Lecture 6. 2D Transformations

Reading

Required:

- Hearn and Baker, Sections 5.1–5.4, 5.6, 6.1–6.3, 6.5

Optional:

- Foley *et al.*, Chapter 5.1–5.5
- David F. Rogers and J. Alan Adams, *Mathematical Elements for Computer Graphics*, Second edition, McGraw-Hill, New York, 1990, Chapter 2.

2D drawing

Think of a program like PowerPoint, Illustrator, MacDraw...

- Interactively create a number of primitives, e.g., polygons and circles.
- Indicate a front-to-back ordering.
- Scale, translate, and rotate objects, as well as group them together.
- Scroll or zoom the "canvas" to look at different parts of the drawing.
- Generate an image and displays it on the screen.

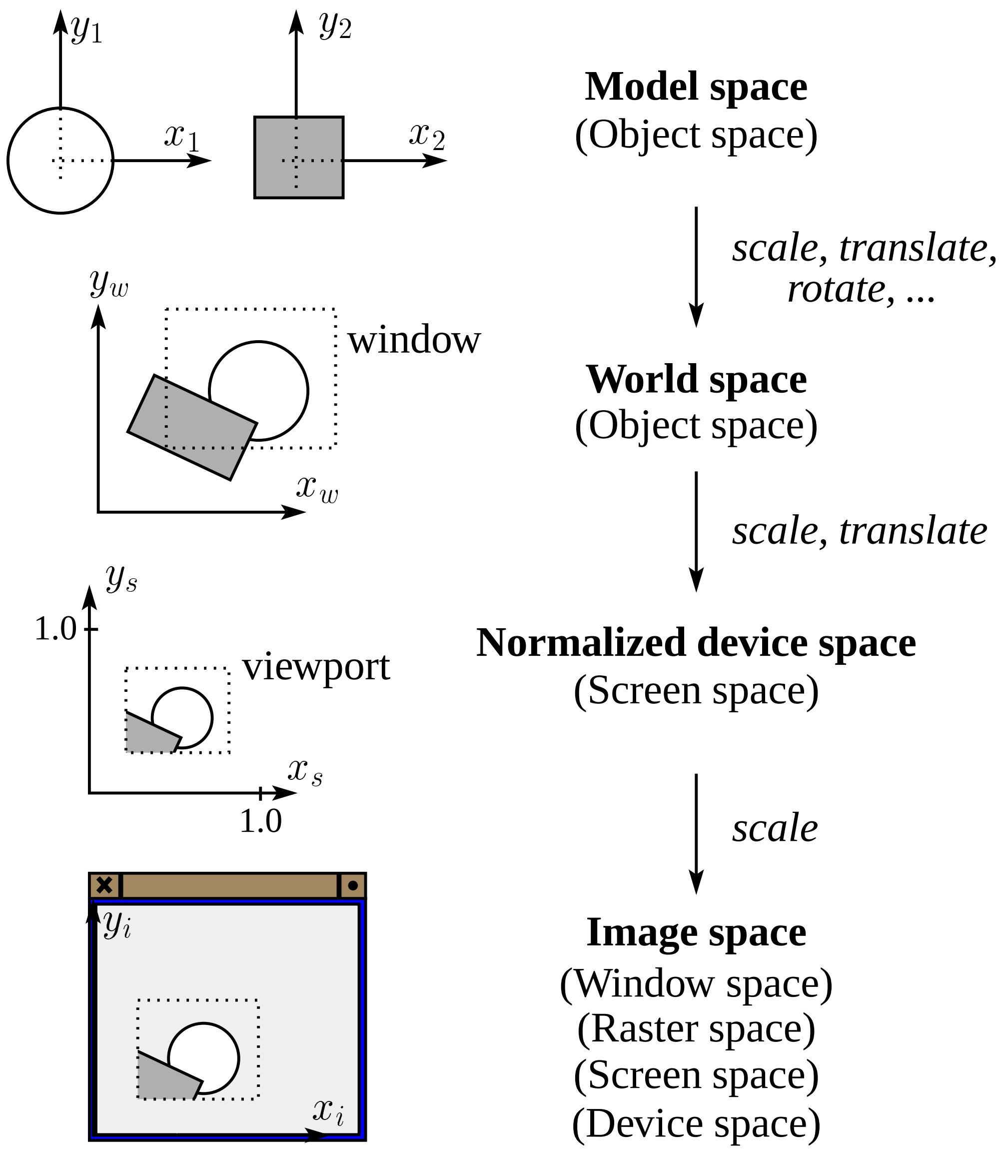
2D drawing, cont'd

What are some of the key ingredients needed to make this work?

- Specification of the front-to-back ordering.
- A sequence of geometric transformations, some of them stored in hierarchies corresponding to groups of primitives.
- Definition of the "visible" portion of the canvas.
- A mapping from the visible portions of the canvas to pixels on the screen.
- Software or hardware that is able to "rasterize" the primitives, i.e., draw the pixels corresponding to the primitives.

2D geometry pipeline

Let's think about this in terms of a set of coordinate systems:



Clipping

To avoid drawing primitives or parts of primitives that do not appear in the viewport, we perform "clipping".

Clipping includes:

- Removal of primitives wholly outside of the viewport (a.k.a., "culling")
- Intersection of the viewport with primitives that straddle the viewport boundary.

Clipping can happen:

- In world space
- In normalized device space
- In image space

A simple OpenGL example

Here's an example of an OpenGL program that will draw a black square over a white background:

```
makeADrawingWindow();
glOrtho(xw_min, xw_max, yw_min, yw_max, -1.0, 1.0);
glViewport(xi_min, yi_min, width_i, height_i);
glClearColor(1.0, 1.0, 1.0, 0.0);
glClear(GL_COLOR_BUFFER_BIT);
glColor3f(0.0, 0.0, 0.0);
glBegin(GL_POLYGON);
  glVertex2f(0.0, 0.0);
  glVertex2f(1.0, 0.0);
  glVertex2f(1.0, 1.0);
  glVertex2f(0.0, 1.0);
glEnd();
glFlush();
```

For the remainder of this lecture, we will focus on 2D geometric transformations...

Representation

We can represent a **point** $p = (x, y)$ in the plane

- as a column vector $\begin{bmatrix} x \\ y \end{bmatrix}$
- as a row vector $\begin{bmatrix} x & y \end{bmatrix}$

Representation, cont.

We can represent a **2-D transformation** M by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If p is a column vector, M goes on the left:

$$p' = M\,p$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

If p is a row vector, M^{T} goes on the right:

$$p' = p\,T$$

$$\begin{bmatrix} x' & y' \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

We will use **column vectors**.

Two-dimensional transformations

Here's all you get with a 2×2 transformation matrix M:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$

We will develop some intimacy with the elements a, b, c, d. . . .

Identity

Suppose we choose $a = d = 1$, $b = c = 0$:

- Gives the "identity" matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Doesn't move the points at all

Scaling

Suppose we set $b = c = 0$, but let a and d take on any positive value:

- Gives a "scaling" matrix

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

- Provides differential scaling in x and y:

$$\begin{aligned} x' &= ax \\ y' &= dy \end{aligned}$$

Suppose we keep $b = c = 0$, but let a and d go negative.

Examples:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Now let's leave $a = d = 1$ and experiment with c. . . .

The matrix

$$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$$

gives:

$$\begin{aligned} x' &= x \\ y' &= cx + y \end{aligned}$$

Effect is called a "________."

Effect on unit square

Let's see how a general 2×2 transformation M affects the unit square:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{bmatrix}$$

Effect on unit square, cont.

Observe:

- Origin invariant under M
- M can be determined just by knowing how the corners $(1, 0)$ and $(0, 1)$ are mapped
- a and d give x- and y-scaling
- b and c give x- and y-shearing

Rotation

From our observations of the effect on the unit square, it should be easy to write down a matrix for "rotation about the origin":

- $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow$
- $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow$

Limitations of the 2×2 matrix

A 2×2 matrix allows

- Scaling
- Rotation
- Reflection
- Shearing

Q: What important operation does that leave out?

Homogeneous coordinates

Idea is to loft the problem up into 3-space, adding a third component to every point:

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

And then transform with a 3×3 matrix:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

. . . Gives **translation**!

Rotation about arbitrary points

Until now, we have only considered rotation about the origin.

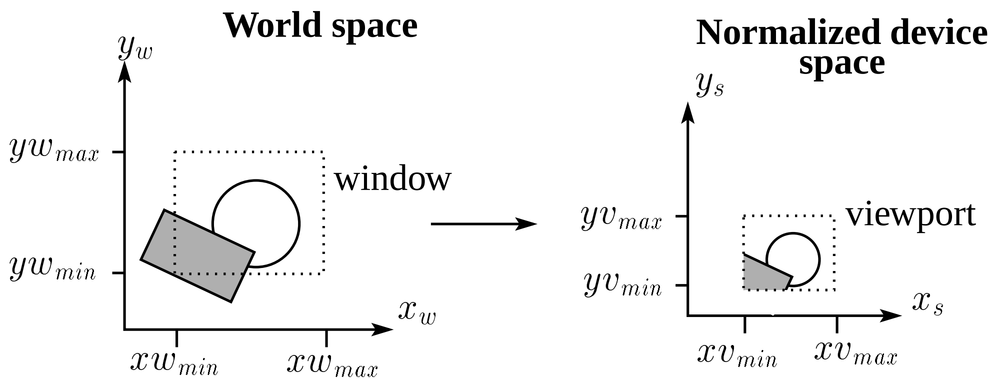
With homogeneous coordinates, you can specify rotations about any point q with a matrix:

1. Translate q to origin
2. Rotate
3. Translate back to q

Note: Transformation order is important!

Window-to-viewport transformation

How do we transform from the window in world coordinates to the viewport in screen space?



Mathematics of affine transformations

All of the transformations we've looked at so far are examples of "affine transformations."

Here are some useful properties of affine transformations:

- Lines map to lines
- Parallel lines remain parallel
- Midpoints map to midpoints
  (in fact, ratios are always preserved)

Summary

What to take away from this lecture:

- All the underlined names and names in quotations.
- How points and transformations are represented.
- What all the elements of a 2×2 transformation matrix do.
- What homogeneous coordinates are and how they work for affine transformations.
- How to concatenate transformations.
- The mathematical properties of affine trasnformations.