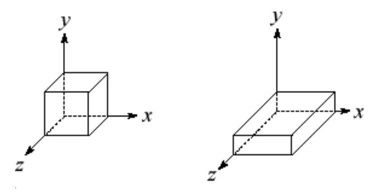
Q: What does this describe?

Basic 3-D transformations: scaling

Some of the 3-D transformations are just like the 2-D ones.

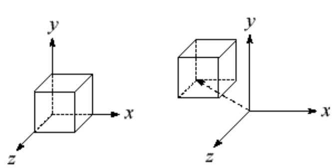
For example, scaling:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Translation in 3D

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



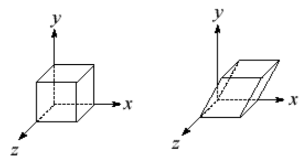
Rotation in 3D

How many degrees of freedom are there in an arbitrary 3D rotation?

Shearing in 3D

Shearing is also more complicated. Here is one example:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

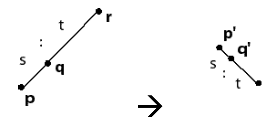


We call this a shear with respect to the x-z plane.

Properties of affine transformations

Here are some useful properties of affine transformations:

- Lines map to lines
- Parallel lines remain parallel
- Midpoints map to midpoints (in fact, ratios are always preserved)



$$\text{ratio} = \frac{\|\mathbf{pq}\|}{\|\mathbf{qr}\|} = \frac{s}{t} = \frac{\|\mathbf{p'q'}\|}{\|\mathbf{q'r'}\|}$$

Affine transformations in OpenGL

OpenGL maintains a “modelview” matrix that holds the current transformation **M.**

The modelview matrix is applied to points (usually vertices of polygons) before drawing.

It is modified by commands including:

- `glLoadIdentity()` **M ← I**
  – set **M** to identity
- `glTranslatef(`t_x`, `t_y`, `t_z`)` **M ← MT**
  – translate by (t_x, t_y, t_z)
- `glRotatef(θ, x, y, z)` **M ← MR**
  – rotate by angle θ about axis (x, y, z)
- `glScalef(`s_x`, `s_y`, `s_z`)` **M ← MS**
  – scale by (s_x, s_y, s_z)

Note that OpenGL adds transformations by *postmultiplication* of the modelview matrix.